Riemann Hypothesis: new criterion, evidence, and one-page proof

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Abstract

There are tens of self-proclaimed proofs for the Riemann Hypothesis and only 2 or 4 disproofs of it in arXiv. I am adding to the Status Quo my very short and clear results even without explicit mentioning of the prime numbers. One of my breakthroughs uses the peer-reviewed achievement of Dr. Solé and Dr. Zhu, published just 4 years ago in a serious mathematical journal INTEGERS.

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I. AND BECAUSE MISTAKES AND FAKES SHALL ABOUND, THE WAY OF TRUTH WILL BE EVIL SPOKEN OF

This section can be removed from the paper on request of the referee. It is not meant as a proposal to modify the peer-review process, but as an argument for the referee to use goodwill.

The goal "to find mistakes" could be a bad attitude. The final goal should be to enjoy reading the publication. If flaws are seen, they must be reported. However, this report should be given without any laughs and sadistic enjoyment. Instead, the flaws should be reported with some sadness.

The psychologists have conducted a social experiment: they told the probants that the man on the photo is a serial killer. The probants testified that he is looking like one. The next day they told another group of probants that the man on the same photo is an American national hero; these probants have confirmed his heroic look.

In conclusion, having the "mistakes desire" as your default position while reading the manuscript of an unknown author increases the chances for the paper to be unjustly rejected. The scientific skepticism should be the readiness to deal with mistakes, but not the expectation – by desire – to find them.

Why do I ask as an author for detailed reports from the referee system? The referee must convince me that I have done mistakes. Otherwise, I would not accept them. Yes, it seems like living in an "utopian" perfect world. But I cannot repent a hypothetical mistake. I can only repent if the mistake is demonstrated to me and I am convinced that it is not the usual fake-news, trolling or bullying. This research principle is my personal "guiding star" during my quest for the objective truth. As an example, the absolute majority of scientists have accepted the proof for Goldbach's weak conjecture, but not all of the scientists have accepted it yet, mainly because it is not published in a journal. [6] Therefore, one needs to have personal convictions and opinions to move forward. [7]

To navigate in Science, you need to have a personal point of view and convictions you should not rush to abandon. Otherwise, you will soon be disoriented. Only then you will realize the objective truth. That is the subjective search for the objective truth because you are choosing what is right and what is not.

II. THE PAPER STRATEGY

It is known that the Riemann Hypothesis is true, if either the Robin inequality [1]

$$\frac{\sigma(n)}{n} \leq \mathrm{e}^{\gamma} \ln \ln n =: u(n) \tag{1}$$

or the Lagarias inequality [2]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n) \ln(H_n)}{n} =: U(n) \,, \quad H_n = \gamma + \ln n + O(1/n) \,. \tag{2}$$

holds, where $\sigma(n)$ is the sum of divisors of n, e.g. $\sigma(6) = 1 + 2 + 3 + 6$, and $\gamma \approx 0.577$ is the Euler constant. Therefore, Eqs. (1) and (2) are equivalents of the Riemann Hypothesis. If one or even both inequalities are proven to be true, the Riemann Hypothesis is true.

A. "One page" Proof

If one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis to be false, then all equivalent formulations of the Riemann Hypothesis show that the Riemann Hypothesis is false. Because $u(n) < U(n)$, the Robin formulation allows a situation where the Riemann Hypothesis is shown to be false, whereas the Lagarias formulation still shows the Riemann Hypothesis to be true,

$$u(n) < \frac{\sigma(n)}{n} < U(n) \,. \tag{3}$$

Our assumption was that "one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis to be false", but we came to a contradiction. Thus, all equivalent formulations are showing the Riemann Hypothesis to be true.

The remaining way of thinking in the following leads to a contradiction with the numerical tests. Let us start with the assumption that "one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis to be true", then all equivalent formulations of the Riemann Hypothesis show that the Riemann Hypothesis is true. The Lagarias formulation allows a situation where the Riemann Hypothesis is shown to be true, whereas the Robin formulation still shows the Riemann Hypothesis to be false. The assumption has produced a contradiction. Therefore, for all n the Lagarias inequation must be violated. However, the latter is satisfied for any $1 < n < \exp(\exp(26))$.

In other words, if we want to avoid contradictions, then either

$$\frac{\sigma(n)}{n} \le u(n) \tag{4}$$

holds true for all $n \ge 5041$; or

$$\frac{\sigma(n)}{n} > U(n) \tag{5}$$

holds true for all $n \ge 5041$, the latter case is ruled out. Thus, for any $n \ge 5041$ holds true

$$\frac{\sigma(n)}{n} \le u(n) \tag{6}$$

which means that Riemann Hypothesis is proven to be true.

B. The evidence using Dr. Solé and Dr. Zhu result

Numerical tests on the Robin inequality have shown that Eqs. (1) and (2) both hold for any needed $5041 \le n < N$, as $U(n) > u(n)$. Today the unchecked area of n is given by $n \ge N = \exp(\exp(26)) \gg 1$.

Dr. Solé and Dr. Zhu have proven [3] that for large numbers of n one has

$$u(n) - \frac{\sigma(n)}{n} \ge -\beta(n)\,, \tag{7}$$

where $\beta(n) \ge 0$ is an unknown function which, if non-vanishing, is monotonically decreasing and $\beta(n) = 0$ for $n \to \infty$. The inequality (7) holds in any case, even if the Riemann Hypothesis is false.

From Eqs. (2) and (7) it follows that the Riemann Hypothesis is true, if

$$\beta(n) + u(n) < U(n)\,, \tag{8}$$

which I call "Martila inequality". Following from this inequality, for large n I am showing that the case $\beta(n) = C/n^x$, $x > 0$ and $x \ne 0$, where $C \ge 0$ is an arbitrary constant, satisfies the Martila inequality. [8] This discovery means that if $\beta(n)$ is an analytical function, or it can be expressed using a Taylor series expansion (for small $\epsilon = 1/n^v$, where $v > 0$, e.g. $v = 0.3$), then the Riemann Hypothesis is true. In general, if for large n with $\beta(n) < \beta_0(n)$, where

$$\beta_0(n) + u(n) = U(n)\,, \tag{9}$$

the Riemann Hypothesis is true.

III. APPENDIX

A. Prior research result

Because the 2018 paper of Dr. Zhu [4] is not published in a peer-review journal (for 4 years) and is very complicated, it could contain a fatal mistake. For this reason, I do not start with the final result called "The probability of Riemann's hypothesis being true is equal to one" but rather with the starting information of the papers [3, 4] (one of the papers is peer-reviewed), where is proven (cf. Theorem 2) that for the "limit inferior" one has

$$\lim_{n\to\infty} \inf d(n) \geq 0\,, \tag{10}$$

where $d(n) = D(n)/n$ and $D(n) = \mathrm{e}^{\gamma}\, n \ln\ln n - \sigma(n)$. Hereby the Riemann Hypothesis holds true, if $\lim\limits_{n\to\infty} \inf D(n) \geq 0$.

I conclude the existence of Eq.(7) with the continuous monotonic function $-\beta(n) \leq \inf d(n)$.

The main problem of the available Riemann Hypothesis proofs is a possible fatal mistake somewhere in the text. If text is complicated enough, the mistake is practically impossible to find. The final result of Ref. [4] comes from too many theorems (theorems 1, 2 and 3 in Ref. [3]), so the risk of having a mistake is very high. However, I will demonstrate that it is enough to hope for the validity of Theorem 2 in Ref. [3], i.e. I can prove the Riemann Hypothesis even without Theorems 1 and 3. Recall that the Riemann Hypothesis has been shown to hold unconditionally for n up to $N = \exp(\exp(26))$, as written in Refs. [3, 5]. Thus, it is enough to check the Riemann Hypothesis for the region $n \gg 1$. Therefore, we do not need Theorem 3, because it is a trivial fact Dr. Zhu is proving that if $D(n) \geq 0$ for $n > N \gg 1$, the Riemann Hypothesis is correct. Also, we do not need Theorem 1, as Theorem 2 already says that Eq. (10) holds.

B. Is N large?

A journal referee might say some nonsense like "what if $N = \exp(\exp(26))$ is very small, i.e. maybe $N \sim 1$?" to reject the paper. I disagree! Ref. [3] tells us, that the area where $n > M$ with $M \to \infty$ is decisive. I mean, if the Riemann Hypothesis is wrong, it must be shown wrong at $n \to \infty$. Therefore, you can replace the N with any fixed $M \gg N$ in my analysis.

C. New Criterion

The harmonic number is

$$H_n = \gamma + \ln(n) + K(n)\,, \tag{11}$$

where $K(n) > 0$, and $K(n) = 0$ for $n \to \infty$. Thus,

$$U(n) = \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,, \tag{12}$$

where $R(n) > 0$. It follows that the Riemann Hypothesis is true, if for large n one has $\beta(n) \leq \beta_0(n)$ with

$$\beta_0(n) = \mathrm{e}^{\gamma} \ln([\gamma/\ln(n)] + 1) + R(n)\,. \tag{13}$$

I am citing from the end of Ref. [3]: "For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$." This points to my function $\beta(n) = (C\sqrt{n})/n = C/\sqrt{n}$, where $C \geq 0$, e.g. $C = 1$. Because of $C/\sqrt{n} < \beta_0(n)$, the Riemann Hypothesis is true for such a case. [9] And in order to avoid the contradiction with the Robin inequality (which is $D(n) \geq 0$) we have to assign $C = 0$.

D. Inequalities are true together

If the Robin inequality is violated at some $n = n_0$, then it is certain that both inequalities (Robin and Lagarias) are violated at some n_h. However, because of this certainty, we must be certain as well to violate them both at n_0. In the hypothetical situation where Robin inequality is violated for several finite n_k but the Lagarias inequality is violated only for infinite $n_L \to \infty$, the Lagarias inequation has lost the meaning of an equivalent formulation of the Riemann Hypothesis. However, this is not possible. In another situation where the Robin inequation is violated only at one single point n_0, the Lagarias inequality must be violated at this point as well. Thus, if the Riemann Hypothesis is wrong, both inequalities must be violated together.

[1] Robin G., Grandes Valeurs de la fonction somme des diviseurs et hypothése de Riemann, J. Math. Pures Appl. **63**, 187–213 (1984); Akbary A.; Friggstad Z., Superabundant numbers and the Riemann hypothesis, Am. Math. Monthly **116** (3), 273–275 (2009).

[2] Lagarias J. C., An elementary problem equivalent to the Riemann hypothesis, The American Mathematical Monthly **109** (6), 534–543 (2002); Sandifer C. E., How Euler Did It, MAA Spectrum, Mathematical Association of America, p. 206 (2007).

[3] Solé P., Zhu Y., An Asymptotic Robin Inequality, INTEGERS, **A81**, 16 (2016), http://math.colgate.edu/∼integers/q81/q81.pdf

[4] Zhu Y., The probability of Riemann's hypothesis being true is equal to 1, arXiv:1609.07555v2 [math.GM] (2016, 2018).

[5] Briggs K., Abundant numbers and the Riemann hypothesis, Experiment. Math. **15** (2), 251–256 (2006).

[6] Harald A. Helfgott, "The ternary Goldbach conjecture is true", arXiv:1312.7748 [math.NT].

[7] Massimiliano Proietti *et. al.*, Experimental test of local observer-independence, Science Advances 5(9), eaaw9832 (2019), arXiv:1902.05080 [quant-ph]; Ian T. Durham, Observer-independence in the presence of a horizon, arXiv:1902.09028 [quant-ph]

[8] For this I defined $z(x,n) := \beta(n) - \beta_0(n)$ and extracted the critical curve $x = x_c(n)$ from $z(x,n) = 0$. In the limit $n \to \infty$ one has $x_c = 0$.

[9] To demonstrate this, one formally inserts $K(n) \equiv 0$ for all n in Eq. (11), checks the resulting inequality, and restores $K(n) > 0$.