Differentials as Tensors and the Taylor Series

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Abstract

The article considers tensor transformations of the differential elements pertaining to the coordinates to derive that the nth order differential behaves as a tensor. This stands in contradiction to accepted notions. It also derives by considering the Taylor series, a significant and an unexpectedly conflicting result that the nth order partial derivatives of scalar functions with respect to the coordinates behave like rank ‘n‘covariant tensors.

Introduction

It is known to us from standard theory that first order differentials of the coordinates behave as contravariant tensors of rank one. Derivatives of scalar functions with respect to the coordinates behave as rank one covariant tensors. In this article we derive the stunning facts that higher order differentials[pertaining to the coordinates] do behave as rank one contravariant tensors. By considering the Taylor series we prove that the nth order partial derivative of scalar functions[with respect to the coordinates] behave as rank n covariant tensors

Higher order Differentials as Tensors

We consider the definition of the rank one contravariant tensor[1]

$$d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha} \ (1)$$

$$d^2\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^2x^{\alpha} + d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) dx^{\alpha} \ \ (2)$$

Each term with equation (2) is a second order differential. But this is a delusion since due to cancellation effect between the term to the left and the first term on the right side we have only one second order differential with equation (2). Let us check

Equation (1) may be written as

$$\bar{x}^{\mu\prime} - \bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} (x^{\alpha\prime} - x^{\alpha}) \ (3)$$

[prime dos not denote differentiation]

Taking differentials on either side we have,

$$d(\bar{x}^{\mu\prime} - \bar{x}^{\mu}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d(x^{\alpha\prime} - x^{\alpha}) + (x^{\alpha\prime} - x^{\alpha}) d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$$

$$d\bar{x}^{\mu\prime} - d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha\prime} - \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha} + dx^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) \text{ (4)}$$

Applying equation (1) on (4) we have,

$$d\bar{x}^{\mu\prime} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha\prime} + dx^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) \text{ (5)}$$

In the above $dx^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$ is a second order differential while the other terms are first order differentials. Therefore

$$d\bar{x}^{\mu\prime} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} dx^{\alpha\prime} \text{ (6)}$$

From (1) and (6) we have

$$d\bar{x}^{\mu\prime} - d\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} (dx^{\alpha\prime} - dx^{\alpha}) \text{ (7)}$$

$$\Rightarrow d^2\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^2 x^{\alpha} \text{ (8)}$$

Taking differentials on either side of (7) we have,

$$d\left(d\bar{x}^{\mu\prime} - d\bar{x}^{\mu}\right) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d(dx^{\alpha\prime} - dx^{\alpha}) + (dx^{\alpha\prime} - dx^{\alpha}) d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$$

$$d^2\bar{x}^{\mu\prime} - d^2\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \left(d^2 x^{\alpha\prime} - d^2 x^{\alpha}\right) + d^2 x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$$

In the above equation $d^2 x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$ is a third order differential while the other terms are second order differentials. Therefore, we may ignore $d^2 x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$ [especially in the limit]

$$d^2\bar{x}^{\mu\prime} - d^2\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \left(d^2 x^{\alpha\prime} - d^2 x^{\alpha}\right)$$

$$d^3\bar{x}^{\mu\prime} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^3 x^{\alpha\prime} \text{ (9)}$$

If

$$d^{n-1}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}x^{\alpha} \quad (10)$$

then

we may show that

$$d^{n}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n}x^{\alpha} \ (11)$$

Proof:

Taking differentials on either side of (10) we obtain

$$d^{n}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n}x^{\alpha} + d^{n-1}x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right)$$

$$d^{n-1}\bar{x}^{\mu\prime} - d^{n-1}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\left(d^{n-1}\bar{x}^{\alpha\prime} - d^{n-1}\bar{x}^{\alpha}\right) + d^{n-1}x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) \ (12)$$

$$d^{n-1}\bar{x}^{\mu\prime} - d^{n-1}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}\bar{x}^{\alpha\prime} - \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}\bar{x}^{\alpha\prime} + d^{n-1}x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) \ (13)$$

Applying (10) on (12) we obtain

$$d^{n-1}\bar{x}^{\mu\prime} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}\bar{x}^{\alpha\prime} + d^{n-1}x^{\alpha} d\left(\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\right) \quad (14)$$

With equation (14) the second term on the right side is an nth order differential while the others are of n-1 th order. Therefore we do have

$$d^{n-1}\bar{x}^{\mu\prime} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}\bar{x}^{\alpha\prime} \ (15)$$

$$d^{n-1}\bar{x}^{\mu\prime} - d^{n-1}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}\bar{x}^{\alpha\prime} - \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n-1}x^{\alpha}$$

From equations (10) and (15) we have

$$d^{n}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\left(d^{n-1}\bar{x}^{\alpha\prime} - d^{n-1}x^{\alpha}\right)$$

Thus we arrive at our intended equation that is at (11):

$$d^{n}\bar{x}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} d^{n}x^{\alpha} \quad (16)$$

$$\frac{d^{n}\bar{x}^{\mu}}{h^{n}} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{d^{n}x^{\alpha}}{h^{n}}$$

$h = d\lambda$ where λ is an invariant parameter like the line element. In the difference table for $'\lambda'$ the various values of λ may or may be unequally spaces. For subsequent considerations we consider equally space t. The various values of x^αcorrespond to the several values of 'λ'. Line element squared $ds^2 = c^2 d\tau^2 \Rightarrow \tau$ is an invariant parameter even in the n-dimensional case

Since h is a constant $\frac{d^n x^\alpha}{h^n} \leftrightarrow \frac{d^n \bar{x}^\mu}{h^n}$ is also a tensor.

$$lim_{h\to 0} \frac{\Delta^n \bar{x}^\mu}{h^n} = lim_{h\to 0} \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\Delta^n x^\alpha}{h^n}$$

$$lim_{h\to 0} \frac{d^n \bar{x}^\mu}{h^n} = \frac{d^n \bar{x}^\mu}{d\bar{x}^{\mu n}}; \frac{d^n x^\alpha}{h^n} = lim_{h\to 0} \frac{d^n x^\alpha}{dx^{\alpha n}} \quad (17)$$

Now we have,

$$\frac{d^n \bar{x}^\mu}{d\tau^n} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{d^n x^\alpha}{d\tau^n} \quad (18)$$

Therefore

$$\frac{d^n x^\alpha}{d\tau^n} \leftrightarrow \frac{d^n \bar{x}^\mu}{d\tau^n}$$

is a tensor.From (1)

$$\frac{d\bar{x}^\mu}{d\tau} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{dx^\alpha}{d\tau}$$

Again by differentiation

$$\frac{d^2 \bar{x}^\mu}{d\tau^2} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{d^2 x^\alpha}{d\tau^2} + \frac{d}{d\tau} \left[\frac{\partial \bar{x}^\mu}{\partial x^\alpha} \right] \frac{dx^\alpha}{d\tau}$$

The last equation stands in contradiction to (18) for n=2 if the second term on the right side is non zero.

Tensors and Taylor Series

We start with scalars and their differentiation

$$\frac{\partial \bar{\varphi}}{\partial \bar{x}^\mu} = \frac{\partial \bar{\varphi}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \quad (19)$$

Since by scalar property $\bar{\varphi}(\bar{x}^\mu) = \varphi(x^\alpha)$ we have the standard result

$$\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial\varphi}{\partial x^{\alpha}}\,(20)$$

which embodies the information that the derivative of a scalar is a rank one covariant tensor.

By differentiation (20) with respect to $\bar{x}^{\nu}$ we obtain

$$\frac{\partial}{\partial\bar{x}^{\nu}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\right)=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial}{\partial\bar{x}^{\nu}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)+\frac{\partial}{\partial\bar{x}^{\nu}}\left(\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\right)\frac{\partial\varphi}{\partial x^{\alpha}}\;\;(21)$$

$$\frac{\partial}{\partial x^{\gamma}}\left[\frac{\partial}{\partial\bar{x}^{\nu}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\right)\right]$$
$$=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial}{\partial x^{\gamma}}\left[\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)\right]+\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\right)\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)+\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\right)\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)$$
$$+\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\right)\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)\;\;(22)$$

Taylor expansion of the scalar $\bar{\varphi}(\bar{x}^{\mu})=\varphi(x^{\alpha})$

$$\varphi(x^{\alpha}+\Delta x^{\alpha}:\alpha=1,2,3\,\ldots.n)$$
$$=\varphi(x^{\alpha})+\frac{1}{1!}\frac{\partial\varphi}{\partial x^{\alpha}}\Delta x^{\alpha}+\frac{1}{2!}\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}\Delta x^{\alpha}\Delta x^{\beta}+\frac{1}{3!}\frac{\partial^{2}\varphi}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}\;\ldots..(23)$$

$$\bar{\varphi}(\bar{x}^{\mu}+\Delta\bar{x}^{\mu}:\mu=1,2,3\,\ldots.n)$$
$$=\bar{\varphi}(\bar{x}^{\mu})+\frac{1}{1!}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\Delta\bar{x}^{\mu}+\frac{1}{2!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\Delta\bar{x}^{\mu}\Delta\bar{x}^{\nu}+\frac{1}{3!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\Delta\bar{x}^{\mu}\Delta\bar{x}^{\nu}\Delta\bar{x}^{\rho}\;\ldots..(24)$$

$$\bar{\varphi}(\bar{x}^{\mu}+\Delta\bar{x}^{\mu}:\mu=1,2,3\,\ldots.n)$$
$$=\bar{\varphi}(\bar{x}^{\mu})+\frac{1}{1!}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\Delta x^{\alpha}+\frac{1}{2!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\Delta x^{\alpha}\Delta x^{\beta}$$
$$+\frac{1}{3!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}\;\ldots..(25)$$

Since

$$\varphi(x^{\alpha}+\Delta x^{\alpha}:\alpha=1,2,3\,\ldots.n)=\bar{\varphi}(\bar{x}^{\mu}+\Delta\bar{x}^{\mu}:\mu=1,2,3\,\ldots.n)$$

we have from (24) and (25)

$$\varphi(x^{\alpha})+\frac{1}{1!}\frac{\partial\varphi}{\partial x^{\alpha}}\Delta x^{\alpha}+\frac{1}{2!}\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}\Delta x^{\alpha}\Delta x^{\beta}+\frac{1}{3!}\frac{\partial^{2}\varphi}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}$$
$$=\bar{\varphi}(\bar{x}^{\mu})+\frac{1}{1!}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\Delta x^{\alpha}+\frac{1}{2!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\Delta x^{\alpha}\Delta x^{\beta}$$
$$+\frac{1}{3!}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}$$

$$\frac{1}{1!}\left(\frac{\partial\varphi}{\partial x^{\alpha}}-\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\right)\Delta x^{\alpha}+\frac{1}{2!}\left(\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}-\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\right)\Delta x^{\alpha}\Delta x^{\beta}+\frac{1}{3!}\left(\frac{\partial^{2}\varphi}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}}-\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\right)\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}+\cdots\ldots\ldots=0$$

Keeping in the mind that $\Delta x^{\alpha}, \Delta x^{\beta}. \Delta x^{\gamma}$ …. .. are arbitrary to the extent equations (23) or (25) do not diverge we should have

$$\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}=\frac{\partial\varphi}{\partial x^{\alpha}}\quad(26.1)$$

$$\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\ (26.2)$$

$$\frac{\partial^{2}\varphi}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial\bar{x}^{\rho}}{\partial x^{\gamma}}\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\rho}\partial\bar{x}^{\mu}\partial\bar{x}^{\nu}}\quad(26.3)$$

etc etc, which means that $\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}, \frac{\partial^{2}\varphi}{\partial x^{\gamma}\partial x^{\beta}\partial x^{\alpha}}$,…. are respectively rank two, rank three….. covaraiant Tensors. That is not true

Even if the range of the convergence for 23 or (25) is narrow but of a continuous nature we could think of an infinite number of possible values for $\Delta x^{\alpha}, \Delta x^{\beta}, \Delta x^{\gamma}$……..

Equation (26.1) holds ; it expresses the fact that the first order derivative is a covariant vector of rank one.

$$\frac{\partial\varphi}{\partial x^{\alpha}}=\frac{\partial\varphi}{\partial\bar{x}^{\mu}}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}$$

Let us try verifying (26.2)

$$\frac{\partial\varphi}{\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}$$

$$\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial\varphi}{\partial x^{\alpha}}\right)=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\right)+\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\right)\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}$$

$$\frac{\partial^{2}\varphi}{\partial x^{\beta}\partial x^{\alpha}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial}{\partial\bar{x}^{\nu}}\left(\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}\right)\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}+\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\right)\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}$$

$$\frac{\partial^{2}\varphi}{\partial x^{\alpha}\partial x^{\beta}}=\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\left(\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\right)+\frac{\partial^{2}\bar{x}^{\mu}}{\partial x^{\beta}\partial x^{\alpha}}\frac{\partial\bar{\varphi}}{\partial\bar{x}^{\mu}}$$

$$\frac{\partial^{2}\varphi}{\partial x^{\alpha}\partial x^{\beta}}\neq\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}\left(\frac{\partial^{2}\bar{\varphi}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\mu}}\right)$$

Equation (26.2) fails in the process of verification

Conclusion

As claimed at the outset we have proved the stunning facts that higher order differentials of the coordinates do behave as rank one contravariant tensors. By considering the Taylor series we have proved that the nth order partial derivative of scalar functions behave as rank n covariant tensors. These notions stand against accepted beliefs.

References

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