Sedeonic Generalization of Navier-Stokes Equation

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Abstract

We present a generalization of the equations of hydrodynamics based on the noncommutative algebra of space-time sedeons. It is shown that for vortex-less flow the system of Euler and continuity equations is represented as a single non-linear sedeonic second-order wave equation for scalar and vector potentials, which is naturally generalized on viscous and vortex flows. As a result we obtained the closed system of four equations describing the diffusion damping of translational and vortex motions. The main peculiarities of the obtained equations are illustrated on the basis of the plane wave solutions describing the propagation of sound waves.

1 Introduction

The analogy between the equations of hydrodynamics and electrodynamics has been actively discussed for a long time. Apparently first, some similarity between vortex dynamics of fluid and electromagnetic phenomena induction was pointed out by H. Helmholtz in [1]. Subsequently, several attempts were made to describe the fluid dynamics by vector fields (similar to electric and magnetic fields) satisfying some Maxwell-like equations [2]-[10]. However a common drawback of the approach used in these works is that the equation for the vortex component of the fluid motion is obtained simply by taking the "curl" operator from the Euler equation for velocity and therefore it is not independent. In particular, in [4] the linearized equations for a free isentropic compressible fluid reduce to the following form:

$$
\begin{aligned}
& c^2[\nabla \times \mathbf{H}] - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}, \\
& [\nabla \times \mathbf{E}] + \frac{\partial \mathbf{H}}{\partial t} = 0, \\
& (\nabla \cdot \mathbf{E}) = g, \\
& (\nabla \cdot \mathbf{H}) = 0,
\end{aligned}
\tag{1}
$$

where vector fields $\mathbf{E}$ and $\mathbf{H}$ are defined by the following expressions:

$$
\begin{aligned}
& \mathbf{E} = -\frac{\partial \mathbf{v}}{\partial t} - \nabla h, \\
& \mathbf{H} = [\nabla \times \mathbf{v}],
\end{aligned}
\tag{2}
$$

and the field sources

$$
\begin{aligned}
& g = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) - \triangle h, \\
& \mathbf{J} = \frac{\partial^2 \mathbf{v}}{\partial t^2} + \frac{\partial}{\partial t}(\nabla h) + c^2[\nabla \times [\nabla \times \mathbf{v}]].
\end{aligned}
\tag{3}
$$

Here $\mathbf{v}$ is the velocity of the fluid, h is the enthalpy per unit mass, c is the speed of sound [4].

In form, the system (1) coincides with Maxwell's equations. However, these equations do not have any predictive power, since the field sources are determined through the quantities $\mathbf{v}$ and h, which themselves must be found from the equations. In addition, by substituting the definition of fields (2) and sources (3) into equation (1), we obtain the identity. A similar situation is observed in the works of other authors.

During the past decades the essential progress is observed in the reformulation of the equations for electromagnetic field and fluid motion based on the different algebras of hypercomplex numbers such as quaternions [11]-[14] and octonions [15]-[18], which take into account the symmetry of physical values with respect to operation spatial inversion. A natural generalization of this approach is the inclusion of time reversal symmetry in an algebraic structure, which requires consideration of extended sixteen-component algebras such as sedenions [19], [20]. However, a significant disadvanage of the sedeons is their non-associativity. Recently, we proposed a suitable associative algebra of sixteen-component sedeons, which takes into account the properties of physical quantities with respect to space-time inversion and implements a scalar-vector representation of the Poincare group [21]. This formalism has been successfully applied to describe classical and quantum fields [21]-[24]. In the present paper we discuss the application of sedeonic algebra to the generalization of the equations describing dynamics of viscous fluid.

2 Algebra of space-time sedeons

In physics, the change or preservation of the sign of scalar and vector quantities under the operations of space-time inversion is determined by a priori physical considerations. Our proposed sedeonic algebra takes into account the space-time properties of scalar and vector quantities in explicit form.

The algebra of sedeons encloses four groups of values, which differe with respect to spatial and time inversion.

1. Absolute scalars (A) and absolute vectors ($\mathbf{A}$) are invariant under spatial and time inversion.

2. Time scalars ($B_{\mathbf{t}}$) and time vectors ($\mathbf{B_t}$) change sign under time inversion and are invariant under spatial inversion.

3. Space scalars ($C_{\mathbf{r}}$) and space vectors ($\mathbf{C_r}$) are changed under spatial inversion and are invariant under time inversion.

4. Space-time scalars ($D_{\mathbf{tr}}$) and space-time vectors ($\mathbf{D_{tr}}$) change sign under spatial and time inversion.

The indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{S}}$, which is defined by the following expression:

$$\tilde{\mathbf{S}} = A + \mathbf{A} + B_{\mathbf{t}} + \mathbf{B_t} + C_{\mathbf{r}} + \mathbf{C_r} + D_{\mathbf{tr}} + \mathbf{D_{tr}}. \tag{4}$$

The system of sedeons is based on the Macfarlane's quaternion algebra [25]. Any vector can be represented as

$$\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3, \tag{5}$$

where the elements $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, are the unit absolute vectors, which generate the right Cartesian basis. These unit vectors have the following rules of multiplication

$$\mathbf{a}_n\mathbf{a}_m = \delta_{nm} + i\varepsilon_{nmk}\mathbf{a}_k, \tag{6}$$

where δ_{nm} is Kronecker delta, ε_{nmk} is Levi-Civita symbol ($n, m, k \in \{1, 2, 3\}$) and i is imaginary unit ($i^2 = -1$). For clarity, the rules of multiplication and commutation for the unit vectors are summarised in Table 1.

The space-time properties of physical values can be taken into account using an additional basis $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$, where $\mathbf{e_t}$ is the time scalar unit; $\mathbf{e_r}$ is the spatial scalar unit; $\mathbf{e_{tr}}$ is the space-time scalar unit. Further we will use digital tensor notations using the following correspondences $\mathbf{e}_1 = \mathbf{e_t}$, $\mathbf{e}_2 = \mathbf{e_r}$ and $\mathbf{e}_3 = \mathbf{e_{tr}}$. The units $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ have the same rules of multiplication [21]:

$$\mathbf{e}_n\mathbf{e}_m = \delta_{nm} + i\varepsilon_{nmk}\mathbf{e}_k. \tag{7}$$

Table 1: The rules of multiplication for absolute unit vectors

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2: The rules of multiplication for space-time units

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

For clarity, the rules of multiplication and commutation for space-time units $\mathbf{e}_m$ are presented in Table 2.

The unit vectors $\mathbf{a}_n$ commute with space-time units $\mathbf{e}_m$

$$\mathbf{a}_n\mathbf{e}_m = \mathbf{e}_m\mathbf{a}_n. \tag{8}$$

In general, the algebra of sedeons is the tensor product of two algebras of Macfarlane quaternions [25] $\{\mathbf{a_n}\}$ and $\{\mathbf{e}_\alpha\}$. It is associative algebra, which is isomorphic to the algebra of (4×4) Dirac matrices [26].

Using the space-time basis we can rewrite the sedeon (4) in terms of absolute scalars and absolute vectors as follows:

$$\tilde{\mathbf{S}} = A + \mathbf{A} + \mathbf{e}_1 B + \mathbf{e}_1\mathbf{B} + \mathbf{e}_2 C + \mathbf{e}_2\mathbf{C} + \mathbf{e}_3 D + \mathbf{e}_3\mathbf{D}. \tag{9}$$

Thus the sedeon $\tilde{\mathbf{S}}$ is a compound space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

The main advantage of this algebra over ordinary vector algebra is the Clifford multiplication of vectors. Indeed, for two absolute vectors $\mathbf{A}$ and $\mathbf{B}$ in accordance with the rules of multiplication (Table 1) we have

$$\mathbf{AB} = (\mathbf{A}\cdot\mathbf{B}) + i\,[\mathbf{A}\times\mathbf{B}]\,, \tag{10}$$

where we denote the scalar product of two vectors by symbol "$\cdot$" and round brackets

$$(\mathbf{A}\cdot\mathbf{B}) = A_1B_1 + A_2B_2 + A_3B_3,$$

and vector product by symbol "$\times$" and square brackets

$$[\mathbf{A}\times\mathbf{B}] = (A_2B_3 - A_3B_2)\,\mathbf{a}_1 + (A_3B_1 - A_1B_3)\,\mathbf{a}_2 + (A_1B_2 - A_2B_1)\,\mathbf{a}_3.$$

Based on relation (10), the algebra of sedeons enables writing physical equations in a compact highly symmetric form and performing intermediate calculations simultaneously with quantities of various space-time types. On the other hand, separating the results of sedeon calculations in accordance with the space-time properties of different quantities, the final results are represented in the usual terms of ordinary vector algebra [24]. The application of sedeonic algebra for the formulation of highly symmetric equations of hydrodynamics is demonstrated in the next sections.

3 Symmetric form of equations for ideal fluid

The dynamics of an ideal vortex-less fluid is described by the following well known system of equations

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{q}, \\
&\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho + \rho(\nabla \cdot \mathbf{v}) = \mathbf{0}, \\
&[\nabla \times \mathbf{v}] = 0,
\end{aligned} \tag{11}$$

where $\mathbf{v}$ is local flow velocity of fluid, ρ is a density, p is a pressure, $\mathbf{q}$ is a force for unit mass [27]. This system includes the Euler equation, the continuity equation and the condition of the absence of vortices.

The solutions of equations (11) depend on the specific type of fluid motion. In the cases of barotropic, isothermal and isentropic motion under the additional assumption of constant velocity of sound the system (11) becomes simplified. Below we show that in all these specific cases the equations (11) can be rewritten in a universal symmetric form, which allows the natural generalization on the basis of the sedeonic approach.

3.1 The barotropic fluid motion

In the case of a simple model of barotropic fluid, the the pressure depends only on density, so the state equation takes the following form:

$$p = p(\rho). \tag{12}$$

We assume that the speed of sound in the medium is constant:

$$c_B^2 = \frac{\partial p}{\partial \rho} = const. \tag{13}$$

Then equations (11) take the form:

$$
\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{c_B^2}{\rho}\nabla\rho = \mathbf{q},\\
&\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}(\mathbf{v}\cdot\nabla)\rho + (\nabla\cdot\mathbf{v}) = \mathbf{0},\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned}
\tag{14}
$$

We introduce new notation:

$$
\begin{aligned}
u_B &= c_B \ln(\rho),\\
\mathbf{f}_B &= \frac{1}{c_B}\mathbf{q},
\end{aligned}
\tag{15}
$$

then the system of equations for ideal fluid becomes symmetric:

$$
\begin{aligned}
&\frac{1}{c_B}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u_B = \mathbf{f}_B,\\
&\frac{1}{c_B}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u_B + (\nabla\cdot\mathbf{v}) = 0,\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned}
\tag{16}
$$

3.2 The isothermal fluid motion

For the isothermal fluid we assume the constant speed of sound

$$
c_T^2 = \left(\frac{\partial p}{\partial \rho}\right)_T = const,
\tag{17}
$$

and use the thermodynamic relation for the Gibbs potential

$$
dz = -s dT + \frac{1}{\rho}dp,
\tag{18}
$$

where z is the Gibbs potential [28], s is the entropy referred to the unit mass, T is the temperature. In the case of $T = const$ we have:

$$
dz = \frac{1}{\rho}dp = \frac{c_T^2}{\rho}d\rho,
\tag{19}
$$

and therefore

$$
\begin{aligned}
&\frac{1}{\rho}\nabla p = \nabla z,\\
&\frac{\partial \rho}{\partial t} = \frac{\rho}{c_T^2}\frac{\partial z}{\partial t},\\
&\nabla\rho = \frac{\rho}{c_T^2}\nabla z = 0.
\end{aligned}
\tag{20}
$$

Then equations (11) take the form:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla z = \mathbf{q},\\
&\frac{1}{c_T^2}\frac{\partial z}{\partial t} + (\mathbf{v}\cdot\nabla)z + (\nabla\cdot\mathbf{v}) = \mathbf{0},\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{21}$$

Introducing the notations:

$$\begin{aligned}
u_T &= \frac{1}{c_T} z,\\
\mathbf{f}_T &= \frac{1}{c_T}\mathbf{q},
\end{aligned} \tag{22}$$

we derive the system of equations for ideal fluid, which again becomes symmetric:

$$\begin{aligned}
&\frac{1}{c_T}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u_T = \mathbf{f}_T,\\
&\frac{1}{c_T}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u_T + (\nabla\cdot\mathbf{v}) = 0,\\
&[\nabla\times\mathbf{v}] = 0.
\end{aligned} \tag{23}$$

3.3 The isentropic fluid motion

We denote the speed of sound in the case of isentropic motion as

$$c_S^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = const, \tag{24}$$

and use the thermodynamic relation for enthalpy

$$dh = Tds + \frac{1}{\rho}dp, \tag{25}$$

where h is the enthalpy per the unit mass [28]. In the case of $s = const$ we have:

$$dh = \frac{1}{\rho}dp = \frac{c_S^2}{\rho}d\rho, \tag{26}$$

and therefore

$$\begin{aligned}
&\frac{1}{\rho}\nabla p = \nabla h,\\
&\frac{\partial \rho}{\partial t} = \frac{\rho}{c_S^2}\frac{\partial h}{\partial t},\\
&\nabla\rho = \frac{\rho}{c_S^2}\nabla h = 0.
\end{aligned} \tag{27}$$

Then equations (11) take the form:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla h=\mathbf{q},\\
&\frac{1}{c_S^2}\frac{\partial h}{\partial t}+(\mathbf{v}\cdot\nabla)h+(\nabla\cdot\mathbf{v})=\mathbf{0},\\
&[\nabla\times\mathbf{v}]=0.
\end{aligned}\tag{28}$$

We introduce new notation:

$$\begin{aligned}
u_S&=\frac{1}{c_S}h,\\
\mathbf{f}_S&=\frac{1}{c_S}\mathbf{q}.
\end{aligned}\tag{29}$$

The system of equations for an ideal fluid takes the following symmetric form:

$$\begin{aligned}
&\frac{1}{c_S}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\mathbf{v}+\nabla u_S=\mathbf{f}_S,\\
&\frac{1}{c_S}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)u_S+(\nabla\cdot\mathbf{v})=0,\\
&[\nabla\times\mathbf{v}]=0.
\end{aligned}\tag{30}$$

4 The sedeonic equation for a vortex-free flow

Using the algebra of sedeons, equations (16), (23) and (30) can be represented as a single generalized first-order wave equation in the following form:

$$\left(i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)-\mathbf{e}_2\nabla\right)(\mathbf{e}_3\mathbf{v}-u)=\mathbf{e}_2\mathbf{f},\tag{31}$$

where the set

$$\{c,u,\mathbf{f}\}\in\{\{c_B,u_B,\mathbf{f}_B\},\{c_T,u_T,\mathbf{f}_T\},\{c_S,u_S,\mathbf{f}_S\}\}\tag{32}$$

depending on the type of fluid motion. Indeed, after the action of the operator on the left side of equation (31), we have

$$\begin{aligned}
&\mathbf{e}_2\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\mathbf{v}-i\mathbf{e}_1(\nabla\cdot\mathbf{v})+\mathbf{e}_1[\nabla\times\mathbf{v}]\\
&-i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)u+\mathbf{e}_2\nabla u=\mathbf{e}_2\mathbf{f}.
\end{aligned}\tag{33}$$

Separating the quantities with different space-time properties, we obtain the following system of equations:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)\mathbf{v}+\nabla u=\mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t}+(\mathbf{v}\cdot\nabla)\right)u+(\nabla\cdot\mathbf{v})=0,\\
&[\nabla\times\mathbf{v}]=0.
\end{aligned}\tag{34}$$

As can be seen, equations (34) coincide with equations (16), (23) and (30).

By analogy with electrodynamics, a generalized equation describing the dynamics of a fluid can be represented in the form of a sedeonic wave equation for some potential functions. Let us introduce scalar φ and vector $\mathbf{A}$ potentials according to the following relations:

$$\begin{aligned} u &= \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\varphi + (\nabla\cdot\mathbf{A}), \\ \mathbf{v} &= -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{A} - \nabla\varphi, \\ [\nabla &\times \mathbf{A}] = 0, \end{aligned} \tag{35}$$

and denote the operator

$$\widehat{\nabla} = \left\{ i\mathbf{e}_1 \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right) - \mathbf{e}_2\nabla \right\}, \tag{36}$$

then equations (34) are equivalent to the following second-order wave equation:

$$\widehat{\nabla}\widehat{\nabla}\left(i\mathbf{e}_1\varphi + \mathbf{e}_2\mathbf{A}\right) = \mathbf{e}_2\mathbf{f}. \tag{37}$$

Indeed after the first operator action we have

$$\widehat{\nabla}\left(i\mathbf{e}_1\varphi + \mathbf{e}_2\mathbf{A}\right) = \mathbf{e}_3\mathbf{v} - u \tag{38}$$

and equation (37) is rewritten as

$$\widehat{\nabla}\left(\mathbf{e}_3\mathbf{v} - u\right) = \mathbf{e}_2\mathbf{f}, \tag{39}$$

that coincides with equation (31). The symmetric form of equations for potentials appears to be convenient for the description of the vortex motion of the fluid.

5 Sedeonic equations for vortex flow

The equation (31) can be generalized for the vortex motion. Let us introduce the vector $\mathbf{w}$ as

$$\mathbf{w} = [\nabla \times \mathbf{A}]. \tag{40}$$

Here $\mathbf{w}(\mathbf{r}, t)$ is vector field of vortex lines [1] in the fluid

$$\mathbf{w} = c\,2\boldsymbol{\Theta}, \tag{41}$$

where $\boldsymbol{\Theta}$ is the vector of angle of rotation for vortex line. It connected with speed of vortex line rotation $\boldsymbol{\omega}$ [1] as

$$\frac{1}{c}\frac{d\mathbf{w}}{dt} = 2\boldsymbol{\omega}. \tag{42}$$

In this case the relations for potentials are changed as following:

$$\begin{aligned}
&u = \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\varphi + \nabla\cdot\mathbf{A},\\
&\mathbf{v} = -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{A} - \nabla\varphi,\\
&\mathbf{w} = [\nabla\times\mathbf{A}].
\end{aligned} \tag{43}$$

Then we have

$$\widehat{\nabla}\,(i\mathbf{e_t}\varphi + \mathbf{e_r}A) = -u + \mathbf{e_{tr}}\mathbf{v} + i\mathbf{w} \tag{44}$$

and the generalized wave equation (31) is rewritten as

$$\widehat{\nabla}\,(-u + \mathbf{e_{tr}}\mathbf{v} + i\mathbf{w}) = \mathbf{e_r}\mathbf{f}. \tag{45}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u + (\nabla\cdot\mathbf{v}) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = \mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{w} - [\nabla\times\mathbf{v}] = 0,\\
&(\nabla\cdot\mathbf{w}) = 0.
\end{aligned} \tag{46}$$

Here the third equation is well known relation between velocity of vortex line rotation $\boldsymbol{\omega}$ and vorticity of linear velocity [1]:

$$2\boldsymbol{\omega} = [\nabla\times\mathbf{v}]. \tag{47}$$

6 Sedeonic equations for vortex flow

The equation (31) can be generalized for vortex motion. Let us introduce the vector $\mathbf{w}$ as

$$\mathbf{w} = -[\nabla\times\mathbf{A}]. \tag{48}$$

Here $\mathbf{w}(\mathbf{r},t)$ is vector field of vortex lines [1] in the fluid

$$\mathbf{w} = c\,2\boldsymbol{\Theta}, \tag{49}$$

where $\boldsymbol{\Theta}$ is the vector of angle of rotation for vortex line. It connected with speed of vortex line rotation $\boldsymbol{\omega}$ [1] as

$$\frac{1}{c}\frac{d\mathbf{w}}{dt} = 2\boldsymbol{\omega}. \tag{50}$$

In this case the relations for potentials are changed as following:

$$\begin{aligned} u &= \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\varphi + (\nabla\cdot\mathbf{A}), \\ \mathbf{v} &= -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{A} - \nabla\varphi, \\ \mathbf{w} &= -[\nabla\times\mathbf{A}]. \end{aligned} \tag{51}$$

Then we have

$$\widehat{\nabla}\,(i\mathbf{e}_1\varphi + \mathbf{e}_2 A) = -u + \mathbf{e}_3\mathbf{v} + i\mathbf{w} \tag{52}$$

and the generalized wave equation (31) is rewritten as

$$\widehat{\nabla}\,(-u + \mathbf{e}_3\mathbf{v} + i\mathbf{w}) = \mathbf{e}_2\mathbf{f}. \tag{53}$$

This equation is equivalent to the following system:

$$\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)u + (\nabla\cdot\mathbf{v}) = 0. \tag{54}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{w} - [\nabla\times\mathbf{v}] = 0, \tag{55}$$

$$(\nabla\cdot\mathbf{w}) = 0, \tag{56}$$

$$\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla)\right)\mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = \mathbf{f}, \tag{57}$$

Let us consider these equations in detail. Equation (54) is the well known condition of flow continuity. Equation (55) is the relation between velocity of vortex line rotation $\boldsymbol{\omega}$ and vorticity of linear velocity [1]:

$$2\boldsymbol{\omega} = [\nabla\times\mathbf{v}]. \tag{58}$$

Often, this relation is interpreted as the definition of a vortex of linear velocity. However, it actually describes the effect of inducing by vortex line the fluid motion at the periphery [1]. As an example, we consider the simplest model of a rectilinear cylindrical vortex [29]. This object is a vortex tube of radius R. The field of the angular velocity vector $\boldsymbol{\omega}$ is uniformly distributed inside the tube and is equal to zero outside (Fig. 1 (a)). Integration of equation (55) with the application of Stokes' theorem gives us

$$2\int \boldsymbol{\omega}\, dS = \oint (\mathbf{v}\cdot\mathbf{dl}). \tag{59}$$

From here, by virtue of cylindrical symmetry, we obtain

$$\begin{aligned} |\mathbf{v}| &= |\boldsymbol{\omega}|r \quad r < R, \\ |\mathbf{v}| &= \frac{|\boldsymbol{\omega}|R^2}{r} \quad r > R. \end{aligned} \tag{60}$$

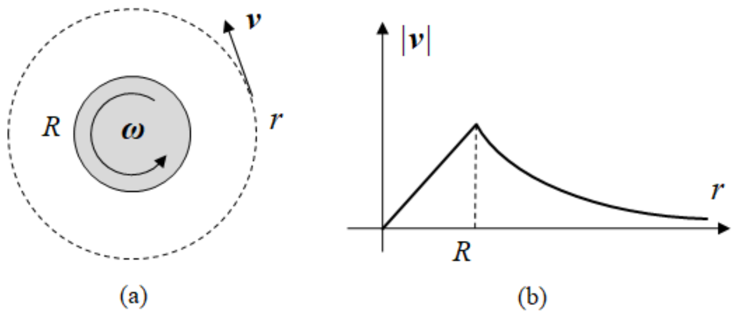

Figure 1: (a) Cross-section of circle cylindrical vortex of radius R rotating with angle speed $\boldsymbol{\omega}$. (b) Radial distribution of linear velocity module.

The distribution of the linear velocity module is shown in Fig. 1 (b). The speed reaches a maximum at the edge of the vortex core and decreases away from the core according to the hyperbolic law. This distribution of the induced velocity explains the phenomenon of rotation of two vortices around their center of mass in the case when the vorticity of the vortices has the same sign, and a rectilinear movement in the direction perpendicular to the line connecting the centers of the vortices in the case when vorticities have different signs [1].

Equation (56) reflects the fact that the vector field of vortex lines $\mathbf{w}(\mathbf{r}, t)$ has no sources and vortex lines can either be closed, or begin and end at the boundary of the fluid [1].

Equation (57) is the Euler equation with a new term $[\nabla \times \mathbf{w}]$ that describes the dynamic resistance associated with the generation of vorticity. In particular, this term can be used for the description of toroidal vortex generation at a circular hole. Of course, a toroidal vortex is a soliton and its formation is non-linear process, however some estimates can be made in the linear approximation. In case of free incompressible fluid when $\rho = const$ and as a consequence $u = const$, equation (57) is rewritten as

$$\mathbf{a} + 2c^2[\nabla \times \boldsymbol{\Theta}] = 0, \tag{61}$$

where $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$. This expression relates the flow of fluid moving with acceleration to the circulation of the vortex line vector. Let us consider a uniform flow moving with the growing acceleration towards the non-transparent screen with the circle hole (Fig. 2 (a)). After passing the plane of the screen the flow acceleration changes its direction (Fig. 2(b)). Integrating equation (61) over the area of the hole gives us

$$\int \mathbf{a}\, dS + 2c^2 \oint (\boldsymbol{\Theta} \cdot d\mathbf{l}) = 0, \tag{62}$$

and we have

$$|\boldsymbol{\Theta}| = \frac{|\mathbf{a}| r}{4c^2}. \tag{63}$$

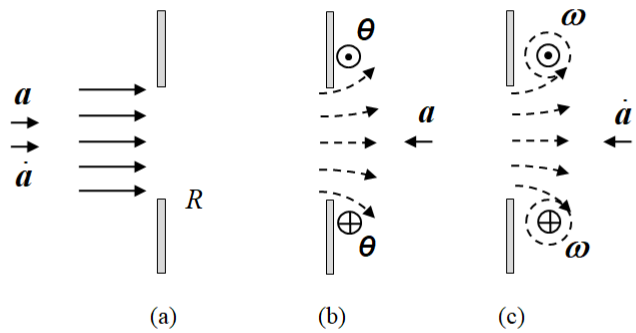

Figure 2: Sketch of the mechanism responsible for the generation of toroidal vortices. (a) The uniform flow moving with the growing acceleration towards the non-transparent screen with the circle hole. (b) Formation of vorticity $[\nabla \times \boldsymbol{\Theta}]$ under the flow deceleration. (c) Formation of a toroidal vortex $[\nabla \times \boldsymbol{\omega}]$ near the periphery of the hole during braking of a strongly accelerated flow.

As can be seen, the maximum deviation of the flow is at the edge of the hole. However, for the vortex generation the flow must be strongly accelerated. For this purpose strike pressure is usually applied with the left side of the screen. Then a toroidal vortex forms near the periphery of the hole and the kinetic energy of the flow is partly transformed into the energy of vortex motion (Fig. 2 (c)). In the linear approximation equation (62) can be rewritten for time derivatives as follows

$$\dot{\mathbf{a}} + 2c^2[\nabla \times \boldsymbol{\omega}] = 0, \tag{64}$$

where $\dot{\mathbf{a}} = \dfrac{d\mathbf{a}}{dt}$. This equation relates the speed of rotation of a vortex line with a change in the acceleration of fluid. In particular, this formula allows one to estimate the speed of rotation in a torroidal vortex, which is generated at the edge of a circular hole under the flow deceleration. Indeed, integrating over the area of the hole, we get

$$\int \dot{\mathbf{a}}\, dS + 2c^2 \oint (\boldsymbol{\omega} \cdot d\mathbf{l}) = 0, \tag{65}$$

and

$$|\boldsymbol{\omega}| = \frac{|\dot{\mathbf{a}}|R}{4c^2}, \tag{66}$$

where R is a radius of hole.

Thus equations (54)-(57) represent a closed system, which describes vortex motion of ideal fluid.

7 Sedeonic equations for viscous fluid

The sedeonic equations (37) and (39) can be generalized for the description of viscous fluid. The viscosity can be taken into account by modifying the operator (36) as

$$\widehat{\nabla}_\nu = \left\{ i\mathbf{e}_1 \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right) - \mathbf{e}_2\nabla \right\}, \tag{67}$$

where ν is the coefficient of kinematic viscosity and $\triangle$ is Laplace operator [30]. Then generalized wave equation for the potentials of viscous vortex flow is

$$\widehat{\nabla}_\nu \widehat{\nabla}_\nu \left(i\mathbf{e}_1\varphi + \mathbf{e}_2\mathbf{A}\right) = \mathbf{e}_2\mathbf{f}, \tag{68}$$

where the relations between parameters $u, \mathbf{v}, \mathbf{w}$ and potentials $\varphi, \mathbf{A}$ have the following form:

$$\begin{aligned}
&u = \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\varphi + (\nabla\cdot\mathbf{A}),\\
&\mathbf{v} = -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{A} - \nabla\varphi,\\
&\mathbf{w} = -[\nabla\times\mathbf{A}].
\end{aligned} \tag{69}$$

After the action of one operator in (68), we have

$$\widehat{\nabla}_\nu \left(-u + \mathbf{e}_3\mathbf{v} + i\mathbf{w}\right) = \mathbf{e}_2\mathbf{f}. \tag{70}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)u + (\nabla\cdot\mathbf{v}) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = \mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{w} - [\nabla\times\mathbf{v}] = 0,\\
&(\nabla\cdot\mathbf{w}) = 0.
\end{aligned} \tag{71}$$

The sedeonic relation (70) and equivalent system (71) is the generalization of the Navier-Stokes equation for viscous vortex flow. In system (71) the first equation describes the convection-diffusion [31, 32]. This is a condition of flow continuity taking into account the processes of self-diffusion in viscous fluid. The second equation is describes the diffusion damping of linear momentum. The third equation describes the diffusion damping of the vortex motion.

8 Sound waves in ideal fluid

Let us consider the sound waves in ideal fluid. In this case we can neglect the convective derivative and sedeonic wave equation (37) is equivalent to the following system

$$
\begin{aligned}
&\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\Delta\right)\varphi=0,\\
&\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\Delta\right)\mathbf{A}=\mathbf{f},
\end{aligned}
\tag{72}
$$

which is similar to the wave equations for electromagnetic field. The parameters $u, \mathbf{v}, \mathbf{w}$ are expressed through the potentials as

$$
\begin{aligned}
u&=\frac{1}{c}\frac{\partial\varphi}{\partial t}+(\nabla\cdot\mathbf{A}),\\
\mathbf{v}&=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}-\nabla\varphi,\\
\mathbf{w}&=-[\nabla\times\mathbf{A}],
\end{aligned}
\tag{73}
$$

and the system (54)-(54) is rewritten as

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial u}{\partial t}+(\nabla\cdot\mathbf{v})=0,\\
&\frac{1}{c}\frac{\partial\mathbf{v}}{\partial t}+\nabla u+[\nabla\times\mathbf{w}]=\mathbf{f},\\
&\frac{1}{c}\frac{\partial\mathbf{w}}{\partial t}-[\nabla\times\mathbf{v}]=0,\\
&(\nabla\cdot\mathbf{w})=0.
\end{aligned}
\tag{74}
$$

If we neglect the changes of enthalpy ($u=0$), which is equivalent to the condition

$$
\frac{1}{c}\frac{\partial\varphi}{\partial t}+(\nabla\cdot\mathbf{A})=0
\tag{75}
$$

similar to Lorentz gauge, then the system (74) is reduced to the Maxwell-like equations

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial\mathbf{v}}{\partial t}+[\nabla\times\mathbf{w}]=\mathbf{f},\\
&\frac{1}{c}\frac{\partial\mathbf{w}}{\partial t}-[\nabla\times\mathbf{v}]=0,\\
&(\nabla\cdot\mathbf{v})=0,\\
&(\nabla\cdot\mathbf{w})=0.
\end{aligned}
\tag{76}
$$

Multiplying the first two equations by $\mathbf{v}$ and $\mathbf{w}$, respectively, and adding, we obtain

$$
\frac{1}{2c}\frac{\partial}{\partial t}(\mathbf{v}^2+\mathbf{w}^2)+(\nabla\cdot[\mathbf{v}\times\mathbf{w}])=(\mathbf{v}\cdot\mathbf{f}).
\tag{77}
$$

This expression is an analogue of the Poynting relation for sound waves in the uncompressible fluid. Here $\mathbf{v}^2$ is the kinetic energy of the translational motion of a unit mass of liquid, and $\mathbf{w}^2$ is the energy of rotational motion. The value $[\mathbf{v}\times\mathbf{w}]$ is the energy flux.

9 Equations for twisted vortex flow

The sedeonic wave equation (68) can be naturally generalized for the twisted vortex flow. Using additional pseudoscalar ϕ and pseudovector $\mathbf{B}$ potentials the wave equation for spiral flow is writing as

$$\widehat{\nabla}_\nu \widehat{\nabla}_\nu \left(i\mathbf{e}_1\varphi + \mathbf{e}_2\mathbf{A} + i\mathbf{e}_2\phi - \mathbf{e}_1\mathbf{B} \right) = \mathbf{e}_2\mathbf{f}. \tag{78}$$

Let us introduce the following definitions:

$$\begin{aligned}
u &= \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\varphi + (\nabla\cdot\mathbf{A}),\\
\mathbf{v} &= -\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{A} - \nabla\varphi + [\nabla\times\mathbf{B}],\\
\mathbf{w} &= -\frac{\mathbf{1}}{\mathbf{c}}\left(\frac{\partial}{\partial \mathbf{t}} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{B} - \nabla\phi - [\nabla\times\mathbf{A}],\\
n &= \frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\phi + (\nabla\cdot\mathbf{B}).
\end{aligned} \tag{79}$$

Here we introduce the value n, which characterizes the twisting of vortex tube [33]. We suppose that this parameter can be presented as

$$n = c\beta(\boldsymbol{l}_0\cdot\boldsymbol{\omega}_0). \tag{80}$$

Here β is the angle of twisting (Fig. 3), $\boldsymbol{l}_0 = \dfrac{\boldsymbol{l}}{|\boldsymbol{l}|}$ is the unit vector of spiral ($l = |\boldsymbol{l}|$ is the pitch of spiral), $\boldsymbol{\omega}_0 = \dfrac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}$ is the unit vector of tube rotation. The projection $(\boldsymbol{l}_0)_x$ is positive ($(\boldsymbol{l}_0)_x > 0$) for right-handed twisting and negative ($(\boldsymbol{l}_0)_x < 0$) for left-handed twisting. The projection $(\boldsymbol{\omega}_0)_x$ is positive ($(\boldsymbol{\omega}_0)_x > 0$) for right-handed rotation and negative ($(\boldsymbol{\omega}_0)_x < 0$) for left-handed rotation. For angle of twisting we have

$$\tan\beta = \frac{2\pi R}{l}, \tag{81}$$

where R is the radius of vortex tube.

Taking into account (79) the wave equation (78) is reduced to

$$\widehat{\nabla}_\nu \left(-u - i\mathbf{e}_3 n + \mathbf{e}_3\mathbf{v} + i\mathbf{w} \right) = \mathbf{e}_2\mathbf{f}. \tag{82}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)u + (\nabla\cdot\mathbf{v}) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = \mathbf{f},\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)\mathbf{w} + \nabla n - [\nabla\times\mathbf{v}] = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla) - \nu\triangle\right)n + (\nabla\cdot\mathbf{w}) = 0.
\end{aligned} \tag{83}$$

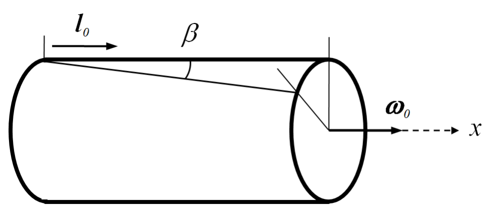

Figure 3: Twisted vortex tube.

The set of equations (83) is an absolutely symmetric closed system consisting of eight scalar equations for eight scalar variables, which describes the twisted vortex flow of viscous fluid.

10 Sound waves in viscous fluid

Let us now consider the free sound waves in viscous fluid. In this case neglecting the convective derivative the equation (82) is rewritten as

$$\left\{ i\mathbf{e}_1 \frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) - \mathbf{e}_2\nabla \right\}(-u - i\mathbf{e}_3 n + \mathbf{e}_3\mathbf{v} + i\mathbf{w}) = 0. \tag{84}$$

This equation is equivalent to the following system:

$$\begin{aligned}
&\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) u + (\nabla\cdot\mathbf{v}) = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) \mathbf{v} + \nabla u + [\nabla\times\mathbf{w}] = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) \mathbf{w} + \nabla n - [\nabla\times\mathbf{v}] = 0,\\
&\frac{1}{c}\left(\frac{\partial}{\partial t} - \nu\triangle\right) n + (\nabla\cdot\mathbf{w}) = 0.
\end{aligned} \tag{85}$$

The equation (84) has the plane wave solution. Let us find the solutions in the following form:

$$\begin{aligned}
u &= u_0 \exp(i\omega t - i(\mathbf{k}\cdot\mathbf{r})),\\
n &= n_0 \exp(i\omega t - i(\mathbf{k}\cdot\mathbf{r})),\\
\mathbf{v} &= \mathbf{v}_0 \exp(i\omega t - i(\mathbf{k}\cdot\mathbf{r})),\\
\mathbf{w} &= \mathbf{w}_0 \exp(i\omega t - i(\mathbf{k}\cdot\mathbf{r})),
\end{aligned} \tag{86}$$

where $u_0, n_0, \mathbf{v}_0, \mathbf{w}_0$ are amplitudes that are independent of coordinates and time, ω is frequency, $\mathbf{k}$ is wave vector. The dispersion relation for the equation (84) is

$$\omega^2 - i2\nu k^2\omega - \nu^2 k^4 - c^2 k^2 = 0, \tag{87}$$

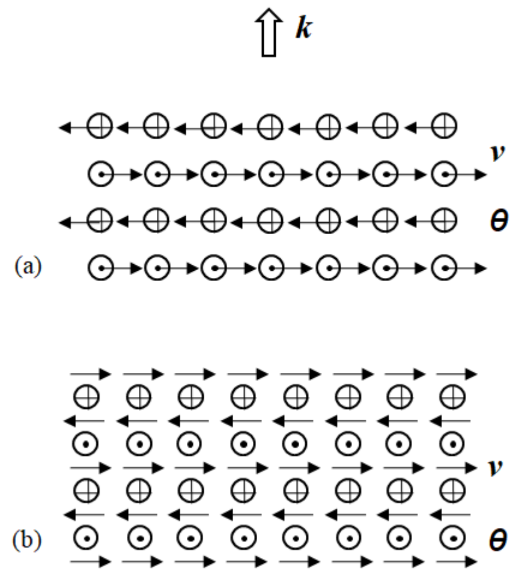

Figure 4: (a) Transverse vortex sound wave. (b) Transverse standing vortex sound wave.

and consequently

$$\omega = \pm ck + i\nu k^2. \tag{88}$$

Here $k = |\mathbf{k}|$. Substituting (86) into the system (85) we have

$$\begin{aligned} u_0 &= (\mathbf{m} \cdot \mathbf{v}_0), \\ \mathbf{v}_0 &= u_0\mathbf{m} + [\mathbf{m} \times \mathbf{w}_0], \\ \mathbf{w}_0 &= n_0\mathbf{m} - [\mathbf{m} \times \mathbf{v_0}], \\ n_0 &= (\mathbf{m} \cdot \mathbf{w}_0), \end{aligned} \tag{89}$$

where $\mathbf{m} = \mathbf{k}/k$. In case of vortex-less motion ($n_0 = 0, \mathbf{w}_0 = 0$) we have

$$\begin{aligned} u_0 &= (\mathbf{m} \cdot \mathbf{v}_0), \\ \mathbf{v}_0 &= u_0\mathbf{m}, \end{aligned} \tag{90}$$

and in sound wave the vector $\mathbf{v}_0$ is parallel to the vector $\mathbf{k}$. In case of incompressible fluid ($u_0 = 0$) with non-twisteed flow ($n_0 = 0$) the system (89) is reduced to

$$
\begin{aligned}
&(\mathbf{m}\cdot\mathbf{v}_0)=0,\\
&\mathbf{v}_0=[\mathbf{m}\times\mathbf{w}_0],\\
&\mathbf{w}_0=-[\mathbf{m}\times\mathbf{v}_0],\\
&(\mathbf{m}\cdot\mathbf{w}_0)=0,
\end{aligned}
\tag{91}
$$

and we have transverse sound wave, where $\mathbf{v}_0 \perp \mathbf{w}_0 \perp \mathbf{m}$ (Fig. 4(a)). In standing sound wave the fronts with oppositely directed speed $\mathbf{v}$ alternate with vortex planes with opposite vorticity of $\mathbf{w}$ (Fig. 4(b)).

11 Conclusion

In our theoretical constructions, we used the associative algebra of space-time sedeons. This algebra takes into account the complete symmetry of physical quantities with respect to spatial rotations and space-time inversions. In the sedeonic algebra the wave equation is written in a very simple and compact form that, on the one hand, enables easy its generalization for a wide class of scalar-vector fields described by various multicomponent potentials, and on the other hand, separating quantities with different space-time properties, allows to obtain systems of Maxwell-type equations, formulated in terms of commonly used vector algebra.

In application to hydrodynamics we have shown that the barotropic, isothermal and isenntropic vortex-less flows can be described by universal symmetric system of equations, which is represented as a single sedeonic non-linear wave equation for scalar and vector potentials. Including into consideration the vector of vorticity $\mathbf{w}$ enables generalizing this equation for vortex flows. As a result we obtained the equation describing vortex motion and additional term in Euler equation describing the dynamical damping of translational motion caused by vorticity generation. Moreover, generalizing the sedeonic wave equation to the case of eight-component potentials, we managed to obtain a completely symmetric system of equations describing vortex motion with twisting.

In a simple model, viscosity was included in the equations using the modification of the time differentiation operator by the term $\nu\triangle$. As a result we obtained the closed system of four equations describing the diffusion damping of translational and vortex motions.

The linearization of the equations enables describing sound waves in vortex viscous fluid. In case of vortex-less fluid these equations describe the longitudional plane waves with vector of velocity parallel to the wave vector ($\mathbf{v} \parallel \mathbf{k}$). In case of incompressible fluid without twisting these equations describe the transverse plane waves with mutually perpendicular vectors ($\mathbf{v}\perp\mathbf{w}\perp\mathbf{k}$). Additionally we have shown that for viscous-less incompressible fluid with non-twisted motion the system of linearized equations coincides in form with Maxwell equations in electrodynamics.

We believe that the proposed approach based on sedeonic equations of hydrodynamics may become a convenient theoretical platform for further analysis of complex vortex dynamics and turbulent flows.

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