Is the Course of the Planck's Radiation-Function the Result of the Existence of an Upper Cut-Off Frequency of the Vacuum?

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Abstract

This work is based on the model published in viXra:1310.0189. Because the CMBR follows the PLANCK's radiation rule more or less exactly, it should, because of the indistinguishability of individual photons, apply to a whatever black emitter. Therefrom arises the guess, that the existence of an upper cut-off frequency of the vacuum could be the cause for the decrease in the upper frequency range. Since the lower-frequent share of the curve correlates with the frequency response of an oscillating circuit with the Q-factor 1/2, it is examined, whether it succeeds to approximate the Planck curve by multiplication of the initial curve with the dynamic, time-dependent frequency response of the above mentioned model. Reason of the time-dependence is the expansion of the universe.

This version considers the correction of a calculating error in viXra:1310.0189, effecting the frequency- and phase-response as well as the phase- and group delay. Furthermore, an updated value of H_0 is used, based on the electron mass specified in viXra:2201.0122. Deutsche Version verfügbar in viXra:2008.0139. Titel der deutschen Version: "Ist der Verlauf der Planckschen Strahlungsfunktion das Resultat der Existenz einer oberen Grenzfrequenz des Vakuums".

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» Ist der Verlauf der Planckschen Strahlungsfunktion das Resultat der Existenz einer oberen Grenzfrequenz des Vakuums? «

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1. Fundamentals

This article is based on a model I published in [1], the idea stems from Prof. Cornelius LANCZOS, outlined in a lecture on the occasion of the Einstein-symposium 1965 in Berlin. The lecture is put in front the work in [1]. It defines the expansion of the universe as a consequence of the existence of a metric wave-field. The temporal function of that field is based on the hypergeometric function ${}_0F_1 = J_0\sqrt{2\kappa_0 t/\varepsilon_0}$, used in form of the Hankel-function. The particular qualities of the function lead to an increase of the wavelength. In this connection the phase angle $2\omega_0 t = Q_0$ plays an important role, being identical with the frame of reference, affecting all proportions within the system. The value ω_0 corresponds to the PLANCK frequency. This version considers the correction of a calculating error in [1], effecting the frequency- and phase-response as well as the phase- and group delay. Furthermore, an updated value of H_0 is used, based on the electron mass specified in [6]. In the annex the new Concerted International System of Units from [6] is used, but it doesn't have any effect to the result.

A special solution of the MAXWELL equations was found for the Hankel function with overlaid interference function, which describes the wave-propagation in the vacuum and co-includes the expansion. This special solution owns an inherent propagation-velocity in reference to the empty space (subspace) which is almost zero to the current point of time.

One conclusion from the model is the existence of an upper cut-off frequency of the vacuum, which could not be detected until now, because its value is about magnitudes greater than the technically feasible one. Another conclusion of the model is the supposition that each photon is connected really or/and virtually with an origin at $Q_0 = 1/2$ That is the frequency, at which the excessive energy after the shape of the metric wave-function has been coupled into the very same one, as an overlaid wave, where it can be observed until now as cosmic background-radiation. Furthermore could be determined, that the bandwidth in the lower frequency range exactly matches the one of an oscillatory circuit with the Q-factor 1/2, which equals the conditions to the point of time of the input coupling. Hence the intention of this article is, to determine, whether the PLANCK's graph can be approximated by application of the frequency response given by the model, upon the spectrum of an oscillatory circuit with the Q-factor 1/2, furthermore to compare the calculated radiation temperature with the measured one.

Since the cosmic background-radiation exactly follows the PLANCK's radiation-rule more or less, it should, because of the indistinguishability of individual photons, apply to a whatever black emitter. Therefrom arises the guess, that the existence of an upper cut-off frequency of the vacuum could be the cause for the decrease in the upper frequency range. In [1] already a simple attempt of an approximation has been taken up, at which point several values of the time-dependent frequency response $A(\omega)\cdot\cos\varphi$ have been multiplied with the source-function, which led to a quite good match, as measured by the simple procedure.

Another aim of this article is, to improve the proceeding any farther in order to make more precise statements. With the model attention should be paid to the fact, that with some many exceptions (c, μ_0, ε_0, κ_0, k), most of the fundamental physical constants are time- and reference-frame-dependent (~). And there is a conductivity of subspace κ_0 different from zero. The model is based on the PLANCK units, which can be determined by the locally measurable values (e.g. ω_0). On the one hand, it suggests the values of the universe as a whole (e.g. H_0), on the other hand, the values of the so called subspace (e.g. $r_1 = \text{const}$). That's the medium the metric wave field is propagating in. The proportionality factor is the phase angle of the temporal function $Q_0 = 2\omega_0 t$.

2. The WIEN displacement law and the source-function

During the examination of the WIEN displacement law meets the eye, that the displacement happens exactly at the lower wing pass of the PLANCK's radiation-function, which coincides with the wing pass of an oscillatory circuit with the Q-factor 1/2 in this section. Quite often in publications the curve is shown in another manner. I prefer the duplicate logarithmic presentation, then the curve turns into a straight line.

Considering the WIEN displacement law (902)[1] more exactly, the factor $\tilde{x} = 2.821439372$ attracts attention particularly. With an oscillatory circuit of the Q-factor 1/2 rather the factor $2\sqrt{2}$ would be applicable for this, at which point the 2 stems from the source-frequency $2\omega_1$. The expression $\sqrt{2}$ arises from the rotation of the coordinate-system about $\pi/4$.

Now the validity of the WIEN displacement law in the time short after BB does not have been examined yet and neither PLANCK's radiation-rule nor the WIEN displacement law contain any information about the way, temperature varies, when it varies. In [1] I found the following relations for the calculation of temperature:

$$T_k = \frac{\hbar\omega_k}{\tilde{x}\mathrm{k}} = \frac{\varepsilon_v}{\tilde{x}}\frac{\hbar_1\omega_1}{6\mathrm{k}}Q^{-\frac{5}{2}} = 0.055693\frac{\hbar_1\omega_1}{\mathrm{k}}Q^{-\frac{5}{2}} \qquad \tilde{x} = \begin{cases} 2.821439372 & \text{Exactly} \\ 2\sqrt{2} & \text{Approximation} \end{cases} \qquad ([1]\ 405)$$

$$T_k = \frac{\hbar\omega_k}{\tilde{x}\mathrm{k}} \approx \frac{1}{3}\frac{\hbar_1\omega_1}{6\mathrm{k}}Q^{-\frac{5}{2}} = \frac{\hbar_1\omega_1}{18\mathrm{k}}Q^{-\frac{5}{2}} \qquad \varepsilon_v = \frac{2}{3}\sqrt{2} = 0.9428090416$$

$$T_k = \frac{\hbar_1\omega_1}{18\mathrm{k}}Q_0^{-\frac{5}{2}} = \frac{\hbar\omega_0}{18\mathrm{k}}Q_0^{-\frac{1}{2}} \qquad \omega_1 = \frac{\kappa_0}{\varepsilon_0} \qquad ([1]\ 902)$$

Expression ε_v is the *vacuum coefficient of absorption*. The calculation of T_k according to [1] turns out a value of 2.79146K, which is 0.06598K higher than the measured temperature of the CMBR (2.7250K). See also section 4.

During an investigation in the Internet, I found a detailed deduction of the WIEN displacement law [2]. The value of the proportionality-factor can be obtained by the identification of the maximum of PLANCK's radiation-rule as follows. We start from (382):

$$d\mathbf{S_k} = \frac{1}{4\pi^2}\frac{\hbar\omega^3}{c^2}\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}\mathbf{e_s}\,d\omega \qquad \text{PLANCK's radiation rule} \qquad ([1]\ 382)$$

$$d\mathbf{S_k} = \frac{1}{4\pi^2}\frac{k^3T^3}{\hbar^2c^2}\left(\frac{\hbar\omega}{kT}\right)^3\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}\mathbf{e_s}\,d\omega \qquad x = \frac{\hbar\omega}{kT} \qquad d\omega = \frac{kT}{\hbar}dx \qquad (1)$$

$$d\mathbf{S_k} = \frac{1}{4\pi^2}\frac{k^4T^4}{\hbar^3c^2}\frac{x^3}{e^x-1}\mathbf{e_s}\,dx \qquad \frac{d}{dx}\frac{x^3}{e^x-1} = 0 \qquad (2)$$

$$3\frac{x^2}{e^x-1} - \frac{x^3e^x}{(e^x-1)^2} = \frac{3x^2(e^x-1)-x^3e^x}{(e^x-1)^2} = 0 \qquad (3)$$

$$3x^2(e^x-1)-x^3e^x = 0 \qquad x^3e^x = 3x^2(e^x-1) \qquad (4)$$

$$e^x(x-3) = -3 \qquad y = x-3 \qquad x = 3+y \qquad (5)$$

[1] Three-digit numerations always refer to [1]

$$ye^{y+3} = ye^y e^3 = -3 \qquad ye^y = -3e^{-3} \tag{6}$$

$$x = 3 + \mathrm{lx}(-3e^{-3}) = 2.821439372 \qquad \mathrm{lx}(xe^x) = x \tag{7}$$

lx is LAMBERT's W-function (ProductLog [#]). Finally, after insertion into the middle expression of (1) WIEN's displacement law turns out:

$$\hbar\omega_{max} = 2.821439372\ kT \qquad \text{WIEN's displacement law} \tag{8}$$

On success in doing the same even for the source-function with Q=½, obtaining the same result, we would be a step forward in answer to the question: Is the course of the Planck's radiation-function the result of the existence of an upper cut-off frequency of the vacuum? First of all however, we have to bring the output-function into a form, suitable for further processing. We start with (380) with the substitution:

$$P_v = \frac{P_s}{1+v^2Q^2} \qquad v = \frac{\omega}{\omega_s} - \frac{\omega_s}{\omega} \qquad \omega_s = 2\omega_1 \qquad \Omega = \frac{\omega}{\omega_s} = \frac{1}{2}\frac{\omega}{\omega_1} \tag{9}$$

The expression stems from electrotechnics describing the power dissipation P_v of an oscillatory circuit with the Q-factor Q and the frequency ω (see [3]), v is the detuning. The Q-factor is known and amounts to Q=1/2 at $\omega_S=2\omega_1$. The right-hand expression results directly from the sampling-theorem. The cut-off frequency of the subspace ω_1 is the value ω_0 at Q=1. After substitution, we get the following expressions:

$$v = \Omega - \Omega^{-1} \qquad v^2 = \Omega^2 + \Omega^{-2} - 2 \qquad v^2Q^2 = \frac{1}{4}\Omega^2 + \frac{1}{4}\Omega^{-2} - \frac{1}{2} \tag{10}$$

$$P_v = \frac{P_s}{\frac{1}{4}\Omega^2 + \frac{1}{4}\Omega^{-2} + \frac{1}{2}} \cdot \frac{4\Omega^2}{4\Omega^2} = 4P_s \frac{\Omega^2}{\Omega^4 + 2\Omega^2 + 1} = 4P_s \left(\frac{\Omega}{1+\Omega^2}\right)^2 \tag{11}$$

You can find that expression more often in [1], among other things even with the group delay T_{Gr} however for a frequency ω_1. For a frequency $2\omega_1$ applies for T_{Gr} and the energy W_v:

$$T_{Gr} = \frac{dB(\omega)}{d\omega} = \frac{1}{\omega_1}\left(\frac{\Omega}{1+\Omega^2}\right)^2 \qquad W_v = \frac{1}{6}P_s T_{Gr} = \frac{2}{3}\frac{P_s}{\omega_1}\left(\frac{\Omega}{1+\Omega^2}\right)^2 \tag{12}$$

The factor ⅙ comes from the splitting of energy onto 4 line-elements, as well as from the multiplication with the factor ⅔ because of refraction during the in-coupling into the metric transport lattice. It oftenly occurs in thermodynamic relations, which doesn't astonish. Thus, total-energy of the CMBR during input coupling is equal to the product of power dissipation and group delay, that is the average time, the wave stays within the MLE, but only for what it's worth. With the help of (11) we obtain:

$$P_v = 4b\ P_s \left(\frac{\Omega}{1+\Omega^2}\right)^2 \qquad P_v = 512b\ \hbar_1\omega_1^2 \left(\frac{\Omega}{1+\Omega^2}\right)^2 \tag{13}$$

b is a factor, we want to determine later on. Let's equate it to one at first. We determined the value Ps with the help of (394) using the values of the point of time Q=1/2. Interestingly enough, the HUBBLE-parameter H_0 at the time $t_{0.5}$ is greater than ω_1 and ω_0. For an individual line-element applies:

$$_{0.5} = \frac{\omega_1}{Q_{0.5}} = \frac{\omega_1}{\frac{1}{2}} = 2\omega_1 \qquad\qquad H_{0.5} = \frac{\omega_1}{Q_{0.5}^2} = \frac{\omega_1}{\frac{1}{4}} = 4\omega_1 \tag{14}$$

$$P_s = \frac{\hat{\hbar}_i}{4\pi t_{0.5}^2 Q_{0.5}^4} = \frac{\hat{\hbar}_i}{2\pi}\frac{2^5}{4t_{0.5}^2} = 32\hbar_1 H_{0.5}^2 = 128\hbar_1\omega_1^2 \qquad \frac{\hat{\hbar}_i}{2\pi} = \hbar_1 = \frac{\hbar_{0.5}}{2} \tag{15}$$

Expression (13) is very well-suited for the description of the conditions at the signal-source. Here, the power makes more sense than the POYNTING-vector $\mathbf{S}_k$. But for a comparison with (382) we just need an expression for $\mathbf{S}_k$, quasi a sort of PLANCK's radiation-rule for technical signals with the bandwidth $2\omega_1/Q_{0.5}=4\omega_1$. Then, this would look like this approximately:

$$d\mathbf{S_k} = 4bA\left(\frac{\Omega}{1+\Omega^2}\right)^2 \mathbf{e_s}\, d\Omega \tag{16}$$

We determine the factor A by a comparison of coefficients (3). We assume, the WIEN displacement law (8) would apply and substitute as follows:

$$A = \frac{1}{4\pi^2}\frac{k^4 T^4}{\hbar^3 c^2} \qquad\qquad c = \omega_1 Q^{-1} r_1 Q \tag{17}$$

We put in $2\sqrt{2}\omega_1$ as initial-frequency into the expression k^4T^4 That's advantageous, as we will already see. This frequency is not a metric indeed ($\omega_x \sim Q^{-1}$), but an overlaid frequency ($\omega \sim Q^{-3/2}$). During red-shift of the source-signal, likewise not the factor 2.821439372 but the factor $2\sqrt{2}$ becomes effective. Thus applies:

$$k^4 T^4 = \frac{(2\sqrt{2})^4}{(2\sqrt{2})^4}\hbar_1^4 Q^{-4}\omega_1^4 Q^{-6} = \hbar_1^4\omega_1^4 Q^{-10} \qquad\qquad Q^{-10} = \frac{Q^{-8}}{Q^2} \tag{18}$$

$$A = \frac{1}{4\pi^2}\frac{\hbar_1^4\omega_1^4 Q^{-8}}{\hbar_1^3 Q^{-3}\omega_1^2 Q^{-2} r_1^2 Q^4} = \frac{1}{4\pi^2}\frac{\hbar^4\omega_0^4}{\hbar^3\omega_0^2 r_1^2 Q^4} = \frac{1}{\pi}\frac{\hbar\omega_0^2}{4\pi R^2} \tag{19}$$

$$4A = \frac{4}{\pi}\frac{\hbar\omega_0^2}{4\pi r_0^2 Q^2} = \frac{4}{\pi}\frac{\hbar\omega_0^2}{4\pi R^2} \qquad\qquad R \text{ for } Q \gg 1 \tag{20}$$

$$d\mathbf{S_k} = \frac{4b}{\pi}\frac{\hbar\omega_0^2}{4\pi R^2}\left(\frac{\Omega}{1+\Omega^2}\right)^2 \mathbf{e_s}\, d\Omega \qquad\qquad R \text{ for } Q \gg 1 \tag{21}$$

Indeed, that submits only the expression without consideration of red-shift. We determine the actual values to the point of time of input coupling, in that we apply the values for Q=1/2 in turn. It applies:

$$A = \frac{1}{4\pi^2}\frac{\hbar_1^4\omega_1^4 Q^{-8}}{\hbar_1^3 Q^{-3}\omega_1^2 Q^{-2} r_1^2 Q^4} = \frac{2^{8-3-2+4}}{4\pi^2}\frac{\hbar_1^4\omega_1^4}{\hbar_1^3\omega_1^2 r_1^2} = \frac{128}{\pi}\frac{\hbar\omega_1^2}{4\pi r_1^2} \tag{22}$$

$$4A = \frac{512}{\pi}\frac{\hbar_1\omega_1^2}{4\pi r_1^2} \qquad\qquad d\mathbf{S_k} = \frac{512b}{\pi}\frac{\hbar_1\omega_1^2}{4\pi r_1^2} Q^{-7}\left(\frac{\Omega}{1+\Omega^2}\right)^2 \mathbf{e_s}\, d\Omega \tag{23}$$

b will be determined later on. It shows, the POYNTING-vector is equal to the quotient of a power P_k resp. P_S and the surface of a sphere with the radius R (world-radius), exactly as per definition. Omitting the surface, we would get the transmitting-power P_V directly. In the above-mentioned expressions the parametric attenuation of 1Np/R, which occurs during propagation in space, is unaccounted for. This must be considered separately if necessary.

Now we have framed the essential requirements and can dare the next step, the proof of the validity of the WIEN displacement law in strong gravitational-fields. The basic-idea was just, that the Planck's radiation-rule (382) should emerge as the result of the application of the metrics' cut-off frequency (302) to the function of power dissipation P_v of an oscillatory circuit with the Q-factor Q=1/2 (13) We proceed on the lines of (2), in that we equate the first derivative of the bracketed expression (23) to zero. A substitution like in (1) is not necessary, because the expression is already correct. It applies:

$$\frac{d}{d\Omega}\left(\frac{\Omega}{1+\Omega^2}\right)^2 = \frac{2\Omega}{(1+\Omega^2)^2} - \frac{4\Omega^3}{(1+\Omega^2)^3} = \frac{2\Omega\left(1-\Omega^2\right)}{(1+\Omega^2)^3} = 0 \tag{24}$$

$$2\Omega\left(1-\Omega^2\right) = 0 \qquad \Omega_1 = 0 \quad \text{Minimum} \qquad \Omega_{2,3} = \pm 1 \quad \text{Maximum} \tag{25}$$

The first solution is trivial, the second and third is identical, if we tolerate negative frequencies (incoming and outgoing vector). Now, we must only find a substitution for Ω, with which (382) and (23) come to congruence in the lower range. This would be the displacement law for the source-signal then (22). Since the ascend of both functions has the same size in the lower range, there is theoretically an infinite number of superpositions, whereat only one of them is useful. Therefore, as another criterion, we introduce, that both maxima should be settled at the same frequency. The displacement law for the source-signal would be then as follows:

$$\hbar\omega_{max} = a\,kT \qquad \text{Displacement law source-signal} \tag{26}$$

at which point we still need to determine the factor a. As turns out, we still have to multiply even the output-function itself, with a certain factor b, in order to achieve a congruence. The 4 we had already pulled out. We apply the value $2\sqrt{2}$and 2.821439372 for a one after the other and determine b numerically with the help of the relation and the function FindRoot[#] using the substitution 2x=ay:

$$\frac{\left(a\frac{y}{2}\right)^3}{e^{a\frac{y}{2}}-1} - 4b\left(\frac{\frac{y}{2}}{1+\left(\frac{y}{2}\right)^2}\right)^2 = 0 \quad y=10^{-5} \qquad \begin{matrix} b \to 2 & \text{for } a = 2\sqrt{2} \\ b \to 2.009918917 & \text{for } a = 2.821439372 \end{matrix} \tag{27}$$

The maxima overlap accurately in both cases. The lower value a is equal to the factor in (903). Thus it seems, that with references, except for those to the origin of each wave with $2\omega_1$, multiplied with $\sqrt{2}$, which is caused by the rotation of the coordinate-system about π/4, rather the approximative solutions with the factor $2\sqrt{2}$ apply. With lower frequencies, the factor 2.821439372 of the WIEN displacement law applies then again.

But to the exact proof of the validity of the WIEN displacement law in the presence of strong gravitational-fields this ansatz is not enough. We must also show that the maximum of the PLANCK's radiation-function behaves exactly according to the WIEN displacement law, that means the approximation and the target-function must come accurately to the congruence. Since the difference between a factor $2\sqrt{2}$ and 2.821439372 amounts to 0.5% after all, we will execute the examination with both values. Only the relations for b=$2\sqrt{2}$ are depicted. Now, we can set about to write down the individual relations:

$$\hbar\omega_{max} = 2\sqrt{2}\,kT \qquad \text{Displacement law source-signal} \tag{28}$$

$$\Omega = \frac{1}{2}\frac{\omega}{\omega_1} = \frac{1}{2\sqrt{2}}\frac{\hbar\omega}{kT_k} = \frac{x}{a} = \frac{y}{2} \qquad y = \frac{\omega}{\omega_1} \qquad b=2 \tag{29}$$

Thus, we have found our source-function. In y it reads as follows:

$$dS_k = \frac{16}{\pi}\frac{\hbar\omega_0^2}{4\pi R^2}\left(\frac{\frac{y}{2}}{1+\left(\frac{y}{2}\right)^2}\right)^2 \mathbf{e}_s\,dy \qquad R \text{ for } \qquad Q \gg 1 \tag{30}$$

But we aren't interested in the absolute value but in the relative level only:

$$dS_1 = 8\left(\frac{\frac{y}{2}}{1+\left(\frac{y}{2}\right)^2}\right)^2 dy \tag{31}$$

We want to mark the approximation with dS_2. For the target-function dS_3 we obtain:

$$dS_3 = \frac{\left(2.821439\frac{y}{2}\right)^3}{e^{2.821439\frac{y}{2}}-1}\,dy \tag{32}$$

In figure 1 are presented the course of the source-function and the PLANCK's graph.

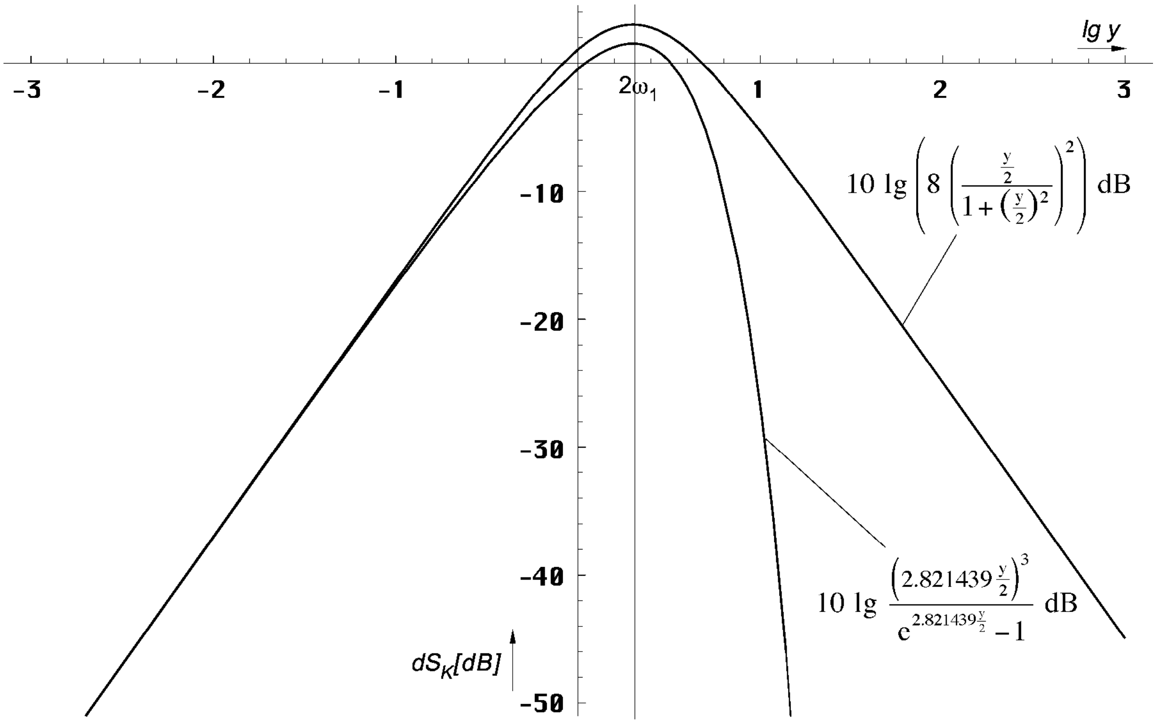

Figure 1
Planck's radiation-rule and source-function
in the superposition (logarithmic, relative level)

3. Solution and analysis

Of course, there is no shift-information y(Q) contained in these relations. Since the considered system is a minimum phase system, we now have to multiply the source-function dS_1 with the amplitude response $A(\omega)$. The result is our approximation dS_2. It is merely applied to a single line-element, which is traversed by the signal in the time r_0/c. Thereat r_0 is equal to the PLANCK's length and identical to the wavelength of the above-mentioned metric wave-function. That means, we have to execute the multiplication with $A(\omega)$ as often as we like, unless the result (almost) no longer changes.

But thereat as well the frequency of the source-function as the cut-off frequency (frequency response) decrease continuously. Therefore it's opportune, to take up the displacement (frequency and amplitude) later on with the result dS_2 (approximation), instead of shifting on and on the location of the source-function. For the proof of our hypothesis indeed this last shift is not of interest, so that we won't take up it in this place.

There is another problem with the amplitude response A(ω) and with the phase-angle φ. Since the cut-off frequency $\omega_0 = f(Q, \omega_1)$ and the frequency ω are varying according to different functions, it causes difficulties to formulate a practicable algorithm. Thus we use the fact that there is no difference, whether we reduce the frequency of the input-function with constant cut-off frequency or if we shift upward the cut-off frequency with constant input-frequency. But this corresponds to a *transposition of integration limits*. We choose this second way incl. the displacement of the approximation at the end of calculation. This all the more, since we would be concerned with two time-dependent quantities (input-frequency and cut-off frequency) otherwise. To the approximation applies:

$$dS_2 = 8\left(\frac{\frac{y}{2}}{1+(\frac{y}{2})^2}\right)^2 \prod_{Q_0}^{1/2} A(y)\cos\varphi(y)\, đy\, dy \qquad (33)$$

Expression (33) looks a little bit strange maybe. It's about a so called product integral, i.e. you have to multiply instead of summate. Then, the letter đ isn't the differential-, but the… let's call it d*ivisional*-operator. I don't want to amplify that, because we anyway have to convert expression (33) to continue. We use $Q_0 = 8.34047113224285 \cdot 10^{60}$ from [6] as the updated value of the Q-factor and the phase-angle of the metric wave-function[1]. It determines the upper limit of the multiplication resp. summation. Fortunately the frequency response can be depicted as e-function, so that the product changes into a sum. We simply have to integrate the exponent quite normally then. We obtain the frequency response inclusive phase-correction with the help of the complex transfer-function (150) to:

$$A(\omega)\cdot\cos\varphi(\omega) = e^{\Psi(\omega)} \qquad \varphi = B(\omega) \qquad \text{Frequency response of a line element} \qquad (34)$$

The fact, that only the real component is transferred, is taken into account by the multiplication of A(ω) with the expression cos φ. We use (302) from [1] for Ψ(ω). Unfortunately, the expression stated there is wrong, because I miscalculated in section 4.3.2. and I could reveal the error only now. After all the function determined there was not referenced in any correspondence table and I was unable to perform the inverse Laplace-transform to the verification until now.

The error is located in (139) and (140). The corrigendum will be published soon in a correctional article [7]. A corrected version of [1] is impossible alas, since my 5 allowed replacements are already exhausted. Fortunately I used a different approach for the rest of the work, without an error. Only concerned is section 4.3.2. With $\omega_1 = 1/(2t_1) = \kappa_0/\varepsilon_0$ expression ([1] 140) reads correctly:

$$y(p) = e^{\int \frac{a-p}{p^2}dp} = \frac{C_1}{p} e^{-\frac{a}{p}} = \frac{a}{p} e^{-\frac{a}{p}+C} = \frac{1}{2pt_1} e^{-\frac{1}{2pt_1}+C} \qquad ([7]\ 140)$$

Because of cos(φ) = cos(–φ) we obtain the following corrected expression (302):

$$\Psi(\omega) = -\frac{1}{2}\ln(1+\Omega^2) + \frac{\Omega^2}{1+\Omega^2} + \ln\cos\left(\arctan\Omega - \frac{\Omega}{1+\Omega^2}\right) \qquad ([7]\ 302)$$

As next, we substitute Ω by y with the help of (29):

$$\Psi(\omega) = -\frac{1}{2}\ln\left(1+\left(\frac{y}{2}\right)^2\right) + \frac{(\frac{y}{2})^2}{1+(\frac{y}{2})^2} + \ln\cos\left(\arctan\frac{y}{2} - \frac{\frac{y}{2}}{1+(\frac{y}{2})^2}\right) \qquad (35)$$

[1] The equality of the Q-factor Q_0 and the phase angle $2\omega_0 t$ is a special property of this function

The value ω in the numerator of y figures the respective frequency of the cosmic background-radiation, for which we just want to determine the amplitude. It is identical to the ω in PLANCK's radiation-rule. Thereat, it's about an overlaid frequency, which is proportional to $Q^{-3/2}$ in the approximation. The frequency ω_0 is exactly proportional to Q^{-1}.

Instead of the value ω_1 in the denominator actually the PLANCK's frequency ω_0 should be written with the frequency response. That is also the cut-off frequency for the transfer from one line-element to another. But with Q=1 the value ω_0 is right equal to ω_1, at which point ω_0 varies with the time; ω_1 on the other hand is strictly defined by quantities of subspace having an invariable value therefore. It applies $\omega_0 = \omega_1/Q$. That means, that even y depends on time, being proportional to $Q^{-1/2}$.

Now however, we want to freeze the value ω, at least up to the end of the calculation, with the consequence, that we must divide y by a supplementary function ξ, which is proportional to $Q^{1/2}$. It applies $\xi = cQ^{1/2}$ and

$$\Psi(\omega) = -\frac{1}{2}\ln\left(1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2\right) + \frac{\left(\frac{y}{2}\frac{1}{\xi}\right)^2}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2} + \ln\cos\left(\arctan\frac{y}{2}\frac{1}{\xi} - \frac{\frac{y}{2}\frac{1}{\xi}}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2}\right) \tag{36}$$

The factor c arises from the initial conditions at Q=1/2 (resonance-frequency $2\omega_1$, cut-off frequency ω_1) to c=4 (In the program cc=y/2):

$$y = \frac{\omega}{\omega_0} \sim \frac{2^{-\frac{3}{2}}}{2^{\frac{1}{2}}} = \frac{1}{4} \qquad \xi = 4\sqrt{Q} \quad \text{Approximation} \tag{37}$$

Thus, together with the 2 of y/2, we acquire exactly the same factor 8 as in the source-function (31). Then, the approximation dS_2 calculates as follows:

$$dS_2 = 8\left(\frac{\frac{y}{2}}{1+\left(\frac{y}{2}\right)^2}\right)^2 e^{-\int_{1/2}^{Q_0}\left[-\frac{1}{2}\ln\left(1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2\right)+\frac{\left(\frac{y}{2}\frac{1}{\xi}\right)^2}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2}+\frac{1}{2}\ln\cos\left(\arctan\frac{y}{2}\frac{1}{\xi}-\frac{\frac{y}{2}\frac{1}{\xi}}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2}\right)\right]dy}\,dy \tag{38}$$

The negative sign before the integral results from the re-exchange of the integration limits. For the determination of the integral, a value of 10^3 as upper limit suffices indeed. Over and above this, it changes very little. Therefore, I worked with an upper limit of $3\cdot 10^3$ in the following representations. The integral only can be determined numerically, namely with the help of the function NIntegrate[f(Q), Q, 1/2, 3×10^3]. The quotient of y/2 and ξ expression (37) however describes the dependency y(Q) in the approximation only. There is an exact solution as well. According to [1] (209), (299) and (509) applies:

$$\xi = \frac{a}{b}\frac{1}{Q}\frac{R(Q)}{R(\tilde{Q})}\sqrt{\frac{\beta_\gamma^4-1}{\tilde{\beta}_\gamma^4-1}} \qquad \text{with } \tilde{Q} = \frac{1}{2} \qquad \text{and} \tag{39}$$

$$R(Q) = 3r_1 Q^{\frac{1}{2}}\int_0^Q \frac{dQ}{\rho_0} \qquad \text{with } \rho_0 = \sqrt[4]{(1-A^2+B^2)^2+(2AB)^2} \tag{40}$$

$$A = \frac{J_0(Q)J_2(Q)+Y_0(Q)Y_2(Q)}{J_0^2(Q)+Y_0^2(Q)} \qquad B = \frac{J_2(Q)Y_0(Q)-J_0(Q)Y_2(Q)}{J_0^2(Q)+Y_0^2(Q)} \tag{41}$$

The factor b arises from the demand, that the exact function ξ and its approximation should be of the same size with larger values of Q. The factor a we will determine later on in turn. The functions in (41) are Bessel functions.

Problematic in (40) and (45) is the integral, which can be determined even only by numerical methods. In order to avoid the numerical calculation of an integral within the numerical calculation of another integral, it's opportune, to re-place the integrand by an interpolation-function (BRQ1), and that inclusive the factor B. The value r_l cancels itself because of (39). We choose sampling points with logarithmic spacing:

```
brq = {{0, 0}};
For[x = -8; i = 0, x < 25, (++i), x += .1;
 AppendTo[brq, {10^x, N[BRQP[10^x]/BGN/(2.5070314770581117*10^x) ]}]]
BRQ0 = Interpolation[brq];
BRQ1 = Function[If[# < 10^15, BRQ0[#], Sqrt[#]]];
```

The function BRQP is equal to the product of Q, root-expression and integral in the denominator of (45). The value BGN is equal to the initial value of the same product at Q=1/2. You'll find the complete program in the appendix. The factor b arises to 2.5(0703). According to (211), (482) and (623) applies further:

$$\beta_\gamma = \frac{\sin\alpha}{\sin\gamma_\gamma} \qquad \gamma_\gamma = \arg\underline{c} + \arccos\left(\frac{c_M}{c}\sin\alpha\right) + \frac{\pi}{4} \tag{43}$$

$$\alpha = \frac{\pi}{4} - \arg\underline{c} = \frac{3}{4}\pi + \frac{1}{2}\arg\left((1 - A^2 + B^2) + j2AB\right) \qquad c_M = \left|\underline{c}\right| \tag{44}$$

$$\xi = \frac{3}{0{,}56408}\frac{a}{b}Q^{-\frac{1}{2}}\sqrt{\beta_\gamma^4 - 1}\int_0^Q \frac{dQ}{\rho_0} = a\frac{3}{2}\sqrt{2}Q^{-\frac{1}{2}}\sqrt{\beta_\gamma^4 - 1}\int_0^Q \frac{dQ}{\rho_0} \tag{45}$$

$\underline{c}$ is the complex propagation-velocity of the metric wave-field. As next, we want to take up a comparison of the two functions $Q^{1/2}$ and BRQ1 (figure 2):

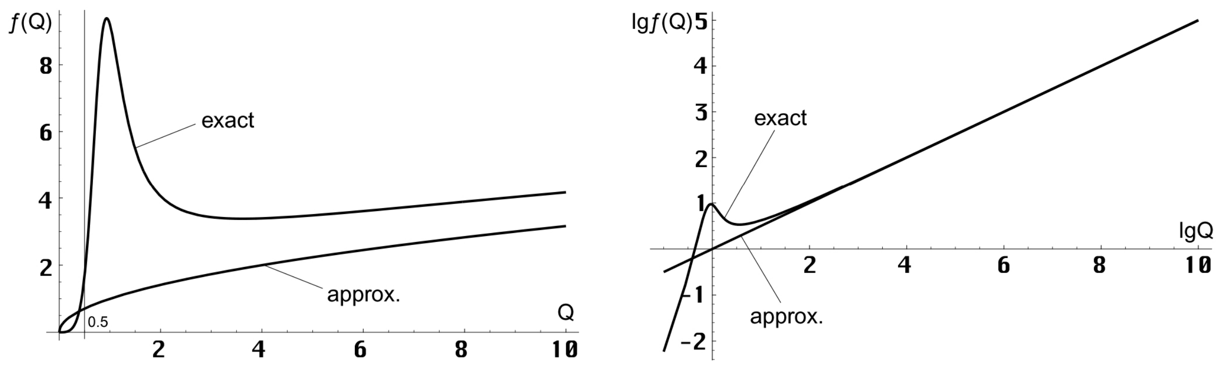

Figure 2
Function BRQ1 exactly and approximation

On the basis of the demand, that the result of both functions must be identical with Q» 1 we choose the factor a to $\sqrt{\pi}$. In this connection is to be remarked that the exact value is $\sqrt{3{,}\bar{5}}$ in fact. But since we finally will not find, in any case, an exact fit in the course of both functions, this small „cheating“ in the initial conditions should be allowed. The value $\sqrt{\pi}$ namely leads to the result with the smallest difference, so that we obtain the following final relation for ξ:

$$\xi = \frac{3}{2}\sqrt{2\pi}\left(Q^{-\frac{1}{2}}\sqrt{\beta_\gamma^4 - 1}\int_0^Q \frac{dQ}{\rho_0}\right) \qquad c = \frac{3}{2}\sqrt{2\pi} = 3.756 \tag{46}$$

For $\sqrt{3{,}\bar{5}}$ a value of c=4 would arise. The bracketed expression corresponds to the factor $Q^{1/2}$ in the approximation. The course of the integral function in (38) as well as of the dynamic cumulative frequency response $A_{ges}(\omega)=e^{-\int\Psi(\omega)dQ}$ you can see in figure 3 and 4. For your information the amount of the complex frequency response $|X_n(j\omega)|$ of subspace is plotted, that's the medium, in which the metric wave field propagates ($\Omega_U=\Omega$).

$$X_n(j\omega) = \frac{1}{2}\,\frac{1}{1+j\Omega}\left(1+\frac{1}{1+j\Omega}\right)$$ Complex spectral function ([1] 459)

That applies to EM−waves propagating simultaneously with the metric wave field but not to the metric wave field itself. They achieve the aperiodic borderline case at Q=½.

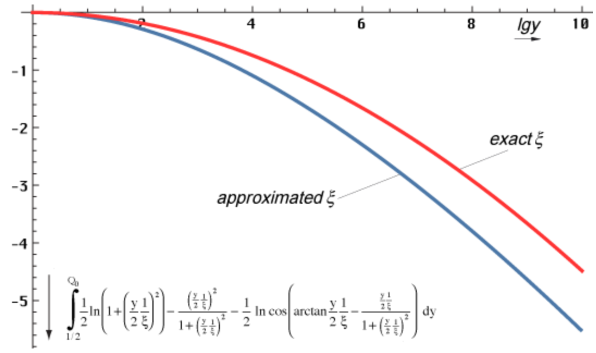

Figure 3
Course of the Integrals Ψ(ω) in (38)
for the approximation and exact function ξ

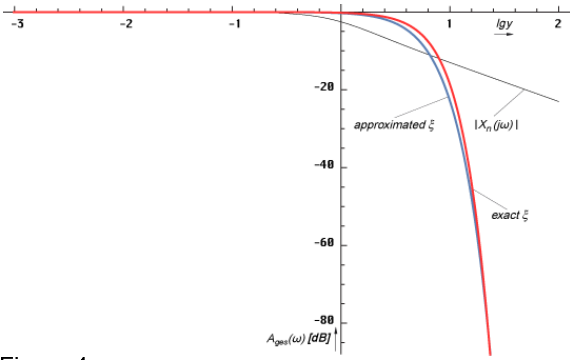

Figure 4
Cumulative frequency response $A_{ges}(\omega)$
and $|X_n(j\omega)|$ of the metric wave field
and subspace

Thus, all requirements are filled and we are able to demonstrate the course of the approximation (38) in comparison with the target-function (32) and that as well for the approximation as for the exact function ξ. We use a logarithmic scale with the unit decibel [dB] and, because it's about power per surface, with the factor 10.

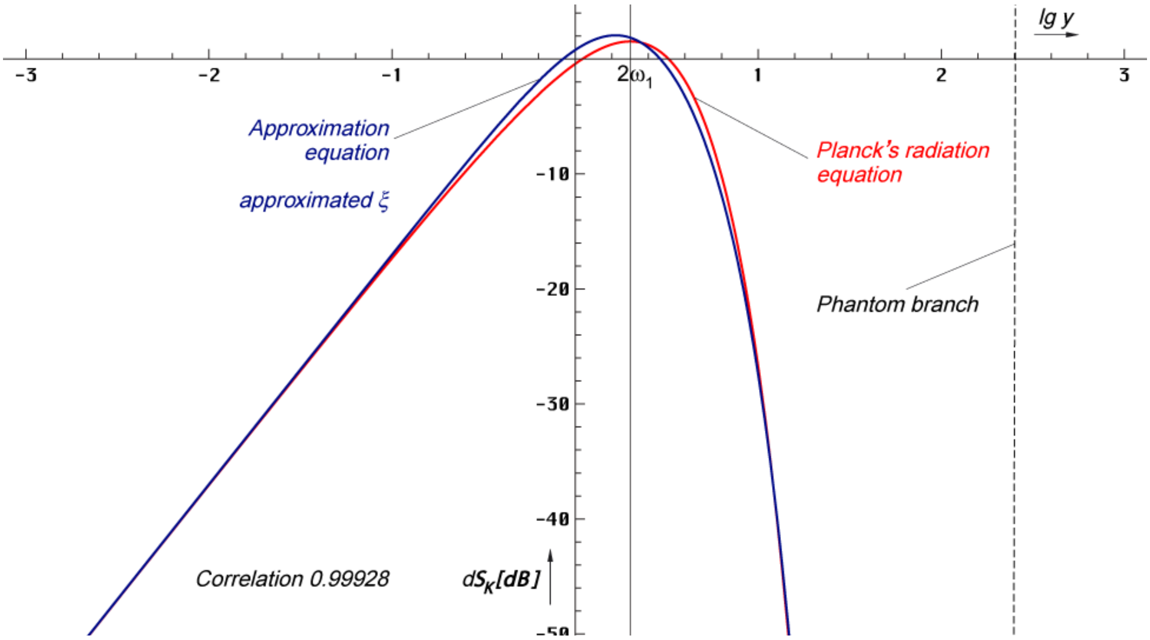

Figure 5
PLANCK's radiation-rule and approximation
with approximation for the function ξ (relative level)

Figure 5 shows the shape of the approximation using the approximation (37) for the function ξ (c=4). The figure shows a phantom branch at the right side due to the down-limited decimal resolution by sign-change according to $e^{1/\pm\to 0}$. It will be removed in the following presentations. Furthermore we can see, both curves doesn't match exactly. The maximum frequency $\Omega_{\hat{m}}$ is downshifted by 18.28% (0.81721). The maximum deviation of the amplitude $\Delta A_{\bar{\wedge}}$ is with −1.78 dB, the difference between both maxima $\Delta A_{\bar{\wedge}}$ +0.42886 dB (+10.38% resp. 1.1038). Altogether the function resembles the shape, shown in [1] section 4.6.4.2.3., obtained by multiplication of the source-function with only 4 choosed values of the frequency response. But there are disparities in the declining branch with higher frequencies.

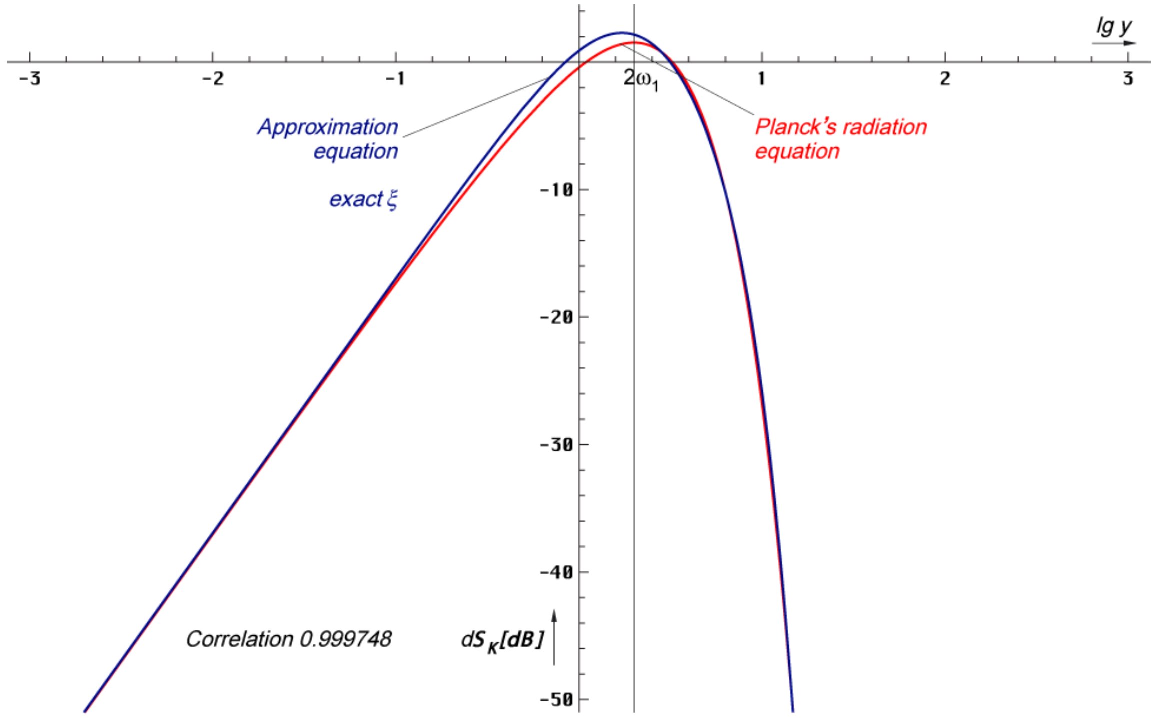

Figure 6
PLANCK's radiation-rule and approximation under application of the exact function ξ (relative level)

Figure 6 presents the course of the approximation under application of the exact function ξ (46) for c=3.756. With it, the best fit (without group delay correction) turns out (With c=4, there is only a minor difference to figure 5). But both functions don't overlap exactly neither in this place. Once again, the maximum frequency $\Omega_{\text{ꝟ}}$ is downshifted by 13.61 % (0.86386) The maximum deviation of amplitude $\Delta A_{\bar{\lambda}}$ is about +1.33 dB, between both maxima $\Delta A_{\bar{\lambda}}$ at +0.75834 dB (+19.07%).

The course of deviation (logarithm of the quotient of approximation and PLANCK's radiation-rule) as a function of y is shown in figure 7. One sees, from ca. $10\omega_1$ on the relative deviation between both functions is strongly growing. But since the absolute level in this range is already microscopic (−54dB at the third zero), nobody will take notice of it. Even there it seems rather to be about a small frequency shift, than about a deformation of the envelope.

Maybe, the downshift of the approximation's maximum could be a reason for the discrepancy between the CMBR-temperature calculated in 7.5.3. [1] to the measured COBE-value with the amount of +2.42086% (−2.36363% in the reciprocal case).

Although, the form of the approximation-graph doesn't correspond to that of a black emitter and the value is too high. But during the COBE-experiment, they just have been ascertained, that the spectrum of the CMBR is exactly? black. Therefore, more forces are required in order to change the form in such a manner, that it equals that of a black emitter. In the next section we will see, which influences may come into consideration for that purpose.

As further considerations [6] show, the above mentioned deviation is less because of the curve shape, but because of the value of the HUBBLE-parameter, determined in [1]. With the value from [6] the calculation exactly fits the limits of the measuring tolerance of the COBE/WMAP-satellite. Read section 4 for details.

In figure 7 we can see that we yield an improvement if we use the exact function ξ. Nevertheless a certain left-over difference remains. If we take a look at the course in the 2nd quadrant, we can see a „gap“ where an already known function, multiplied with the factor ½, could slot right in there.

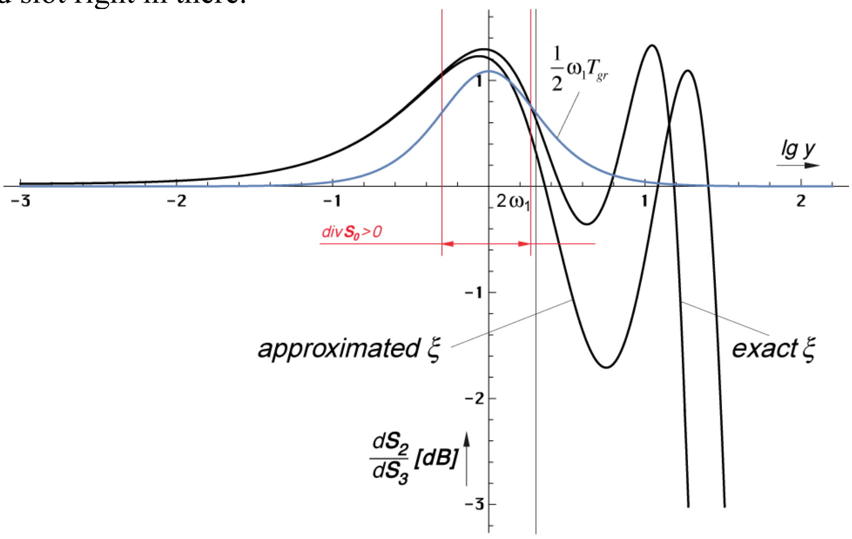

Figure 7
Relative offset between approximation and radiation-rule in dependency of the function ξ used

That's the group delay T_{Gr} of the metric wave field of [1] section 4.3.2. Whereas the phase response affects the form of the carrier frequency (ω_1 resp. ω_0), the group response affects the shape of the envelope curve. Due to the miscalculation, expression ([1] 152) is also wrong. It reads correctly:

$$T_{Gr} = \frac{d}{d\omega}B(\omega) = -\frac{2}{\omega_1}\left(\frac{\Omega}{1+\Omega^2}\right)^2 = -2\frac{\theta^2}{\omega_1} \qquad ([7]\ 152)$$

With $\Omega=\omega/\omega_1$. The factor 2 cancels out, since it's about a spin2-system, with which all temporal constants are 2T instead of T (double phase-/group-velocity). Whereas the group response is constantly equal to zero across nearly all decades, it is not the case close to ω_1 resp. ω_0 nowadays. A frequency dependent group response always leads to a distortion of the envelope curve.

As we can see, the group response is negative. That happens in technology too and is not a violation of causality. See [8] for details. So far we have taken into account the frequency response $A(\omega)$ and the phase response $B(\omega)$, only the group delay correction $\Theta(\omega)=\frac{1}{2}\omega_1 T_{gr}$, is missing, implemented by the function gdc[ω]:

$$\frac{1}{2}\omega_1 T_{Gr} = -\frac{\cancel{2\omega_1}}{\cancel{2\omega_1}}\left(\frac{\Omega}{1+\Omega^2}\right)^2 = -\left(\frac{\Omega}{1+\Omega^2}\right)^2 \qquad (47)$$

$$\Theta(\omega) = e^{-\omega_1 T_{Gr}} = e^{-\left(\frac{\Omega}{1+\Omega^2}\right)^2} = 10^{-\left(\frac{\Omega}{1+\Omega^2}\right)^2 \lg e} = 10^{-0{,}434294\left(\frac{\Omega}{1+\Omega^2}\right)^2} \qquad (48)$$

The decimal power is important, if we want to calculate with dB. The group delay correction $\Theta(\omega)$ on dS_2 is applied only once:

$$dS_2 = 8\left(\frac{\frac{y}{2}}{1+\left(\frac{y}{2}\right)^2}\right)^2 e^{\int_{1/2}^{Q_0} \frac{1}{2}\ln\left(1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2\right) - \frac{\left(\frac{y}{2}\frac{1}{\xi}\right)^2}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2} - \frac{1}{2}\ln\cos\left(\arctan\frac{y}{2}\frac{1}{\xi} - \frac{\frac{y}{2}\frac{1}{\xi}}{1+\left(\frac{y}{2}\frac{1}{\xi}\right)^2}\right) dy - \frac{1}{2}\omega_1 T_{Gr}} dy \qquad (49)$$

The resulting functions with group delay correction for both ξ are shown in figure 8 and 9. There is already a better fit of both graphs in figure 8, as we can see. Now the maximum $\Omega_{\pitchfork}$ of the frequency is downshifted about 12.52% (0.87476). The maximum deviation of amplitude $\Delta A_{\overset{=}{\wedge}}$ amounts to +0.42061 dB. The deviation between both peaks $\Delta A_{\bar{\wedge}}$ is –0.40484 dB or –1.45%.

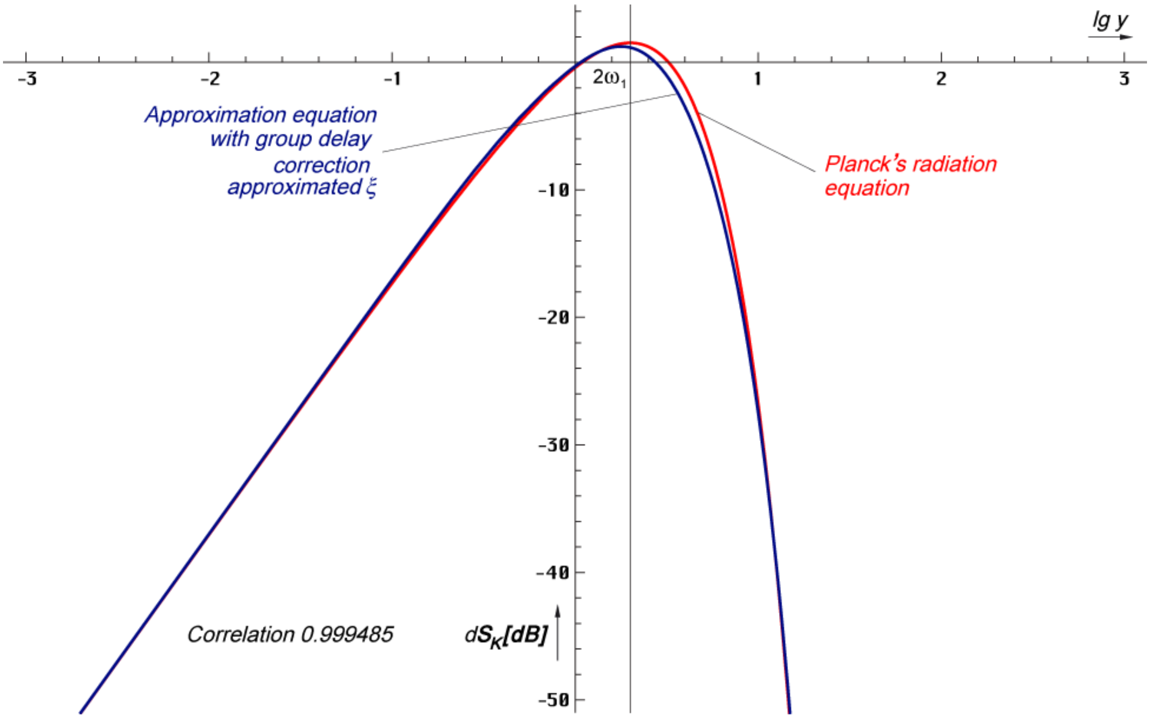

Figure 8
PLANCK's radiation-rule and approximation with group delay correction with approximation of the function ξ (relative level)

A nearly perfect result we have got for the case exact ξ with group delay correction (figure 9). Now the maximum frequency $\Omega_{\pitchfork}$ is downshifted about –7.00% (0.93003) only. That value is far in excess of the –2.36% deviation between measured and calculated CMBR-temperature. The maximum amplitude deviation $\Delta A_{\overline{\overline{\wedge}}}$ is at about –0.58954 dB, between both maxima $\Delta A_{\overline{\wedge}}$ is at –0.02762 dB (–0.64%). Of particular interest is the extremely high correlation coefficient of 0.999835 between both curves.

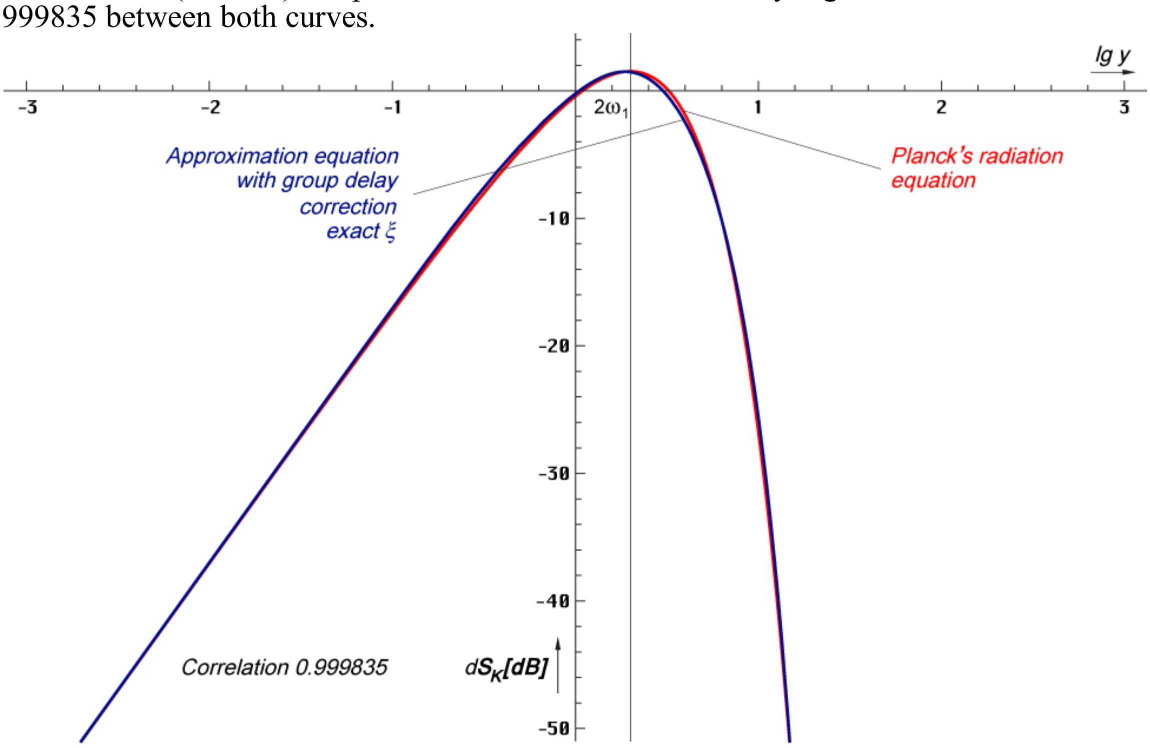

Figure 9
PLANCK's radiation-rule and approximation with group delay correction under application of the exact function ξ (relative level)

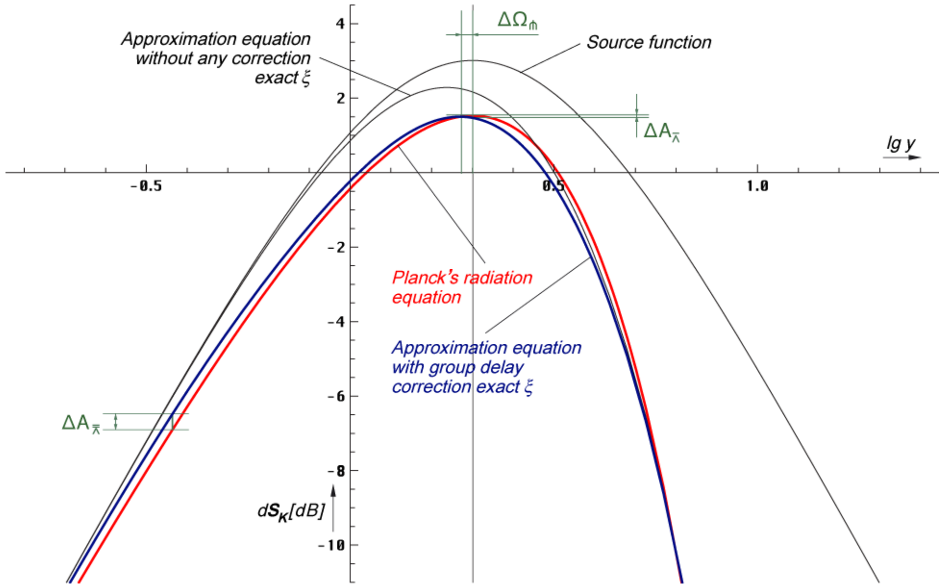

Figure 10
PLANCK's radiation-rule and approximation with group delay correction under application of the exact function ξ (relative level) high resolution

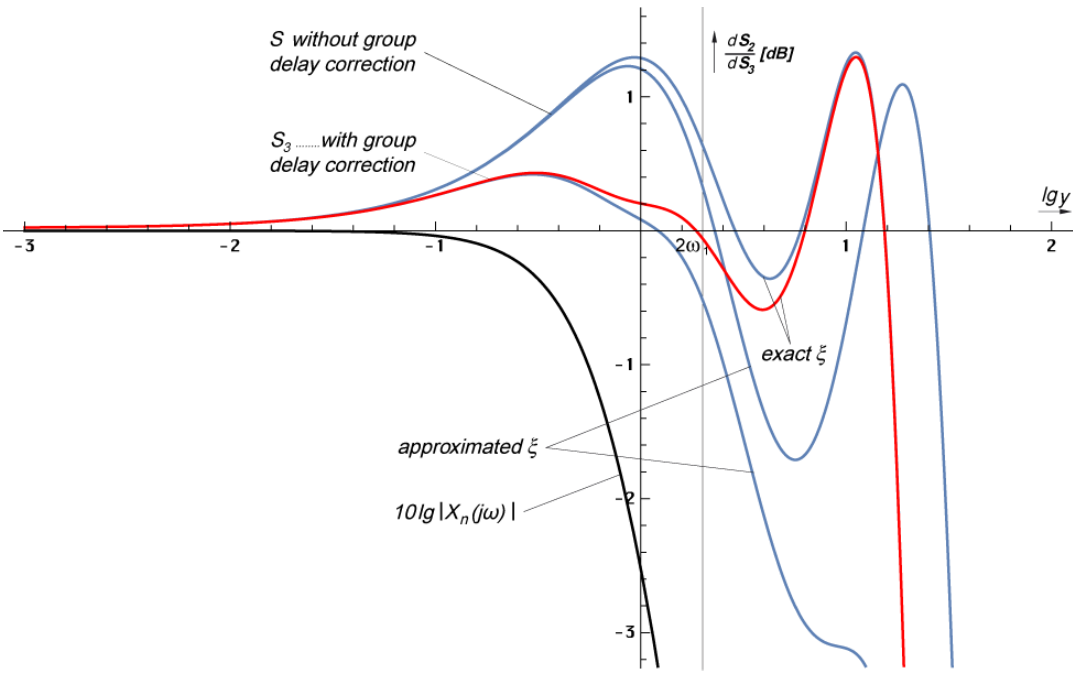

Figure 11
Relative deviation between approximation and radiation-rule according to the function ξ used without and with group delay correction

Value	$\Omega_{\pitchfork}$	$\Delta\Omega_{\pitchfork}$	$A_{\barwedge}$	$\Delta A_{\barwedge}$	$\Omega_{\doublebarwedge}$	$\Delta A_{\doublebarwedge}$	$\Omega_{\doublebarwedge}$	$\Delta A_{\doublebarwedge}$
	[1]	[%]	[dB]	[dB]	[1]	[dB]	[1]	[dB]
Planck	1.00000	± 0.00	+1.52727	±0.00000	––	––	––	––
Figure 5	0.81721	−18.28	+1.95613	+0.42886	0.41944	+1.20008	2.88334	−1.78499
Figure 6	0.86386	−13.61	+2.28561	+0.75834	0.46495	+1.29392	5.55922	+1.32996
Figure 8	0.87476	−12.52	+1.12244	−0.40484	0.14776	+0.42061	––	––
Figure 9	0.93003	− 7.00	+1.49965	−0.02762	0.15421	+0.43171	1.95909	−0.58954

Table 1
Extreme values of PLANCK's radiation-function and approximation according to the function ξ used without and with group delay correction

To the better clarity, the last case is depicted in figure 10 with higher resolution. You can find the exact results in table 1. Figure 11 shows a summary of the relative deviations of all solutions in comparison with the course of the absolute value of the complex frequency response $|X_n(j\omega)|$ of subspace.

4. The WIEN displacement

The solution according to figure 9 seems to fit to the best the observations. As we can see in figure 11, the curve oscillates around the nominal value near the upper cut-off-frequency, a behaviour, as we even know from technical minimum-phase low-pass filters (overshoot). Usually it is being suppressed by an attenuator and there is the parametric damping. Aside from that the level at the third null is already with –50dB, the rest disappears in noise.

Let's suppose, that the $^{+1.3}_{-0.5}$dB are „healed up“ during the many billion years or have been „ironed out“ by other influences not considered here – at the end, we must carry out, as promised, a WIEN-displacement. Starting with the in-coupling frequency $2\omega_1$, with the help of the expressions given in [1] section 2, we are able to calculate the temperature of the CMBR to compare it with the COBE-measuring:

Indeed, it is hard to believe, that we can actually calculate back until a point of time before the phase jump at Q=1. But the contemplations conducted in [6] turned out, that both, photons – these behaved like neutrinos in the beginning – and electrons and protons, had had different properties shortly after BB, banish the usual notions of this period to the realm of imagination.

Albeit with a different value for H_0 (71.9845 km s^{-1}Mpc^{-1}), I succeeded in [1], to calculate a CMBR-temperature of 2.79146K with the model. This was close to the 2.72548K ±0.00057K (±2.09137·10^{-4}), determined by the COBE-satellite. What works in one direction, naturally also works in the other direction. So the 2.72548K of COBE using the values from [1] match an H_0 in the amount of 68.6072 km s^{-1}Mpc^{-1}. Indeed, that's less than I calculated. Now, based on the electron, I determined, a new H_0 with an amount of 68.6241km s^{-1}Mpc^{-1} in this work. And I was not a little surprised, that it was extremely close to the COBE-value. So I assume, that the new value must be more accurate, than the one calculated in [1]. Now to the calculation.

Whereas the temperature of the metric wave field is equal to zero, it's not the case with the CMBR. Since it's about almost black radiation ($\varepsilon_v=0.9428=\tfrac{2}{3}\sqrt{2}$), we are able to calculate the *black temperature* indeed, but we want to work-on with the *grey temperature*. By transposing the WIEN displacement rule with the energetic redshift $z_{22}=12\,\varepsilon_v Q_0^{5/2}$ of ([6] 174) we obtain for $\omega_U=2\omega_1$:

$$T_k = \frac{\hbar\omega_k}{\tilde{x}k} = \frac{\varepsilon_\nu}{\tilde{x}}\frac{\hbar_1\omega_1}{6k}Q_0^{-\frac{5}{2}} = 0.055693\frac{\hbar_1\omega_1}{k}Q_0^{-\frac{5}{2}} \qquad \tilde{x}=\begin{cases} 2.821439372 & \text{Exactly} \\ 2\sqrt{2} & \text{Approx.}\end{cases} \qquad ([6]\,175)$$

$$T_k = \frac{\hbar\omega_k}{\tilde{x}k} \approx \frac{1}{3}\frac{\hbar_1\omega_1}{6k}Q_0^{-\frac{5}{2}} = \frac{\hbar_1\omega_1}{18k}Q_0^{-\frac{5}{2}} \qquad \varepsilon_\nu = \frac{\tilde{x}}{3} = 0.94048 \ \text{Exactly} \qquad ([6]\,176)$$

That's the temperature of the cosmologic background radiation in consideration of the frequency response (see figure 12). I already offered expression ([6] 176) as an approximation in [1], since the value $\tilde{x}=3+\mathrm{lx}(-3e^{-3})$ is only 0.25% below $2\sqrt{2}$, see also section 2. With it, we get an extremely simple expression, which corresponds to a value $\varepsilon_v=\tilde{x}/3$. That would be 4× the 3 in one expression and the subspace slightly greyer, as thought. Since we want to know exactly, we will verify even this approach.

$$T_k = 1.002476662335245\,\frac{\hbar_1\omega_1}{18k}Q_0^{-\frac{5}{2}} \qquad \varepsilon_\nu = \tfrac{2}{3}\sqrt{2} \qquad ([6]177)$$

$$T_k = 0.997209201884998\,\frac{\hbar_1\omega_1}{18k}Q_0^{-\frac{5}{2}} \qquad \varepsilon_\nu = \delta \qquad ([6]178)$$

$$T_k = 1.000016126070630\,\frac{\hbar_1\omega_1}{18k}Q_0^{-\frac{5}{2}} \qquad \varepsilon_\nu = 1.002814779667422 \qquad ([6]179)$$

The last, constructed case exactly brings us to the 2.72548K. Table 2 shows all possible solutions once again.

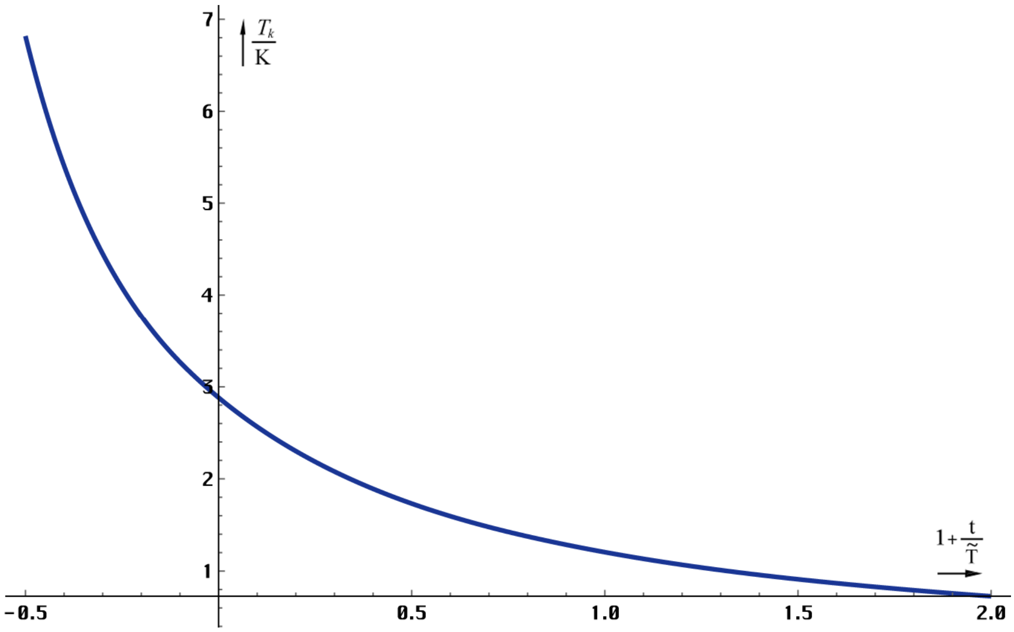

Figure 12
Temporal dependence of the radiation-temperature of the CMBR (linearly)

Value	Q_0	H_0	H_0	CMBR temperature	Absolute difference	Relative deviation
	[1]	[s^{-1}]	[$kms^{-1}Mpc^{-1}$]	[K]	[K]	[%]
(890) [1]	$7.9518\cdot10^{60}$	$2.3328\cdot10^{-18}$	71.9843	2.791460	+0.06598	+2.42086
(177) [6]	$8.3405\cdot10^{60}$	$2.2239\cdot10^{-18}$	68.6241	2.732186	+0.00671	+0.24605
(COBE)+	$8.3397\cdot10^{60}$	$2.2243\cdot10^{-18}$	68.6365	2.726050	+0.00057	+0.02091
$(COBE)_0$	$8.3404\cdot10^{60}$	$2.2239\cdot10^{-18}$	68.6250	2.725480	±0.00000	±0.00000
(179) [6]	$8.3405\cdot10^{60}$	$2.2239\cdot10^{-18}$	68.6241	2.725480	±0.00000	±0.00000
(176) [6]	$8.3405\cdot10^{60}$	$2.2239\cdot10^{-18}$	68.6241	2.725436	-4.4×10^{-5}	-0.00161
$(COBE)_-$	$8.3411\cdot10^{60}$	$2.2236\cdot10^{-18}$	68.6135	2.724910	-0.00057	-0.02091
(178) [6]	$8.3405\cdot10^{60}$	$2.2239\cdot10^{-18}$	68.6241	2.717830	-0.00765	-0.28069

Table 2
Calculated and measured CMBR-temperature in comparison with the values of the HUBBLE-parameter

The Q_0- and H_0-values for the COBE-satellite have been determined with the help of ([6]176). The upper and the lower limits of the COBE-values are yellow highlighted. As we can see, the approximation ([6]176) is very good. The value from [1] is much too high and

([6]177) is outside the measuring precision of COBE. Expression ([6]178) is out of question, since its value is below the measured one. Moreover it's not related to the model. That also applies to ([6]179). The approximation ([6]176) in contrast, seems to hit the nail on the had. Whether that's true, further, more precise measurements will prove. Thus, we define:

$$T_k = \frac{\hbar\omega_0}{18k} Q_0^{-\frac{1}{2}} = \frac{\hbar_1\omega_1}{18k} Q_0^{-\frac{5}{2}} = 2.725436049\mathrm{K} \qquad \Delta = -1.61258\cdot 10^{-5} \qquad ([6]180)$$

The calculated value is within the accuracy limits of the value 2.72548K ±0.00057K measured by the COBE-satellite and is reference frame dependent ($\sim Q_0^{-5/2}$). For the choose of the correct relation to the calculation of T_K I leave the reader room for his own interpretations. In addition, we want to calculate the corresponding frequencies for the technicians too. With the help of WIEN's displacement rule and ([6]180) we get the following relations:

$$\omega_{max} = \frac{1}{18}\tilde{x}\omega_1 Q_0^{-\frac{3}{2}} = 1.0067316\cdot 10^{12}\mathrm{s}^{-1} \qquad \nu_{max} = \frac{1}{36\pi}\tilde{x}\omega_1 Q_0^{-\frac{3}{2}} = 160.2263\,\mathrm{GHz} \qquad ([6]181)$$

5. Possible reasons of a deviation

Next, I wanted to discuss possible causes that could have led to the original discrepancy. However, due to the updated value of H_0 and Q_0, this has been done. However, I don't want to simply delete this section, as it can serve as an example for other cases.

The simplest and most troublesome cause is always that the model is wrong. However, the original result agreed reasonably well with the prediction, so there could have been another cause. Therefore, the most probable shall be discussed as next.

Since the line-element is a minimum phase system, we computed the (inaccurate) approximation function, by an iterative multiplication of the source-function with the just significant amplitude characteristic $A(\omega)$, as long as the result changes essentially. At the point the frequency of the signal-function has dropped far below the cut-off frequency, there is no more change to be observed. The factor $\cos\varphi$ emerged from the fact, that only the real-part is being transferred ($\varphi=B(\omega)$).

That's the procedure with minimum phase systems in general. But according to [3] p. 340 it applies for *stable* minimum phase systems only! Because only with these, an explicit correlation exists between amplitude- and phase response curve, so that we can calculate with the amplitude response exclusively. With the line-element just after input coupling ($Q\approx 1$), that's shortly after big bang, it's about a minimum phase system indeed, but also about a *marginally stable* system (pole and null at the point {0,0}), as I have found after correction of the calculating error.

If we want to get an exact result, we must also introduce a reference between amplitude and phase, quasi a phase-correction, because a phase-lag appears with unstable systems. At the observer the phase-lag manifests itself in the form, that the spectral shares with lower frequency are more redshifted, than the higher frequent ones. Indeed, the lower-frequent shares aren't older than the higher-frequent ones (we observe always the same point of time at the in-coupling with $Q_0=1/2$), but they have covered a longer distance. And that automatically leads to a higher redshift.

But how this longer way can be explained? The lower-frequent shares simply took a different route, than the higher-frequent ones (different angle of emission). That leads to a kind of achromatism at the observer, which is hard to be detected, since the radiation arrives from all directions at once. Even with the propagation-function (306) such a phase-lag occurred, characterized by the term $\Phi(\omega)$. We considered that term and we also took a group delay correction. Hence, it cannot be that.

Let's go to talk about the high dynamics during the in-coupling process. Figure 13 shows the course of the energy flux-density vector div$\mathbf{S_0}$ of the metric wave field at that point of time. One sees, it's positive in the range 0.52549<Q<1.5975. Thus, energy is radiated. The range is depicted even in figure 7. In the range below 0.52549 the field is been established, above 1.5975 the effect of parametric attenuation for overlaid waves can be seen.

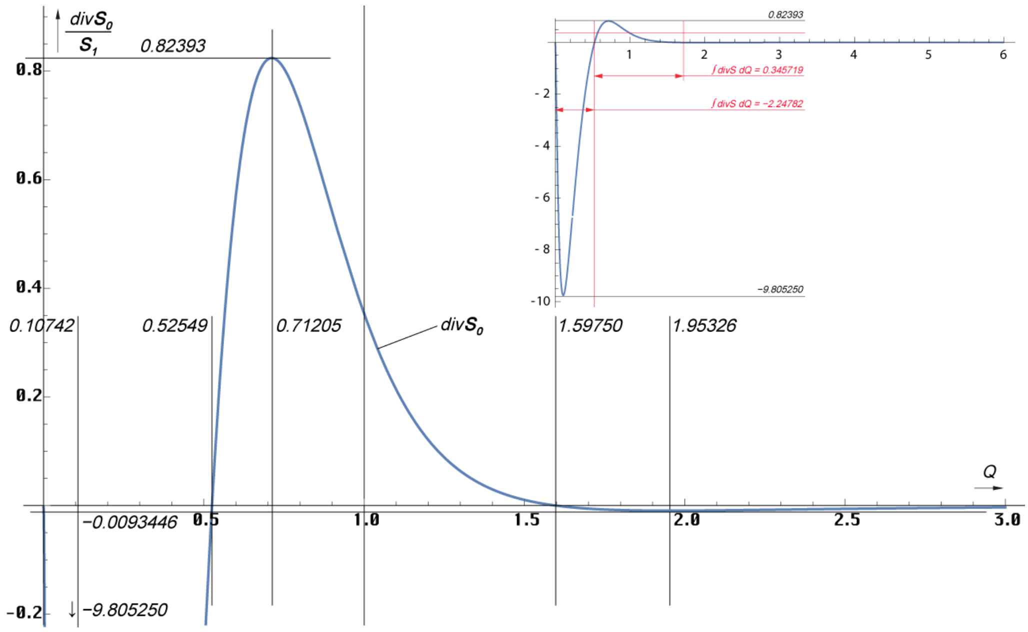

Figure 13
Course of the energy flux-density vector of the metric wave field as a function of Q

Hence, with the in-coupling process it's not about a sudden act with before → after, but it's a dynamic process. Energy is absorbed and partially re-emitted, deferred by the group delay time. At the same time the CMBR is coupled in, according to the frequency at different moments. Concerning the partial re-emission the share of absorbed energy depends on the area ratio of both left-hand sections. The numerical integration yields a value of 2.24784 for the absorbed, as well as of 0.345719 for the re-emitted energy share. The calculation 2.24784/(0.345719+2.24784) a value of 0.866700931 turns out in reference to Q. But we need the value in reference to the time t. Because t^2~Q, we must resolve the substitution t^2 on the x-axis in that we extract the root of the result. We obtain a value of 0.930967739. It corresponds, except for a deviation of 0.0118413026, to our *vacuum coefficient of absorption* ε_v=0,9428090416.

Thus, the deviation has something to do with the *gray body* [4]. Now, once we already considered ε_v indeed, but only as a constant and with the value at the time of in-coupling. But with the *gray body* ε_v depends on the frequency ω. If we want to consider that, we have to calculate an $\varepsilon_T(\omega)$ respectively a correction term $\varepsilon_K(\omega)$ to multiply ([1] 902) with, since ε_v is already included there. In [4] the following is denoted for ε_T: » Thereby ε_T correlates with the weighted averages of ε_ν resp. ε_λ, which are equal:

$$\varepsilon_T = \frac{\int\int_0^\infty \varepsilon_\nu \cdot I(\nu)\cdot \mathrm{d}\nu \cdot \mathrm{d}\Omega}{\int\int_0^\infty I(\nu)\cdot \mathrm{d}\nu \cdot \mathrm{d}\Omega} = \frac{\int\int_0^\infty \varepsilon_\lambda \cdot I(\lambda)\cdot \mathrm{d}\lambda \cdot \mathrm{d}\Omega}{\int\int_0^\infty I(\lambda)\cdot \mathrm{d}\lambda \cdot \mathrm{d}\Omega} \quad \text{from [4] «} \tag{50}$$

But we don't want to make it as quite as complicated. Therefore we assume, that the root of the area ratio should equal the average of ε_v, i.e. be equal to ε_T. It applies: $\varepsilon_T=\varepsilon_v\varepsilon_K$, with $\varepsilon_v=\frac{2}{3}\sqrt{2}$=0.942809 and ε_K=0.987440402. Multiplying the calculated T_k=2.79837K with ε_K, we obtain a value of 2.76322K, which is about +0.0377K above the measured one. But is it correct, to apply ε_K resp. ε_T simply as a factor to WIEN's displacement law? The answer is no. It's about a factor from PLANCK's radiation-rule. Applying ε_T to (1)…(7), it cancels out at the end. Herewith the inclination 2 at WIEN's displacement rule ($\tilde{x}$ is the ratio slope/peak-line) also applies to the *gray body*. But even a constant of integration would be possible here. There are influences on the displacement indeed. But these depend on the shape of the envelope-curve and, with it, on the function $\varepsilon_v(\omega)$, which we do not know. Therefore we must improvise, contriving a function, which well-complies the requirements. Then, at least, we can see, which influence a frequency-dependent ε_v has onto the shape of the curve and with it even onto the displacement itself.

As a start the function before the in-coupling must have the value $\varepsilon_{vmax}=⅔\sqrt{2}=0.942809$. Furthermore it must vary somehow. We choose a simple change from one to another value. As inflection point we choose the moment of in-coupling with $Q=1/2$ resp. $2\omega_1$. Then $y=\Omega$ applies. The 0.930967739 from the area ratio of div$\mathbf{S_0}$ are our $\overline{\varepsilon}_T$. We use the function as per (51). Therefrom a lower limit of $\varepsilon_{vmin}=0.920464$ arises. With it $\overline{\varepsilon}_T$ is a little bit smaller than the average, due to the function used. All that appears plausible on the whole, because the metric wave field mostly picks up energy before the in-coupling. Thus, it has a higher absorption coefficient as thereafter, when a share of energy is re-emitted. Even the offset of the zero-transition of div$\mathbf{S_0}$ of $Q=0.52549$ is mapped very well. If you don't like it, it's only a model and an optimized example function. Whether it really happens in that manner, is another matter.

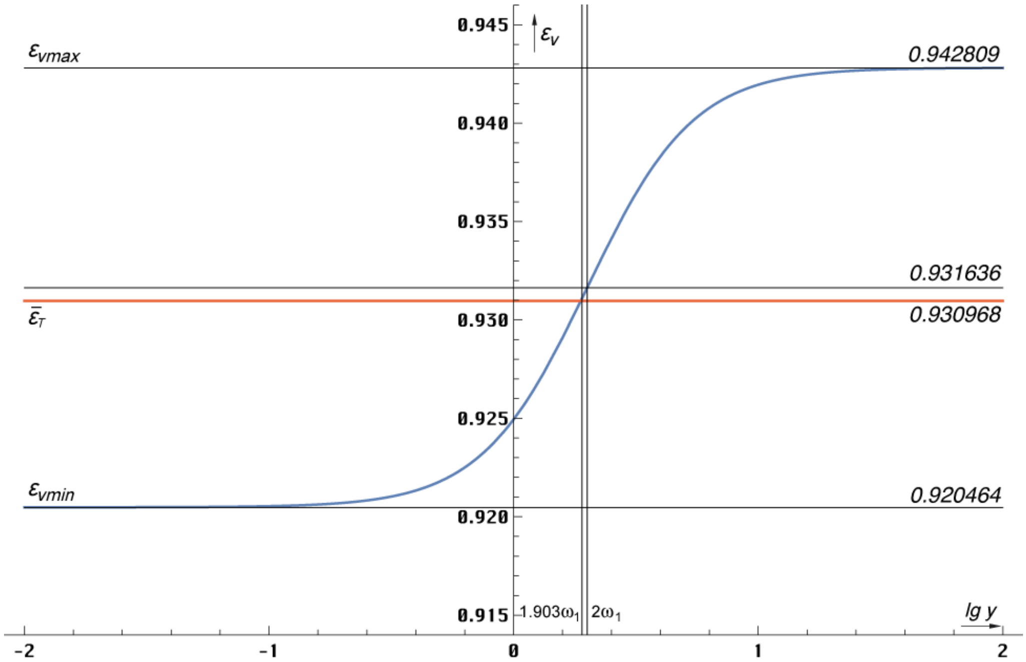

Figure 14
Vacuum coefficient of absorption ε_v as a function of ω

$$\varepsilon_v = \varepsilon_{v\max}\left(1-2\frac{\varepsilon_{v\max}-\overline{\varepsilon}_T}{1+\Omega^2}\right) \qquad \varepsilon_{v\min} = \varepsilon_{v\max}\left(1-2\left(\varepsilon_{v\max}-\overline{\varepsilon}_T\right)\right) \tag{51}$$

$$\varepsilon_T = \frac{2}{3}\sqrt{2}\left(1-\frac{0.02368}{1+\Omega^2}\right) \qquad \varepsilon_K = 1-\frac{0.02368}{1+\Omega^2} \qquad \begin{matrix}\varepsilon_{K\max}=1.00000\\ \varepsilon_{K\min}=0.97630\end{matrix} \tag{52}$$

Now we want to analyze the effect of ε_K on the envelope-curve. We believe in the „*self-healing powers*" of the solution of figure 9 using a clean PLANCK-curve. Since the effect on (51) is hardly to be seen in the graphics, we use an additional, *exaggerated function* ε_{T5} to the better presentation.

$$\varepsilon_{T5} = \frac{2}{3}\sqrt{2}\left(1-\frac{0.5}{1+\Omega^2}\right) \qquad \varepsilon_{K5} = 1-\frac{0.5}{1+\Omega^2} \tag{53}$$

That corresponds to an $\overline{\varepsilon}_{T5}=0.69281$. We obtain the following course with it:

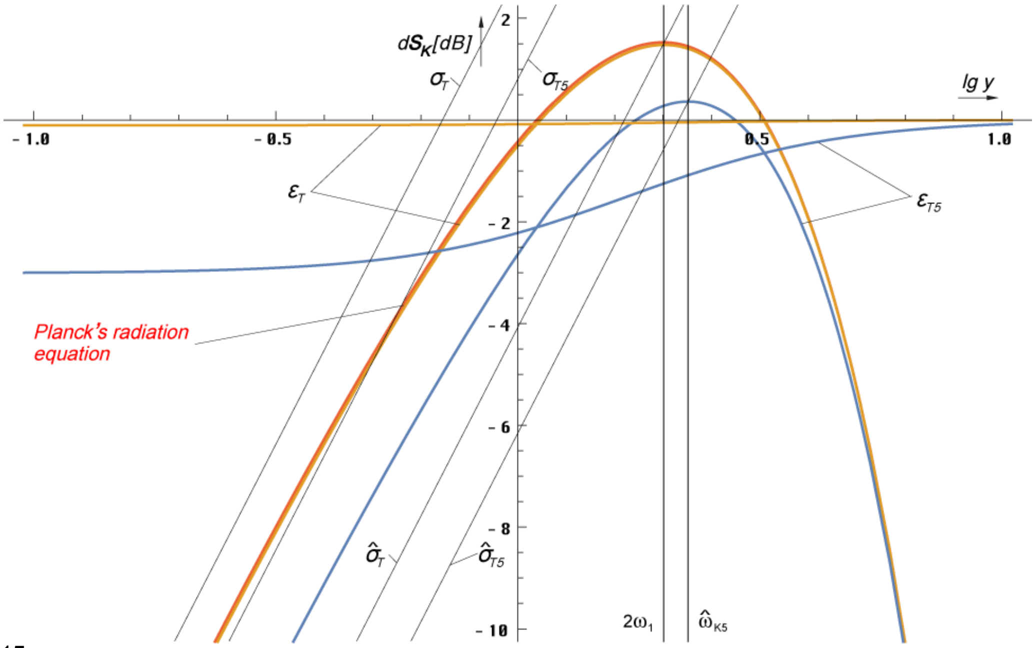

Figure 15
Effect of the absorption coefficient ε_v
onto the envelope-curve, high resolution

One sees, the function (52) mostly affects the lower-frequent part of the envelope-curve. The maximum is up-shifted in frequency. But the inclination in the left part remains constant. That applies as I said to the example function only. Natural materials may distort the envelope-curve significantly even in this region. Then the regression line applies as a function of $\overline{\varepsilon}_T$ according to (50). Then it has the same inclination and even only, it's more or less amplitude-shifted (constant of integration!). B.t.w. the regression line σ_T resp. the lower-frequent slope is also the line, the WIEN displacement happens at. Here we can see the benefit of the duplicate logarithmic presentation, the curve becomes a line then.

The regression line σ_T can be determined by trying out most suitably. It applies $y=\Omega$ too. In the duplicate logarithmic presentation the following functions arise:

$$\sigma_T(\Omega) = 10(2\Omega + \lg(2\varepsilon_{K\,min})) \quad [\text{dB}] \qquad \text{Slope} \qquad (54)$$

$$\hat{\sigma}_T(\Omega) = 10(2\Omega - \lg\tilde{x} + \lg\varepsilon_{K\,min}) \quad [\text{dB}] \qquad \text{Maximum} \qquad (55)$$

$$\sigma_T(\Omega) = 2\varepsilon_{K\,min}10^{2\Omega} = 2\varepsilon_{K\,min}e^{2\ln 10\Omega} = 2\varepsilon_{K\,min}e^{4.60517\Omega} \qquad \text{Slope linearly} \qquad (56)$$

That only applies to the example function used here. The 2 on the right side stems from the definition of Ω according to (9). To the *black body* and with it, even to the PLANCK-curve applies $\varepsilon_{Kmin}=\varepsilon_T=\varepsilon_{Kmax}=1$. With natural materials we must replace ε_{Kmin} by $\overline{\varepsilon}_T$ from (50). The course is shown in figure 15. Of course even a regression line for the maximum can be defined. With it ($\tilde{x}$), the circle closes to WIEN's displacement law. However expression (55) isn't very accurate and the line may miss the maximum with smaller ε_{vmin}. But it applies exactly to the *black body* and to our example function. With natural materials even more than one maximum may occur. The more the envelope-curve differs from the ideal, the less reasonable is it, to speak of a radiation temperature.

From (55) arises, that we, nevertheless can define a WIEN's displacement law for the *gray body*, at least for the example function and when the curve-shape do not differ too far from that of a *black body*:

$$T \approx \frac{1}{\tilde{x}\,\varepsilon_{\mathrm{K\,min}}}\frac{\hbar\omega_{\max}}{\mathrm{k}}$$ WIEN's displacement law for the *gray body* (57)

With natural materials $\varepsilon_{\mathrm{Kmin}}$ must be replaced by $\overline{\varepsilon}_{\mathrm{T}}$ again.

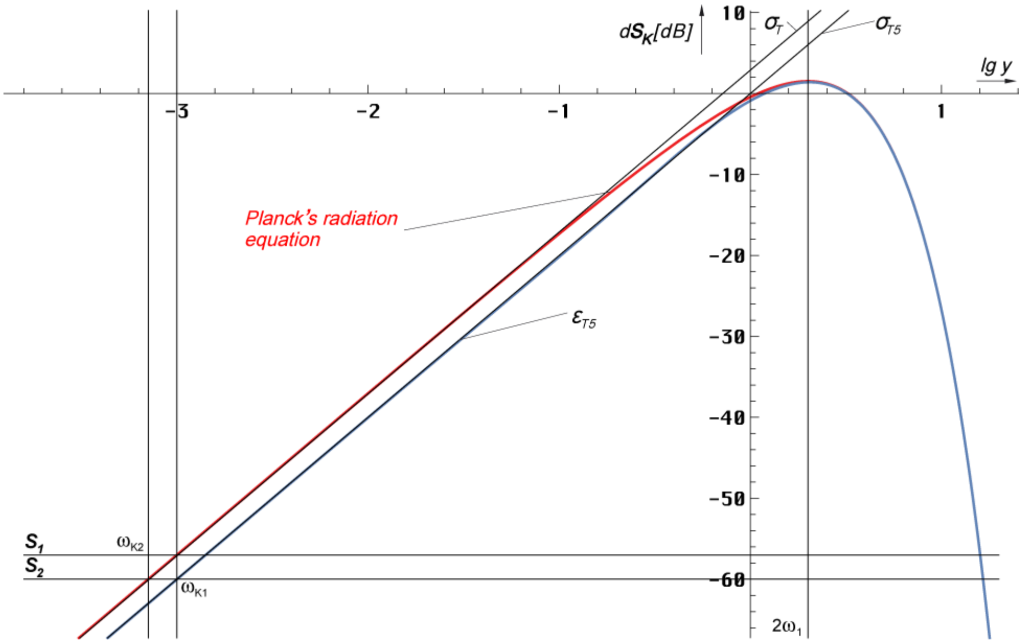

Figure 16
Displacement lines σ_T and σ_{T5}
as well as envelope-curves, low resolution

As next we want to determine the frequency-shift ω_{K2}/ω_{K1}. We choose the *exaggerated function* (53), since we cannot see anything otherwise. We want to navigate in the lower-frequent range, namely at $\omega_{K1}=0.5\cdot10^{-3}\,\omega_{\max}$. Therefore we can employ WIEN's radiation-rule:

$$\mathrm{d}\mathbf{S}_1 \approx \frac{1}{4\pi^2}\frac{\hbar\omega_{\mathrm{K1}}^3}{\mathrm{c}^2}\,\mathrm{e}^{\frac{\hbar\omega_{\mathrm{K1}}}{\mathrm{k}T}}\,\mathbf{e}_s\,\mathrm{d}\omega$$ WIEN's radiation rule (58)

To the amplitude of $\mathrm{d}\mathbf{S}_2$ applies ($T_1=T_2=T$) :

$$\mathrm{d}\mathbf{S}_2 \approx \frac{\varepsilon_{\mathrm{K\,min}}}{4\pi^2}\frac{\hbar\omega_{\mathrm{K1}}^3}{\mathrm{c}^2}\,\mathrm{e}^{\frac{\hbar\omega_{\mathrm{K1}}}{\mathrm{k}T}}\,\mathbf{e}_s\,\mathrm{d}\omega = \frac{1}{4\pi^2}\frac{\hbar\omega_{\mathrm{K2}}^3}{\mathrm{c}^2}\,\mathrm{e}^{\frac{\hbar\omega_{\mathrm{K2}}}{\mathrm{k}T}}\,\mathbf{e}_s\,\mathrm{d}\omega \tag{59}$$

By equating we obtain the following expression:

$$\omega_{\mathrm{K2}}^3 = \varepsilon_{\mathrm{K\,min}}\,\omega_{\mathrm{K1}}^3\,\mathrm{e}^{\frac{\hbar\omega_{\mathrm{K1}}}{\mathrm{k}T}-\frac{\hbar\omega_{\mathrm{K2}}}{\mathrm{k}T}} = \varepsilon_{\mathrm{K\,min}}\,\omega_{\mathrm{K1}}^3\,\mathrm{e}^{\frac{\hbar}{\mathrm{k}T}(\omega_{\mathrm{K1}}-\omega_{\mathrm{K2}})} \tag{60}$$

$$\frac{\hbar}{\mathrm{k}T} = \frac{2.821439372}{\omega_{\max}} = \frac{2.821439}{2\cdot10^3\,\omega_{\mathrm{K1}}} = \frac{1.41072\cdot10^{-3}}{\omega_{\mathrm{K1}}} \tag{61}$$

$$\omega_{\mathrm{K2}}^3 = \varepsilon_{\mathrm{K\,min}}\,\omega_{\mathrm{K1}}^3\,\mathrm{e}^{1.41072\cdot10^{-3}\left(1-\frac{\omega_{\mathrm{K2}}}{\omega_{\mathrm{K1}}}\right)} \approx \varepsilon_{\mathrm{K\,min}}\,\omega_{\mathrm{K1}}^3\,\mathrm{e}^0 = \varepsilon_{\mathrm{K\,min}}\,\omega_{\mathrm{K1}}^3 \tag{62}$$

$$\omega_{\mathrm{K2}} = \sqrt[3]{\varepsilon_{\mathrm{K\,min}}}\,\omega_{\mathrm{K1}} = \sqrt[3]{0.97630}\,\omega_{\mathrm{K1}} = 0.992037\,\omega_{\mathrm{K1}} \tag{63}$$

With it the frequency of our example function shifts downward by +0.8027% at the base. The offset of the maximum is +0.4860% (Function FindMaximum[#]). Just for information, with the *exaggerated function* ε_{T5} the base-shift is at +25.99%, at the maximum at +12.64%. Thus, in both cases a narrowing of the envelope-curve occurs, at which point the frequency shift at the base is nearly twice as large, as at the maximum. Because with the real values only fractions of a percent come into effect, it looks like the curve is *black*.

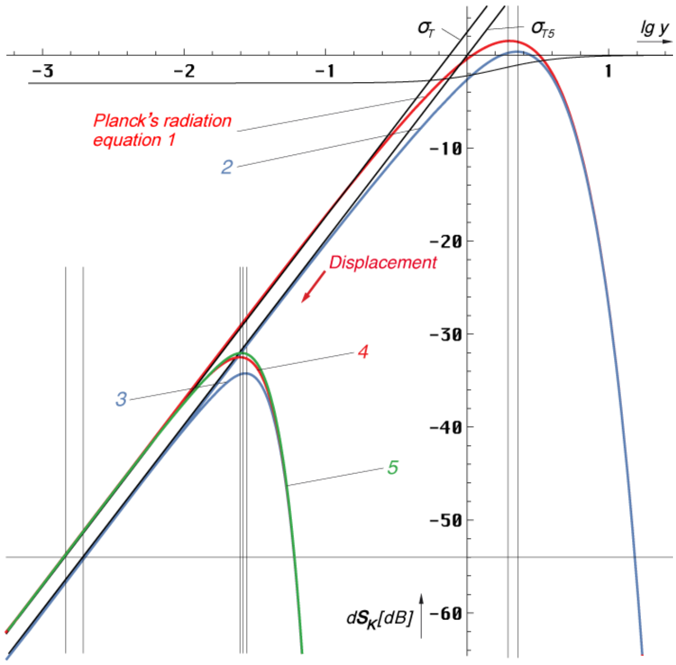

Figure 17
Possible error sources by misinterpretation of the curve-characteristic

Subsequently it's about errors in the interpretation of actual measured data only. The model itself is no issue and it's irrelevant, whether any universal natural constants change over time or not and how. Figure 16 shows what may happen, if we misinterprete the curve-characteristic, by a mistaken application of the *black body* mathematics to a *gray curve*. Curve #1 is the curve of a *black body* at the moment of in-coupling, curve #2 is the *gray curve*. The redshift z (displacement) takes place in the direction of arrow along the displacement line σ_T and σ_{T5}. You can perform it in a graphics program even manually in the following manner: At first duplicate the graph. Then scale it equably by shifting the corner point right above to the bottom left with pressed shift key, maintaining the contact with the displacement line left.

The result are the curves 3 and 4. Now, however the *gray curve* 3 can be „inflated" in such a manner, that it almost fits the *black curve* 4, that's curve 5 (green). This happens, when a too small redshift z is being assumed, a value, which we actually wanted to determine. One sees, it's possible to wangle a nearly perfect covering of the maxima. The difference is, in practice, nearly undetectable with ε_T-values near 1. The result is, that a too small z and a too small radiation temperature T_k is calculated, and that by half the offset at the base.

Presuming the calculated T_k-value in the amount of 2.79837K to be the *gray temperature,* under consideration of the interpretation error at the measured value of 2.72548K, the application of (57) a measured *gray temperature* of 2,79164K turns out. Then the calculated temperature is only +0.0067K above (+0.25%). Thus, in contrast to the hitherto +7.29%, the improvement wouldn't be insignificant. Of course, I could have configured the example function even such, that I hit the measured value exactly. But that would not have been very meaningful.

In any case, the effects of a possible *gray radiation-characteristic* should be considered, especially then, when we want to measure extremely accurate. But then we can forget the declared accuracy of ±0.00057K for the measured value resp. it applies only relatively and not absolutely.

6. Summary

In the course of this article, according to the model in [1], we succeeded in approximating the envelope-curve of PLANCK's radiation-rule as a function of a dynamic frequency response under application of a phase- and group-delay-correction with a residual deviation of $^{+1,3}_{-0,5}$ dB. Furthermore it was shown, that the temperature calculated in [1] is in the proximity of the value measured by the COBE-satellite. With the help of the updated values of H_0 and Q_0, determined in [6] a more recent CMBR-temperature could be calculated, which well fits the accuracy limits of the radiation temperature, measured by the COBE/WMAP-satellite.

The results of the work on hand don't exclude the possibility, that the course of the PLANCK's radiation-rule could be the result of the existence of an upper cut-off frequency of the vacuum. Both, the classic definition formula, and the approximation are compatible and complement each other.

7. Explanatory notes to the annex

The expressions and definitions used in this work are described in the annex and can be calculated. It's about the source code for *Mathematica/Alpha*. The data can be transferred using copy&paste via the clipboard. You can also save it into a text file (UTF8), which can be opened and evaluated directly. Advantageously, you should not copy the whole source code into one single cell but section per section. Calculation may take about one hour.

THE END

8. References

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Institut für Experimentelle Physik, Universität Ulm
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E−mail−adress: GerdPommerenke@arcor.de
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E−Mail−Adresse: GerdPommerenke@arcor.de
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[8] **Manfred Zollner**
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(Zuletzt abgerufen: 22. Februar 2022, 18:03 UTC)

Enveloppe Curve Approximation

Copy Friendly Version
Choose »Evaluate Notebook« from Menu

Declarations

```
Off[General::spell]
Off[General::spell1]
Off[InterpolatingFunction::dmval]
Off[FindMaximum::lstol]
Off[FindRoot::nlnum]
Off[NIntegrate::inumr]
Off[NIntegrate::precw]
Off[NIntegrate::ncvb]
```

Units

```
km=1000;
Mpc=3.08572*10^19 km;
minute=60;
hour=60 minute;
day=24*hour;
year=365.24219879*day;
```

Basic values

```
c=2.99792458*10^8;
my0=4 Pi 10^-7;
ka0=1.3697776631902217*10^93;
hb1=8.795625796565464*10^26;
k=1.3806485279*10^-23;
me=9.109383701528*10^-31;           (*Electron rest mass with Q0 Magic value 1*);
mp=1.6726219236951*10^-27;                    (*Proton rest mass Magic value 2*);
```

Auxilliary values

```
mep=SetPrecision[me/mp,20];
ϵ=ArcSin[0.30282212088197429933345006247691344477]-3Pi/4;
γ=Pi/4-ϵ;
ζ=1/(36Pi^3)(3Sqrt[2])^(-1/3)/mep;
xtilde=3+ProductLog[-3E^-3];
alpha=Sin[Pi/4-ϵ]^2/(4Pi);
delta=4Pi/alpha*mep;
Q0=(9Pi^2 Sqrt[2]delta me/my0/ka0/hb1)^(-3/7);
```

Composed expressions

```
Z0=my0 c;
ep0=1/(my0 c^2);
R∞=1/(72 Pi^3)/r1 Sqrt[2] alpha^2 /delta Q0^(-4/3);
Om1=ka0/ep0;
Om0=Om1/Q0;
OmR∞=2Pi c R∞;
cR∞=c R∞;
H0=Om1/Q0^2;
H1=3/2*H0;
r1=1/(ka0 Z0);
a0=9Pi^2 r1 Sqrt[2] delta/alpha Q0^(4/3);
λbarC=a0 alpha;
λC=2 Pi λbarC;
re= r1 (2/3)^(1/3)/ζ Q0^(4/3);
r0= r1 Q0;
```

```
R= r1 Q0^2;
RR=R/Mpc/1000;
t1=1/(2 Om1);
t0=1/(2 Om0);
T=1/(2 H0);
TT=2T/year;
hb0=hb1/Q0;
h0=2Pi*hb0;
q1=Sqrt[hb1/Z0];
q0=Sqrt[hb1/Q0/Z0];
qe=q0 Sin[Pi/4-ϵ];
M2=my0 ka0 hb1;
M1=M2/Q0;
m0=M2/Q0^2;
mp=4Pi me/alpha/delta;
MH=M2/Q0^3;
G0 =c^2*r0/m0;
G1=G0/Q0^2;
G2=G0/Q0^3;
U0=Sqrt[c^4/4/Pi/ep0/G0];
U1=U0*Q0;
W1=Sqrt[hb1 c^5/G2];
W0=W1/Q0^2;
S1=hb1 Om1^2/r1^2;
S0=S1/Q0^5;
μB=-9/2Pi^2 Sqrt[2 hb1/Z0]delta Sin[γ]/my0/ka0 Q0^(5/6);
μN=-μB*mep;
μe=1.0011596521812818 μB;
Tk1=hb1 Om1/18/k;
Tk0=Tk1/Q0^(5/2);
Tp0=0.;
Tp1=0.;
Φ0=Pi Sqrt[hb1 Z0/Q0 ]/Sin[Pi/4-ϵ];
GQ0=1/Pi/Z0*Sin[Pi/4-ϵ]^2;
KJ=2q0 Sin[Pi/4-ϵ]/h0;
RK=.5my0 c/alpha;
σe=8Pi/3 re^2;
ae=SetPrecision[μe/μB,20]-1;
ge=-2(1+ae);
γe=2 Q0 Abs[μe]/hb1;
σ=Pi^2/60k^4/c^2/hb1^3*Q0^3;
```

Functions needed

```
A=Function[(BesselJ[0,#]*BesselJ[2,#]+BesselY[0,#]*BesselY[2,#])/
       (BesselJ[0,#]^2+BesselY[0,#]^2)];
B=Function[(BesselY[0,#]*BesselJ[2,#]-BesselJ[0,#]*BesselY[2,#])/
       (BesselJ[0,#]^2+BesselY[0,#]^2)];
ThetaQ=Function[2*A[#]*B[#]/(1-A[#]^2+B[#]^2)]; ArgThetaQ=Function[Arg[1-
A[#]^2+B[#]^2+I*2*A[#]*B[#]]]; PhiQ=Function[If[#>10^4,-Pi/4-3/4/#,
 Arg[-2*I/#/Sqrt[1-(HankelH1[2,#]/HankelH1[0,#])^2]]]];
Rho=Function[Abs[-2*I/Sqrt[#]/Sqrt[1-
(HankelH1[2,Sqrt[#]]/HankelH1[0,Sqrt[#]])^2]]];
RhoQ=Function[If[#<10^4,N[Abs[-2*I/#/Sqrt[1-
(HankelH1[2,#]/HankelH1[0,#])^2]]],1/Sqrt[#]]];
RhoQQ=Function[If[#<10^4,Sqrt[Sqrt[(1-
A[#]^2+B[#]^2)^2+(2*A[#]*B[#])^2]],2/Sqrt[#]]];
AlphaQ=Function[N[Pi/4-PhiQ[#]]];
GammaPQ=Function[N[PhiQ[#]+ArcCos[RhoQ[#]*Sin[AlphaQ[#]]]+Pi/4]];
rq={{0,0}};
For[x=-8;i=0,x<4,++i,x+=.01;AppendTo[rq,{10^x,N[1/RhoQQ[10^x]]}]];
RhoQ1=Interpolation[rq];
RhoQQ1=Function[If[#<10^4,RhoQ1[#],1/2Sqrt[#]]];
Rk=Function[If[#<10^4,3*Sqrt[#]*NIntegrate[RhoQQ1[x],{x,0,#}],#^2]];
Rn=Function[Abs[3*Sqrt[#]*NIntegrate[RhoQQ1[x]*Exp[-
I/2*(ArgThetaQ[x]+Pi)],{x,0,#}]]];
RnB=Function[Arg[3*Sqrt[#]*NIntegrate[RhoQQ1[x]*Exp[-
I/2*(ArgThetaQ[x]+Pi)],{x,0,#}]]];
BRQP=Function[Rk[#] Sqrt[(Sin[AlphaQ[#]]/Sin[GammaPQ[#]])^4-1]];
```

```
BGN=Sqrt[2]*BRQP[.5]/3;
brq={{0,0}};
For[x=(-8); i=0,x<25,(++i),x+=.1;
AppendTo[brq,{10^x,N[BRQP[10^x]/BGN/(2.5070314770581117*10^x) ]}]]
BRQ0=Interpolation[brq];
BRQ1=Function[If[#<10^15,BRQ0[#],Sqrt[#]]];
 M1=Function[Abs[HankelH1[0,#]]];
SGenau=Function[Pi/2*Rho[#]^2*Abs[HankelH1[0,Sqrt[#]]^2]];
(*kk=Function[Expp[Sqrt[2]*Log10[E]*#/(1+#^2)]] was wrong*)
gdc=Function[10^(Log10[E]*(-1) (1*#)^2/(1 + 1*#^2)^2)] ;
(*Group Delay Correction*)
AnU=Function[.5*1/Sqrt[1+#^2]*(1+1/Sqrt[1+#^2])];
FG=Function[.5/(1+I*#)*(1+1/(1+I*#))];
Xline=Function[10^33*(#1-#2(*Wert_x*))];
Xlline=Function[33+(10^#1-Log10[#2](*Wert_x*))];
Pom=Function[Print[StringJoin["x = ",ToString[10^Chop[First[xx/.Rest[%]],10^-7]],
" Om1", "  (",ToString[.5*10^Chop[First[xx/.Rest[#]],10^-7]]," OmU)"]]];
Pol=Function[Print["y = "<>ToString[First[#]]<>" dB ("<>
  If[First[#]-zzz>0,"+",""]<>ToString[First[#]-zzz]<>" dB)"]];
Expp=Function[If[#<0,1/Exp[-#],Exp[#]]];
(* Strictly needed to avoid calculation errors *)
```

Functions used for calculations in article

```
cc = xtilde^2;
b  = xtilde;
S1 = 8*(#1/(2*((#1/2)^2 + 1)))^2 & ;
S2 = (b*(#1/2))^3/(Expp[b*(#1/2)] - 1) & ;

Psi1 = NIntegrate[(1/2)*Log[1 + (#1/(cc*Sqrt[Q]))^2] -
       ((#1/(cc*Sqrt[Q]))^2)/(1 + (#1/(cc*Sqrt[Q]))^2) -
       Log[Cos[-ArcTan[#1/(cc*Sqrt[Q])] +
          #1/(cc*Sqrt[Q])/(1 + (#1/(cc*Sqrt[Q]))^2)]],
      {Q, 0.5, 3000}] & ;

Psi2 = NIntegrate[(1/2)*Log[1 + (#1/(cc*BRQ1[Q]))^2] -
       ((#1/(cc*BRQ1[Q]))^2)/(1 + (#1/(cc*BRQ1[Q]))^2) -
       Log[Cos[-ArcTan[#1/(cc*BRQ1[Q])] +
          #1/(cc*BRQ1[Q])/(1 + (#1/(cc*BRQ1[Q]))^2)]],
      {Q, 0.5, 3000}] & ;
```

Approximation

```
(*b = xtilde; Figure1 *)
Plot[{Log10[(b*.5*10^y)^3/(Expp[b*.5*10^y]-1)],
Log10[ 8*(.5*10^y/((.5*10^y)^2+1))^2],
Xline[y,Log10[2]]},{y, -5, 3},PlotRange->{-10.1,.45}]
```

Expansion

```
Plot[{(*Log10[BRQP[10^qqq]/BGN/(2.5070314770581117×10^qqq)], Figure2a *)
Log10[BRQ1[10^qqq]], Log10[Sqrt[10^qqq]]}, {qqq, -1, 10}]
Plot[{(*BRQP[qqq]/BGN/(2.5070314770581117×qqq), Figure2b *)
BRQ1[qqq], Sqrt[qqq]}, {qqq, 0, 10}, PlotRange -> {-0.3, 9.6}]
```

Integral

```
cc=8; (*Factor 8 approx ξ Figure3 *)
Plot[{Psi1[y],Psi2[y]},{y,0,10},
PlotStyle->RGBColor[0.91,0.15,0.25],PlotLabel->None,
LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}]

cc=8; (*Factor 8 approx ξ Skipped *)
Plot[{Expp[Psi1[y]],Expp[Psi2[y]]},{y,-4,4},PlotLabel->None,
LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}]
```

```
cc=8; (*Factor 8 approx ξ Figure4 *)
Plot[{10Log10[Expp[Psi1[10^y]]],10 Log10[Expp[Psi2[10^y]]]},{y,-3,2},
PlotRange->{-88,2},LabelStyle->{FontFamily->"Chicago",12,GrayLevel[0]}];

b4=Plot[{10 Log10[Abs[FG[10^y]]]},{y,-3,2},PlotRange->{-88,2},PlotLabel->None,
PlotStyle->RGBColor[0,0,0],LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}];
Show[%%,b4]
```

Approximation 1

```
cc=8; (* Factor 8 approximated BGN exact Figure5 *)
Plot[{10 Log10[S2[10^y]],10 (Log10[S1[10^y]*Expp[Psi1[10^y]]]),Xline[y,Log10[2]]},
{y,-3,3},PlotRange->{-51,10.5},ImageSize->Full,
LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}] (*All exact error max +1.3dB*)

cc=7.519884824; (* Sqrt[π] exact ξ Figure6 *)
Plot[{10 Log10[S2[10^y]],10
(Log10[S1[10^y]]+Log10[E]*Psi2[10^y]),Xline[y,Log10[2]]},{y,-3,3},
PlotRange->{-51,4.5},ImageSize->Full,
LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}] (*All exact error max +1.3dB*)
```

Extrema 1

```
u=FindMaximum[10 Log10[S2[10^xx]],{xx, 0}];
(* Planck's curve *)
Print[StringJoin["x = ",ToString[(10^First[xx/.Rest[u]])],
" Om1         (1.000000 OmU)"]]
Print[StringJoin["y = ",ToString[zzz = First[u]],
" dB     (±0.000000 dB)"]]

FindMaximum[
  10 (Log10[S1[10^xx]*Expp[Psi1[10^xx]]]) - 10 Log10[S2[10^xx]],
{xx, 0}] ;
(* Maximum deviation 1 Psi1 *)
Pom[%]
Pol[%%]

FindMinimum[
  10 (Log10[S1[10^xx]*Expp[Psi1[10^xx]]/S2[10^xx]]), {xx, 2}];
(* Maximum deviation 2 Psi1 *)
Pom[%]
Pol[%%]

FindMaximum[
  10 (Log10[S1[10^xx]*Expp[Psi2[10^xx]]]) - 10 Log10[S2[10^xx]],
{xx, 0}];
(* Maximum deviation 1 Psi2 *)
Pom[%]
Pol[%%]

FindMaximum[
  10 (Log10[S1[10^xx]*Expp[Psi2[10^xx]]]) - 10 Log10[S2[10^xx]],
{xx, 1}];
(* Maximum deviation 2 Psi2 *)
Pom[%]
Pol[%%]

FindMaximum[10 (Log10[S1[10^xx]] + Log10[E]*Psi1[10^xx]), {xx, 0}];
(* Deviation between maxima Psi1 *)
Pom[%]
Pol[%%]

FindMaximum[10 (Log10[S1[10^xx]] + Log10[E]*Psi2[10^xx]), {xx, 0}];
(* Deviation between maxima Psi2 *)
Pom[%]
Pol[%%]
```

Deviation 1

```
cc=8; (*Factor 8 approx ξ Figure7 *)
b71=Plot[{10 Log10[S1[10^y]*Expp[Psi1[10^y]]/S2[10^y]],Xline[y,Log10[2]]},
{y,-3,2},PlotRange->{-3.1,1.35},ImageSize->Full,
LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}];

cc=7.519884824; (* Sqrt[π] exact ξ *)
b72=Plot[{10 Log10[S1[10^y]*Expp[Psi2[10^y]]/S2[10^y]]},{y,-3,2},
ImageSize->Full,LabelStyle->{FontFamily->"Chicago",10,GrayLevel[0]}];

b73=Plot[{-10 Log10[gdc[10^x]]}, {x, -3, 2.2}, PlotRange -> {-3.02, 1.42},
   PlotStyle -> RGBColor[0.06, 0.52, 0.]];
Show[b71, b72, b73, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 12, GrayLevel[0]}]
```

Approximation 2

```
cc = 8; (* Factor 8 approximated BGN exact Figure8 *)
Plot[{
  10 Log10[S2[10^y]],
  10 (Log10[S1[10^y]*Expp[Psi1[10^y]]]) + 10 Log10[gdc[10^y]],
  Xline[y, Log10[2]]
}, {y, -3, 3}, PlotRange -> {-51, 4.5}, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}]
(* Exakt exakt exakt Fehler max +1.3dB *)

cc = 7.519884824; (* Sqrt[π] exact ξ Figure9 *)
Plot[{
  10 Log10[S2[10^y]],
  10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) + 10 Log10[gdc[10^y]],
  Xline[y, Log10[2]]
}, {y, -3, 3}, PlotRange -> {-51, 4.5}, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}]
(* Exact exact exact deviation max +1dB *)
```

Extrema 2

```
v=FindMaximum[10 Log10[S2[10^xx]],{xx, 0}];
(* Planck's curve *)
Print[StringJoin["x = ",ToString[(10^First[xx/.Rest[v]])],
" Om1         (1.000000 OmU)"]]
Print[StringJoin["y = ",ToString[zzz = First[v]],
" dB     (±0.000000 dB)"]]

FindMaximum[
  10 Log10[(S1[10^xx]*Expp[Psi1[10^xx]]*gdc[10^xx])/S2[10^xx]],
{xx, 0}];
(* Maximum deviation 1 Psi1 *)
Pom[%]
Pol[%%]

FindMaximum[
  10 Log10[(S1[10^xx]*Expp[Psi2[10^xx]]*gdc[10^xx])/S2[10^xx]],
{xx, 0}];
(* Maximum deviation 1 Psi2 *)
Pom[%]
Pol[%%]

FindMinimum[
  10 Log10[(S1[10^xx]*Expp[Psi2[10^xx]]*gdc[10^xx])/
     S2[10^xx]], {xx, .5}];
(* Maximum deviation 2 Psi2 *)
Pom[%]
Pol[%%]

FindMaximum[
  10 Log10[(S1[10^xx]*Expp[Psi2[10^xx]]*gdc[10^xx])/S2[10^xx]],
```

```
{xx, 1}];
(* Maximum deviation 3 Psi2 *)
Pom[%]
Pol[%%]

FindMaximum[10 Log10[S1[10^xx]*Expp[Psi1[10^xx]]*gdc[10^xx]], {xx, 0}];
(* Deviation between maxima Psi1 *)
Pom[%]
Pol[%%]

FindMaximum[10 Log10[S1[10^xx]*Expp[Psi2[10^xx]]*gdc[10^xx]], {xx, 0}];
(* Deviation between maxima Psi2 *)
Pom[%]
Pol[%%]

Plot[{(* Figure10 *)
  10 Log10[S1[10^y]],
  10 Log10[S2[10^y]],
  10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]),
  10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y] + Log10[gdc[10^y]]),
  Xline[y, Log10[2]]
  }, {y, -0.8, 1.4}, PlotRange -> {-11, 4.5},
PlotLabel -> None, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}]
```

Deviation 2

```
cc = 7.519884824; (* Sqrt[π] exact ξ Figure11 *)
b11=Plot[{10 Log10[S1[10^y]*Expp[Psi1[10^y]]/S2[10^y]] +
   10 Log10[gdc[10^y]],
   10 Log10[S1[10^y]*Expp[Psi2[10^y]]/S2[10^y]] +
   10 Log10[gdc[10^y]]}, {y, -3, 2}, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}];
Show[b11, b71, b72, b4, PlotRange -> {-3.02, 1.42}]
```

Nulls

```
n1 = y/. FindRoot[10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) +
     10 Log10[gdc[10^y]] - 10 Log10[S2[10^y]] == 0, {y, 0}]
n2 = y/. FindRoot[10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) +
     10 Log10[gdc[10^y]] - 10 Log10[S2[10^y]] == 0, {y, .75}]
n3 = y/. FindRoot[10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) +
     10 Log10[gdc[10^y]] - 10 Log10[S2[10^y]] == 0, {y, 1.1}]
N[10^n1]     (* Level at 1st null *)
ToString[10 Log10[S2[%]]] <> " dB"
N[10^n2]     (* Level at 2nd null *)
ToString[10 Log10[S2[%]]] <> " dB"
N[10^n3]     (* Level at 3rd null *)
ToString[10 Log10[S2[%]]] <> " dB"
N[10^1.4142](* Level after 3rd null *)
ToString[10 Log10[S2[%]]] <> " dB"

Plot[{(* Skipped *)
  10 Log10[S1[10^y]],
  10 Log10[S2[10^y]],
  10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]),
  10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) + 10 Log10[gdc[10^y]],
  Xline[y, Log10[2]]
  }, {y, -3, 3}, PlotRange -> {-51, 4.5},
PlotLabel -> None, ImageSize -> Full,
LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}]
```

Correlation

Takes a long time

```
FindRoot[10 Log10[S2[10^yy]] + 50 == 0, {yy, 1.15, 1.18}]
cc = 8; (* Factor 8 approximated BGN exact Figure5 *)
```

```
cc = 7.519884824; (* Sqrt[π] exact ξ Figure6 *)
F2 = {};
For[y = -3; i = 0, y < 1.16415, ++i, y += .001;
  AppendTo[F2, N[10 Log10[S2[10^y]]]]];
cc = 8; (* Factor 8 approximated BGN exact Figure5 *)
F5 = {};
For[y = -3; i = 0, y < 1.16415, ++i, y += .001;
  AppendTo[F5, N[10 (Log10[S1[10^y]*Expp[Psi1[10^y]]])]]];
cc = 7.519884824; (* Sqrt[π] exact ξ Figure6 *)
F6 = {};
For[y = -3; i = 0, y < 1.16415, ++i, y += .001;
  AppendTo[F6, N[10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y])]]];
cc = 8; (* Factor 8 approximated BGN exact Figure8 *)
F8 = {};
For[y = -3; i = 0, y < 1.16415, ++i, y += .001;
  AppendTo[F8,
   N[10 (Log10[S1[10^y]*Expp[Psi1[10^y]]]) + 10 Log10[gdc[10^y]]]]];
cc = 7.519884824; (* Sqrt[π] exact ξ Figure9 *)
F9 = {};
For[y = -3; i = 0, y < 1.16415, ++i, y += .001;
  AppendTo[F9,
   N[10 (Log10[S1[10^y]] + Log10[E]*Psi2[10^y]) +
     10 Log10[gdc[10^y]]]]];

{Correlation[F5, F2], Correlation[F6, F2],
 Correlation[F8, F2], Correlation[F9, F2]}
(* Out[157]= {0.99928, 0.999748, 0.999485, 0.999835} *)
```

Energy flow density vector

```
w0g=Function[Sqrt[Pi^3/8]*M1[Sqrt[#]]^3*Rho[#]^3];
w0n=Function[#^-(3/2)];
w0nPunkt2Int=Function[-(w0n[#])^2+.897659];
w0gPunkt=Function[(w0g[#+.00001]-w0g[#])/.00001];
w0gPunkt2=Function[(w0g[#+.00001]^2-w0g[#]^2)/.00001];
w0gPunkt2Int=Function[-(w0g[#])^2+.897659];
ka0g=Function[Pi/4*M1[Sqrt[#]]^2*Rho[#]^2];
ka0g2=Function[Pi^2/12*M1[Sqrt[#]]^4*Rho[#]^4];
ka0g2n=Function[1/3*#^(-2)];
ka0g2Int=Function[NIntegrate[ka0g2[tt],{tt,0,#}]];
ka0g2nInt=Function[-1/(6*#1^(3/2))+1/(6*10^(3/2))+0.345818];

Plot[{-w0gPunkt2[t^2]-ka0g2[t^2]},{t,0,3},PlotRange->{-0.22,0.88}, (* Figure12 *)
PlotLabel->None,ImageSize->Full,LabelStyle->{FontFamily-
>"Chicago",10,GrayLevel[0]}]
```

Displacement line

```
b = xtilde;
Plot[{(* Skipped *)
Log10[S2[10^y]], Log10[S1[10^y]],Xline[y,Log10[2]],
 2*y + Log10[2], 2*y - Log10[xtilde]}, {y, -3.05, 3.05},
  PlotRange -> {0.55, -5.05}, ImageSize -> Full,
  LabelStyle -> {FontFamily -> "Chicago", 10,
  GrayLevel[0]}]

b = 2.821439;
Plot[{(* Skipped *)
N[(b*y)^3/(E^(b*y) - 1)], 10^N[2*Log10[y] + Sin[2]]},
{y, 0, 0.15}, PlotRange -> {0, 0.2}]
```

Grey body

```
x=2.972456 10^-63;
y=8.6556 10^-64;
z=y 2^(1/6)/3^(2/3) Q0^-.5;fff=Function[1/(1+(#1/#2)^2)];
fff=Function[1/(1+(#1/#2)^2)];
ggg=Function[1/(1+((#1/#2)-(#2/#1))^2)];
hhh=Function[2*(#1/#2)/(1+(#1/#2)^2)];
Ek3=Function[1-0.0236820832fff[#1,#2]];
Ek5=Function[1-0.5fff[#1,#2]];  (* Ek5 over-scaled *)

Plot[{
  2/3Sqrt[2]Ek3[10^xxx,2Om0],0.942807,.920464,.930967739,
  Xline[xxx,Log10[2Om0]]},  (* Epsilon T *)
 {xxx,-2+ Log10[Om0],2+ Log10[Om0]},PlotRange->{0.91,0.95}]

Plot[{(* Figure13 *)
  2/3Sqrt[2]Ek3[10^xxx,2],0.942807,.920464, 0.930967739,(0.942807+.920464)/2,
  Xline[xxx,Log10[2]],Xline[xxx,Log10[1.903]]},
 {xxx,-2,2},PlotRange->{0.914,0.946},ImageSize->Full,PlotLabel->None,
LabelStyle->{FontFamily->"Chicago",11,GrayLevel[0]}]
(* Epsilon T *)

aaa = Log10[2];
bbb = xtilde (*2*Sqrt[2]*);
ccc = 1;
Plot[{(* Figure14 *)
  10*Log10[(bbb*10^(zzz - aaa))^3/(E^(bbb*10^(zzz - aaa)) - 1)],
  10*Log10[Ek3[10^(zzz - aaa), ccc]*((bbb*10^(zzz - aaa))^3/
 (E^(bbb*10^(zzz - aaa)) - 1))],
  10*Log10[Ek5[10^(zzz - aaa), ccc]*((bbb*10^(zzz - aaa))^3/
 (E^(bbb*10^(zzz - aaa)) - 1))], 10*Log10[Ek3[10^(zzz - aaa), ccc]],
  10*Log10[Ek5[10^(zzz - aaa), ccc]],
  Xline[zzz, Log10[2]],Xline[zzz,0.35271201428301324],
  10*(2*zzz + Log10[2]), 10*(2*zzz - Log10[xtilde]),
  10*(2*zzz + Log10[2*0.69281]), 10*(2*zzz - Log10[xtilde] + Log10[(0.69281+.5)/2])
  }, {zzz, -1.02, 1.02}, PlotRange -> {-10.25, 3.25}, ImageSize -> Full,
  PlotLabel -> None, LabelStyle -> {FontFamily -> "Chicago", 12, GrayLevel[0]}]
```

Extrema 3

```
FindMaximum[10*Log10[S2[10^zzz]],{zzz,-1.02,1.02}]

FindMaximum[10*Log10[(bbb*10^(zzz-aaa))^3/(Exp[(bbb*10^(zzz-aaa))]-1)],
{zzz,-1.02,1.02}]

FindMaximum[10*Log10[Ek3[10^(zzz-aaa),ccc]*((bbb*10^(zzz-aaa))^3/(E^(bbb*10^(zzz-
aaa))-1))],{zzz,-1.02,1.02}]

FindMaximum[10*Log10[Ek5[10^(zzz - aaa), ccc]*((bbb*10^(zzz - aaa))^3/
 (E^(bbb*10^(zzz - aaa)) - 1))],{zzz,-1.02,1.02}]

aaa = 0*Log10[2];
bbb = xtilde (*2*Sqrt[2]*);
ccc = 0.5 (* Q(max) *);
Plot[{(* Figure15 *)
  10*Log10[S2[10^zzz]],
  10*Log10[Ek5[10^zzz, ccc]*S2[10^zzz]],
  Xline[zzz, Log10[2]], Xline[zzz,-3], 10*Log10[S2[10^-3]],
  10*(2*zzz + Log10[2(1-0.0268)]),
  10*(2*zzz + Log10[2(1-0.5)])
  (* 2 εKmin *)},
  {zzz, -3.8, 1.3}, PlotRange -> {-67.25, 10.25}, ImageSize -> Full,
  PlotLabel -> None, LabelStyle -> {FontFamily -> "Chicago", 12, GrayLevel[0]}]

aaa = 1*Log10[2];
bbb = xtilde;
```

```
ccc = 0.5;
Plot[{(* Figure16 *)
      10*Log10[(bbb*10^(zzz - aaa))^3/(Expp[bbb*10^(zzz - aaa)] - 1)],
      10*Log10[Ek5[10^(zzz - aaa), ccc]*((bbb*10^(zzz - aaa))^3/
  (E^(bbb*10^(zzz - aaa)) - 1))], 10*Log10[Ek5[10^(zzz - aaa), ccc]],
      Xline[zzz, Log10[2]],Xline[zzz,0.35271201428301324]},
      {zzz, -3.8, 3.4}, PlotRange -> {-67.25, 5.25}, ImageSize -> Full,
      LabelStyle -> {FontFamily -> "Chicago", 10, GrayLevel[0]}]
Beep[]
Beep[]
Beep[]
```