A REMARK ON THE STRONG GOLDBACH CONJECTURE

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ABSTRACT. Under the assumption that $\sum\limits_{n\leq N}\Upsilon(n)\Upsilon(N-n)>0$, we show that for all even number $N>6$

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)=(1+o(1))K\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)$$

for some constant $K>0$, and where Υ and Λ_0 denotes the master and the truncated Von mangoldt function, respectively. Using this estimate, we relate the Goldbach problem to the problem of showing that for all $N>6$ ($N\neq 2p$), If $\sum\limits_{p|N}\sum\limits_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)>0$, then $\sum\limits_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)>0$ for each prime $p|N$.

1. INTRODUCTION

The Goldbach problem has been settled for almost all even integers (See [1])- with some exception - but a complete proof for the general case remains elusive. The problem, which states that every even number $N\geq 4$ can be written as the sum of two primes, has a conjectural quantitative formulation

$$\sum_{n}\Lambda(n)\Lambda(N-n)=\Im(N)N+O(N^{\frac{1}{2}+o(1)}),$$

as $N\longrightarrow\infty$ where

$$\Im(N):=2\Pi_2\prod_{\substack{p>2\\p|N}}\frac{p-1}{p-2}\quad\text{and}\quad\Pi_2:=\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

and the singular series $\Im(N)$ vanishes when N is odd and Π_2 is the twin prime constant (See [1]), with

$$\Lambda(n):=\begin{cases}\log p & \text{if}\quad n=p^k\\ 0 & \text{otherwise},\end{cases}$$

and where p is prime and $k\geq 1$. In many ways establishing the positive correlation $\sum\limits_{n\leq N}\Lambda(n)\Lambda(N-n)>0$ for all even numbers $N\geq 4$ settles the Goldbach problem. However the Goldbach problem can be relaxed by saying that every even number $N>6$, excluding those of the form $N=2p$ where p is prime, can be written as a sum of two integers N_1, N_2 having exactly two prime factors ($\Omega(N_1)=\Omega(N_2)=2$).

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In a more quantitative setting, it suffices to show that

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)>0,$$

where

$$\Upsilon(n):=\begin{cases}\log p \quad \text{if} \quad n=p^2\\ \log(p_1p_2) \quad \text{if} \quad n=p_1p_2, \quad p_1\neq p_2\\ 0 \quad \text{otherwise,}\end{cases}$$

and where p, p_1, p_2 runs over the primes. In this paper we show that the correlated sums of these two functions are related. This insight will then precipitates the study of correlations of the master function.

2. NOTATIONS

Through out this paper the lower case letters p, q and all of it's subscripts will always stand for the primes. Any other letter will be clarified. The function $\Omega(n) := \sum_{p||n} 1$ counts the number of prime factors of n with multiplicity. The inequality $|k(n)| \leq Mp(n)$ for sufficiently large values of n will be compactly written as $k(n) \ll p(n)$ or $k(n) = O(p(n))$. The function $\phi(n) := \#\{m \leq n : (m,n) = 1\}$. The limit $\lim_{n\longrightarrow\infty}\frac{k(n)}{p(n)} = 0$ will be represented in a compact form as $k(n) = o(p(n))$ as $n \longrightarrow \infty$.

3. REDUCTION TO CORRELATION OF THE MASTER FUNCTION

In this section we show that indeed studying correlations on the master function is equivalent to studying correlations on the Von mangoldt function . In many ways proving a positive correlation on the master function is a first good path to settling the Goldbach conjecture. We first launch an arithmetic function, which in essense is an indicator on the primes. This function is a modification of the Von mangoldt function where we tend to ignore all higher prime powers and focus only on the primes, since the prime powers are in any case wasteful.

Definition 3.1. (Truncated Von mangoldt function) Let $n \geq 2$, then we set

$$\Lambda_0(n):=\begin{cases}\log p \quad \text{if} \quad n=p\\ 0 \quad \text{otherwise,}\end{cases}$$

where p runs over the primes.

Lemma 3.2. *For $N \geq 2$, we have*

$$\sum_{n\leq N}\phi(n)=(1+o(1))\frac{3}{\pi^2}N^2 \quad (N\longrightarrow\infty).$$

Proof. For a proof see Hildebrand [2]. □

Remark 3.3. Lemma 3.2 has several interpretations, so it not suprising turns up in many applications. The one interpretation of importance to us is that, it gives us the total count of all lattice points (m,n) with $0 < m, n \leq N$ and that $(m,n) = 1$. That is, by symmetry the total count for the number of such lattice points obeying such a property is given by $(1+o(1))\frac{6}{\pi^2}N^2$.

Theorem 3.4. *If* $\sum\limits_{n\leq N}\Upsilon(n)\Upsilon(N-n) > 0$, *then*

$$\alpha\sum_{p|N}\sum_{n\leq \frac{N}{p}}\Lambda_0(n)\Lambda_0(N/p-n)+O\bigg(\frac{\log N\log\log N}{N}\bigg)\geq \sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)$$
$$\geq \beta\sum_{p|N}\sum_{n\leq \frac{N}{p}}\Lambda_0(n)\Lambda_0(N/p-n)+O\bigg(\frac{\log N\log\log N}{N}\bigg),$$

where p runs over all the primes dividing N. In particular, if $\sum\limits_{n\leq N}\Upsilon(n)\Upsilon(N-n) >$ 0, *then*

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n) = (1+o(1))K\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n),$$

for $\beta < K < \alpha$, where $0 < \beta := \beta(N) \leq 1$ and $\alpha := \alpha(N) > 1$.

Remark 3.5. In words, Theorem 3.4 is basically saying that an even number N can be written as $N = n_1+n_2$ with $\Omega(n_1) = \Omega(n_2) = 2$ if and only if some even number of the form $\frac{N}{p}$ can be written as a sum of two primes for all $N > 6$.

Proof. Suppose $N \neq 2p$ and $\sum\limits_{n\leq N}\Upsilon(n)\Upsilon(N-n) > 0$, then in relation to Lemma 3.2 we can write

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n) = \sum_{\substack{n\leq N\\(n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)+\sum_{\substack{n\leq N\\(n,N-n)=1}}\Upsilon(n)\Upsilon(N-n)$$
$$= \sum_{\substack{n\leq N\\(n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)+\bigg(\frac{1}{(1+o(1))\frac{6}{\pi^2}N^2}\bigg)$$
$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)+o(1),$$

since the latter sum on the right contribute less to the correlated sum. It is now incumbent on us to estimate explicitly the main and the error term. We now break the sum in the error term in the following cases: the case the correlated sum runs entirely over a prime square; the case it runs over a prime square and a product of two primes; the case it runs entirely over product of primes. That is

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n) = \sum_{p^2\leq N}\Upsilon(p^2)\Upsilon(N-p^2)+\sum_{pq\leq N}\Upsilon(pq)\Upsilon(N-pq).$$

We estimate each term in the expression above. Clearly, by applying the Cauchy-Swartz inequality, we find that

$$\sum_{\substack{p^2\leq N\\ N-p^2=q^2}}\Upsilon(p^2)\Upsilon(N-p^2)\leq\left(\sum_{p^2\leq N}\Upsilon^2(p^2)\right)^{\frac{1}{2}}\left(\sum_{p^2\leq N}\Upsilon^2(p^2)\right)^{\frac{1}{2}}$$
$$=\left(\sum_{p\leq\sqrt{N}}\log^2 p\right)\ll\sqrt{N}\log N.$$

Again

$$\sum_{\substack{p^2\leq N\\ N-p^2=p_1p_2}}\Upsilon(p^2)\Upsilon(N-p^2)\leq\left(\sum_{p^2\leq N}\Upsilon^2(p^2)\right)^{\frac{1}{2}}\left(\sum_{p_1p_2\leq N}\Upsilon^2(p_1p_2)\right)^{\frac{1}{2}}$$
$$=\left(\sum_{p^2\leq N}\log^2 p\right)^{\frac{1}{2}}\left(\sum_{p_1p_2\leq N}\log^2(p_1p_2)\right)^{\frac{1}{2}}$$
$$\ll\log N\sqrt{N\sqrt{N}\log\log N}.$$

Similarly, we find that

$$\sum_{\substack{pq\leq N\\ N-pq=p_1q_1}}\Upsilon(pq)\Upsilon(N-pq)\ll\left(\sum_{pq\leq N}\Upsilon^2(pq)\right)^{\frac{1}{2}}\left(\sum_{pq\leq N}\Upsilon^2(N-pq)\right)^{\frac{1}{2}}$$
$$\ll\left(\sum_{pq\leq N}\log^2 pq\right)$$
$$\ll N\log N\log\log N.$$

By piecing each of these estimates together we obtain the order of the error term. We now estimate the main term. Clearly the main term can be decomposed as

$$\sum_{\substack{n\leq N\\ (n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)=\sum_{\substack{i,j\\ p_1(p_i+p_j)=N\\ p_i\neq p_j}}2\Upsilon(p_1p_i)\Upsilon(p_1p_j)+\cdots \tag{3.1}$$

$$+\sum_{\substack{i,j\\ p_s(p_i+p_j)=N\\ p_i=p_j}}\Upsilon(p_sp_i)\Upsilon(p_sp_j)+\cdots+\sum_{\substack{i,j\\ p_n(p_i+p_j)=N\\ p_i\neq p_j}}2\Upsilon(p_np_i)\Upsilon(p_np_j).$$

It follows by virtue of the definition of the master function that

$$\sum_{\substack{n\leq N\\ (n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)=\sum_{\substack{i,j\\ p_i+p_j=N/p_1\\ p_i\neq p_j}}2(\log p_1+\log p_i)(\log p_1+\log p_j)$$

$$+\cdots+\sum_{\substack{i,j\\p_i+p_j=N/p_s\\p_i=p_j}}(\log p_s+\log p_i)(\log p_s+\log p_j)+\cdots$$
$$+\sum_{\substack{i,j\\p_i+p_j=N/p_n\\p_i\neq p_j}}2(\log p_n+\log p_i)(\log p_n+\log p_j).$$

It follows that

$$\sum_{\substack{n\leq N\\(n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)\geq\sum_{\substack{i,j\\p_i+p_j=N/p_1\\p_i\neq p_j}}2(\log p_i)(\log p_j)+\cdots+$$
$$\sum_{\substack{i,j\\p_i+p_j=N/p_s\\p_i=p_j}}(\log p_i)(\log p_j)+\cdots+\sum_{\substack{i,j\\p_i+p_j=N/p_n\\p_i\neq p_j}}2(\log p_i)(\log p_j)$$
$$=\beta\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n).$$

where $1\geq\beta:=\beta(N)>0$. On the other hand it follows that

$$\sum_{\substack{n\leq N\\(n,N-n)\neq 1}}\Upsilon(n)\Upsilon(N-n)\leq\alpha_1\sum_{\substack{i,j\\p_i+p_j=N/p_1\\p_i\neq p_j}}2(\log p_i)(\log p_j)+\cdots \tag{3.2}$$
$$+\alpha_s\sum_{\substack{i,j\\p_i+p_j=N/p_s\\p_i=p_j}}(\log p_i)(\log p_j)+\cdots+\alpha_n\sum_{\substack{i,j\\p_i+p_j=N/p_n\\p_i\neq p_j}}2(\log p_i)(\log p_j)$$
$$\leq\alpha\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n).$$

where $\alpha:=\alpha(N)>1$. By combining both inequalities, the result follows immediately. □

4. CONNECTION TO THE GOLDBACH CONJECTURE

The problem of showing that every even number $N>6$ is expressible an the sum of two integers with the property that $\Omega(n_1)=\Omega(n_2)=2$; that is, $N=n_1+n_2$ with $\Omega(n_1)=\Omega(n_2)=2$ has a profound connection with the strong Goldbach conjecture, which states that every even number ≥ 4 can be written as the sum of two primes. Assumming every even number has the property stated, then it follows quantitatively that

$$\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)>0, \tag{4.1}$$

for all even number greater than 6. It follows immediately from Theorem 3.4, that

$$\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)>0. \tag{4.2}$$

The Goldbach conjecture will be completely attacked by showing the following.

Conjecture 4.1. If

$$\sum_{p|N}\sum_{n\leq \frac{N}{p}}\Lambda_0(n)\Lambda_0(N/p-n)>0,$$

then

$$\sum_{n\leq \frac{N}{p}}\Lambda_0(n)\Lambda_0(N/p-n)>0$$

for all primes p dividing N.

Remark 4.1. Conjecture 4.1 is basically saying that a positive average correlation on the Von mangoldt function over an even number implies a positive correlation over numbers short of one distinguished prime factor of the even number. In other words Conjecture 4.1 is saying that a positive correlation on the Von mangoldt function over any even number is uniformly distributed among correlations on it's not too small even factors.

5. CONCLUSION

The above exposition does reveals rather than establishing the asymptotics stated at the outset of the paper to prove the Goldbach conjecture; that is,

$$\sum_{n}\Lambda(n)\Lambda(N-n)=\Im(N)N+O(N^{\frac{1}{2}+o(1)})\quad (N\longrightarrow\infty),$$

we only need to show the following:

(i) For all even numbers $N>6$, $\sum_{n\leq N}\Upsilon(n)\Upsilon(N-n)>0$.

(ii) If $\sum_{p|N}\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)>0$, then $\sum_{n\leq N/p}\Lambda_0(n)\Lambda_0(N/p-n)>0$ for all primes $p|N$.

[1].

References

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