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ON CHOWLA'S CONJECTURE

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ABSTRACT. In this paper, using the area method, we show that there exist some $0 < \epsilon < 1$ such that

$$\sum_{n \leq x} \lambda(n)\lambda(n+1) \ll x^{1+\epsilon} e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}.$$

The Chowla conjecture which asserts that

$$\sum_{n \leq x} \lambda(n)\lambda(n+1) = o(x)$$

as $x \longrightarrow \infty$ then follows from this estimate.

1. Introduction and statement

Let $\lambda : \mathbb{N} \longrightarrow \mathbb{R}$ be the Liouville function, defined by $\lambda(n) = (-1)^{\Omega(n)}$, where

$$\Omega(n) := \sum_{p^{\alpha}||n} \alpha$$

is the number of prime factors counting multiplicity function. It is a well-known conjecture of chowla that

$$\sum_{n \leq x} \lambda(n)\lambda(n+1) = o(x)$$

as $x \longrightarrow \infty$. A recent progress on this conjecture affirms that

$$\sum_{n \leq x} \frac{\lambda(n)\lambda(n+1)}{n} = o(\log x)$$

as $x \longrightarrow \infty$ [1]. Using a different method, we establish the upper bound

$$\sum_{n \leq x} \lambda(n)\lambda(n+1) \ll x^{1+\epsilon} e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}$$

for some $\epsilon \in (0, 1)$. The chowla conjecture therefore follows from this estimate as $x \longrightarrow \infty$.

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2. The area method

In this section we introduce and develop a fundamental method for solving problems related to correlations of arithmetic functions. This method is fundamental in the sense that it uses the properties of four main geometric shapes, namely the triangle, the trapezium, the rectangle and the square. The basic identity we will derive is an outgrowth of exploiting the areas of these shapes and putting them together in a unified manner.

Theorem 2.1. *Let $\{r_j\}_{j=1}^{n}$ and $\{h_j\}_{j=1}^{n}$ be any sequence of real numbers, and let r and h be any real numbers satisfying $\sum\limits_{j=1}^{n} r_j = r$ and $\sum\limits_{j=1}^{n} h_j = h$, and*

$$(r^2+h^2)^{1/2} = \sum_{j=1}^{n}(r_j^2+h_j^2)^{1/2},$$

then

$$\sum_{j=2}^{n} r_j h_j = \sum_{j=2}^{n} h_j\left(\sum_{i=1}^{j} r_i + \sum_{i=1}^{j-1} r_i\right) - 2\sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}.$$

Proof. Consider a right angled triangle, say $< ABC$ in a plane, with height h and base r. Next, let us partition the height of the triangle into n parts, not neccessarily equal. Now, we link those partitions along the height to the hypothenus, with the aid of a parallel line. At the point of contact of each line to the hypothenus, we drop down a vertical line to the next line connecting the last point of the previous partition, thereby forming another right-angled triangle, say $< A_1B_1C_1$ with base and height r_1 and h_1 respectively. We remark that this triangle is covered by the triangle $< ABC$, with hypothenus constituting a proportion of the hypothenus of triangle $< ABC$. We continue this process until we obtain n right-angled triangles $< A_jB_jC_j$, each with base and height r_j and h_j for $j = 1, 2, \ldots n$. This construction satisfies

$$h = \sum_{j=1}^{n} h_j \text{ and } r_j = \sum_{j=1}^{n} r_j$$

and

$$(r^2+h^2)^{1/2} = \sum_{j=1}^{n}(r_j^2+h_j^2)^{1/2}.$$

Now, let us deform the original triangle $< ABC$ by removing the smaller triangles $< A_jB_jC_j$ for $j = 1, 2, \ldots n$. Essentially we are left with a rectangles and squares piled on each other with each end poking out a bit further than the one just above, and we observe that the total area of this portrait is given by the relation

$$\begin{aligned}\mathcal{A}_1 &= r_1h_2 + (r_1+r_2)h_3 + \cdots (r_1+r_2+\cdots+r_{n-2})h_{n-1} + (r_1+r_2+\cdots+r_{n-1})h_n\\ &= r_1(h_2+h_3+\cdots h_n) + r_2(h_3+h_4+\cdots+h_n) + \cdots + r_{n-2}(h_{n-1}+h_n) + r_{n-1}h_n\\ &= \sum_{j=1}^{n-1} r_j \sum_{k=1}^{n-j} h_{j+k}.\end{aligned}$$

On the other hand, we observe that the area of this portrait is the same as the difference of the area of triangle $< ABC$ and the sum of the areas of triangles $< A_jB_jC_j$ for $j = 1, 2, \ldots, n$. That is

$$\mathcal{A}_1 = \frac{1}{2}rh - \frac{1}{2}\sum_{j=1}^{n} r_j h_j.$$

This completes the first part of the argument. For the second part, along the hypothenus, let us construct small pieces of triangle, each of base and height (r_i, h_i) $(i = 1, 2 \ldots, n)$ so that the trapezoid and the one triangle formed by partitioning becomes rectangles and squares. We observe also that this construction satisfies the relation

$$(r^2 + h^2)^{1/2} = \sum_{i=1}^{n} (r_i^2 + h_i^2)^{1/2},$$

Now, we compute the area of the triangle in two different ways. By direct strategy, we have that the area of the triangle, denoted $\mathcal{A}$, is given by

$$\mathcal{A} = 1/2\bigg(\sum_{i=1}^{n} r_i\bigg)\bigg(\sum_{i=1}^{n} h_i\bigg).$$

On the other hand, we compute the area of the triangle by computing the area of each trapezium and the one remaining triangle and sum them together. That is,

$$\mathcal{A} = h_n/2\bigg(\sum_{i=1}^{n} r_i + \sum_{i=1}^{n-1} r_i\bigg) + h_{n-1}/2\bigg(\sum_{i=1}^{n-1} r_i + \sum_{i=1}^{n-2} r_i\bigg) + \cdots + 1/2 r_1 h_1.$$

By comparing the area of the second argument, and linking this to the first argument, the result follows immediately. $\square$

Theorem 2.2. *Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ be an arithmetic function. Suppose there exist some constant $0 < \mathcal{K}(x, l_0) < x$ such that*

$$\frac{\mathcal{K}(x, l_0)}{x} \sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n+j) = \sum_{n<x} f(n)f(n+l_0),$$

where $1 \leq l_0 < x$, then

$$\sum_{n<x} f(n)f(n+l_0) = \frac{\mathcal{K}(x, l_0)}{x} \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).$$

Proof. By Theorem 2.1, we obtain

$$\sum_{n \leq x} f^2(n) = f^2(1) + \sum_{2 \leq n \leq x} f(n)\bigg(\sum_{m \leq n-1} f(m) + \sum_{m \leq n} f(m)\bigg) - 2\sum_{n \leq x-1} f(n) \sum_{s \leq x-n} f(n+s)$$

for $f : \mathbb{N} \longrightarrow \mathbb{R}$ by taking $r_j = h_j = f(j)$. We find from this identity that

$$\sum_{n \leq x-1} \sum_{j \leq x-n} f(n)f(n+j) = \sum_{2 \leq n \leq x} f(n) \sum_{m \leq n-1} f(m).$$

Under the hypothesis, there exist some constant $x > \mathcal{K}(x, l_0) > 0$ for a fixed $1 \leq l_0 < x$ such that

$$\sum_{n<x} f(n)f(n+l_0) = \frac{\mathcal{K}(x,l_0)}{x} \sum_{n\leq x-1} \sum_{j\leq x-n} f(n)f(n+j).$$

It follows that

$$\sum_{n<x} f(n)f(n+l_0) = \frac{\mathcal{K}(x,l_0)}{x} \sum_{2\leq n\leq x} f(n) \sum_{m\leq n-1} f(m),$$

thereby ending the proof. □

3. Application to the chowla conjecture

In this section we apply the area method developed in the previous section to the two-point chowla conjecture. We obtain the following weaker result, in the following sequel.

Theorem 3.1. *There exist some* $0 < \epsilon < 1$ *such that*

$$\sum_{n<x} \lambda(n)\lambda(n+1) \ll x^{1+\epsilon} e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}.$$

Proof. Applying the area method, It follows that there exist some constant $0 < \mathcal{K}(x, 1) < x$ such that we can write

$$\sum_{n<x} \lambda(n)\lambda(n+1) = \frac{\mathcal{K}(x,1)}{x} \sum_{2\leq n\leq x} \lambda(n) \sum_{m\leq n-1} \lambda(m).$$

Using the estimate

$$\sum_{n\leq x} \lambda(n) \ll xe^{-c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}},$$

we find that

$$\sum_{2\leq n\leq x} \lambda(n) \sum_{m\leq n-1} \lambda(m) \ll x^2 e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}.$$

It follows that we can write

$$\sum_{n\leq x} \lambda(n)\lambda(n+1) \ll \mathcal{K}(1,x) x e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}.$$

Since $\mathcal{K}(x, 1) < x$, we can write $\mathcal{K}(x, 1) := x^{\delta}$ for some $0 < \delta < 1$. Thus we find that

$$\sum_{n<x} \lambda(n)\lambda(n+1) \ll x^{1+\delta} e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}},$$

thereby ending the proof. □

An immediate consequence of this result is that of the chowla conjecture, which asserts:

Corollary 1. The estimate

$$\sum_{n\leq x}\lambda(n)\lambda(n+1)=o(x)$$

holds as $x\longrightarrow\infty$.

Proof. The result follows from Theorem 3.1. □

4. **Final remarks**

In this paper we have established an upper bound for the two-point correlation of the Liouville function $\lambda(n)$ given by

$$\sum_{n\leq x}\lambda(n)\lambda(n+1)\ll x^{1+\epsilon}e^{-2c(\log x)^{\frac{4}{5}}(\log\log x)^{\frac{-1}{5}}}$$

for some $0<\epsilon<1$. This allowed us to establish the chowla conjecture. This method may also have an independent interest in relation to other open problems involving two-point correlations.

References

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