

# Nested Radicals and Gray Code

## Simpler methods of converting integers to Gray code

George Plousos

### Abstract

There is a characteristic form of nested radicals that contains many signs (+, -) and is expressed in a closed trigonometric form. When the value of the trigonometric expression is known, we are called to find the values of the signs of the radical representation (the reverse is easier). The solution is provided directly by the Gray code of an integer contained in the known member of this equation of signs. However, the method of converting integers to Gray code differs from the standard method and is easier. The reason is that I found the equivalent method without knowing the Gray code. At the same time, a model emerged that interprets and facilitates the conversion of a group of integers into Gray code.

This is a simplified description of how to deal with such equations.

We consider the following form

$$\underbrace{\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \dots \pm \sqrt{2}}}}}_{n \text{ roots}}$$

This expression is divided into two parts

$$\underbrace{\sqrt{2 + \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \dots \pm \sqrt{2}}}}}}_{n \text{ roots}} = 2 \cos\left(\frac{90^\circ (2a + 1)}{2^n}\right)$$

$$\underbrace{\sqrt{2 - \sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \dots \pm \sqrt{2}}}}}}_{n \text{ roots}} = 2 \sin\left(\frac{90^\circ (2a + 1)}{2^n}\right)$$

where

$$a \in \{0, 1, 2, \dots, 2^{n-2} - 1\}$$

If the values of n and 'a' are known then we can easily find the unknown signs ( $\pm$ ) of the radical representation by the following way.

My method for converting an integer to Gray code (binary) uses successive divisions by powers of 2 and looks at the parity of the rounded quotient.

Example with  $n=8$ ,  $a=29$

$$29/2 = 14.5 \approx 15 \Rightarrow 1$$

$$29/4 = 7.25 \approx 7 \Rightarrow 1$$

$$29/8 = 3.625 \approx 4 \Rightarrow 0$$

$$29/16 = 1.8125 \approx 2 \Rightarrow 0$$

$$29/32 = 0.90625 \approx 1 \Rightarrow 1$$

$$29/64 = 29/2^{n-2} = 0.453... \approx 0 \Rightarrow 0$$

The decimal value 29 has the binary value 10011 in Gray code.

However, since we have to divide 29 by the power  $2^{n-2}$ , we have to include in Gray the code of 29 and the digit 0 of the last division, ie we get the result 010011.

Now we correspond these digits to the signs by setting  $0 = +$ ,  $1 = -$ . We get this:

+ - + + - -

This string is the solution for constructing the radical representation - here let's choose the expression  $\sin()$ :

$$\sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2}}}}}}}} = 2 \sin\left(\frac{90^\circ (2 \cdot 29 + 1)}{2^8}\right)$$

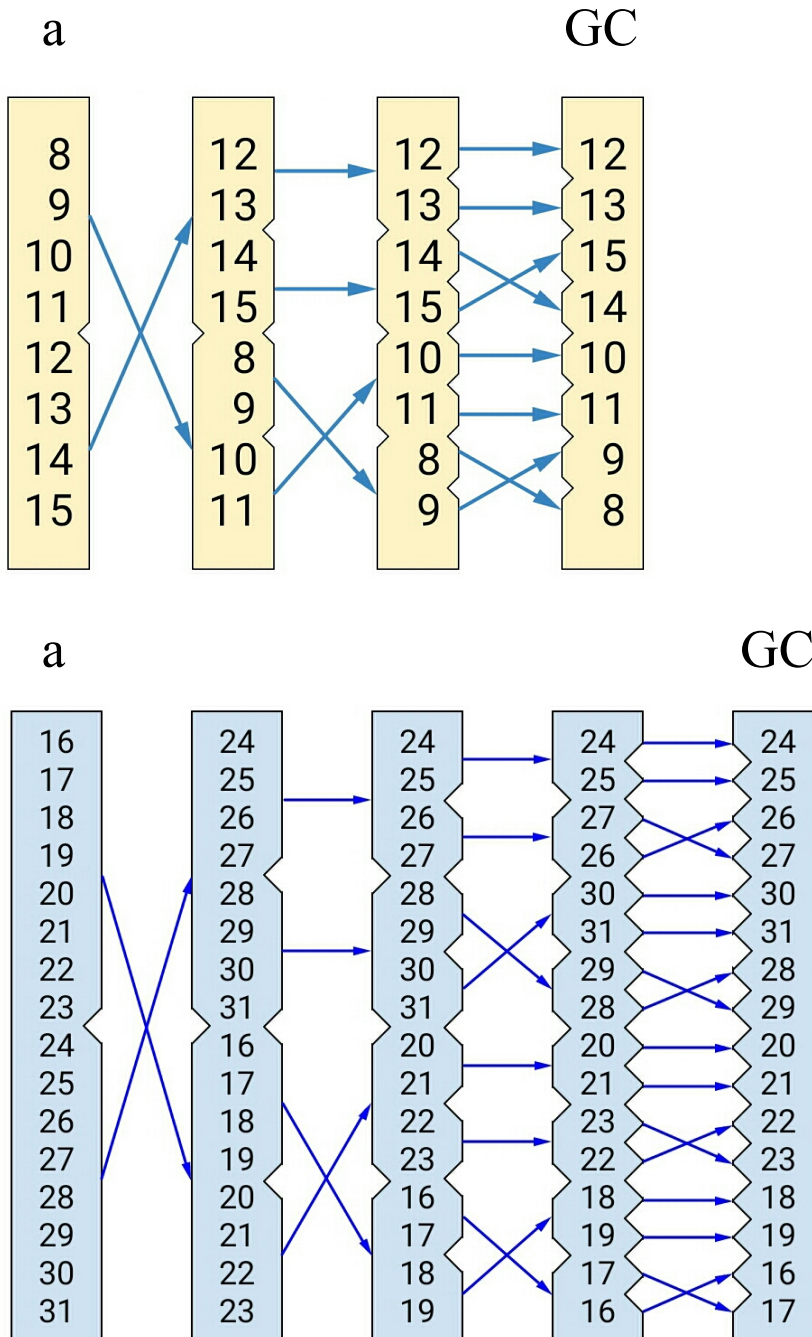
If the radical representation is known, then by definition the value of  $n$  will also be known. Then the value of 'a' can be obtained from the relationship

$$a = \frac{2^8 \sin^{-1}\left(\frac{0.5 \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2}}}}}}}}}{180^\circ}\right)}{180^\circ} - 0.5 = 29$$

We can also find the value of 'a' by decoding the value of the Gray code according to the above.

The following tables show the characteristic matching pattern in the area of four-digit binary numbers (8-15) and five-digit binary numbers (16-31). The numbers that are rearranged in Gray code are arranged in the first column. There are two ways to transfer groups of numbers from column to column, crosswise (×) and parallel (=).

Eg if  $\alpha=11$  then Gray code = 14 (and vice versa).



This type of matching of integers to Gray code should not be known. By the way, if after the second column you change the order of succession "=" and "×" you will end up with a different OEIS sequence in place of the Gray code.

The above also applies to more general expressions of these nested radicals; I have written a relevant article (see references) only that is a bit messy.

## Some observations that can help in further research, especially when 'n' tends to infinity

When the signs of the radical representation form a repetitive pattern, then 'a' will be of the form  $\text{int}(2^n/m)$ , where m is a constant number for each value of n (eg  $m=6, 9.5, 7.25, 9.333\dots$ ). In place of m, other numbers can be placed,  $m_2, m_3, \dots$  for which the same pattern formed by the signs starts from a different point in the radical representation. Often, the numbers  $m_2, m_3, \dots$  are quotient of fractions  $m/1, m/2, m/3, m/4, \dots$ . In all cases, the pattern of the binary representation of 'a' is also periodic.

### Examples

$$++++ \dots a = 0$$

$$---- \dots a = \text{int}\left(\frac{2^n}{6}\right) = 1010\dots_2$$

$$+-+- \dots a = \text{int}\left(\frac{2^n}{10/1}\right) = 110000110000\dots_2$$

$$-+-+ \dots a = \text{int}\left(\frac{2^n}{10/2}\right) = 110000110000\dots_2$$

$$+--+- \dots a = \text{int}\left(\frac{2^n}{14/1}\right) = 100100\dots_2$$

$$--+-+ \dots a = \text{int}\left(\frac{2^n}{14/2}\right) = 100100\dots_2$$

$$-+--+ \dots a = \text{int}\left(\frac{2^n}{14/3}\right) = 110000110000\dots_2$$

$$\text{General form: } a = \text{int}\left(\frac{2^n}{m/q}\right), m, q \in \mathbb{Z}$$

Seth Zimmerman explores the same radical expression where n tends to infinity.

It is fair to clarify a few things.

I have known all this since 2004. Until 2015, I had nothing to do with the internet. That year, in April 2015, I published the first relevant article at the following address (the official website of the Hellenic Mathematical Society).

<https://www.mathematica.gr/forum/viewtopic.php?t=49175>

File name (in Greek): ΡιζΤριγΣυν-I.doc (free)

Work title: "Ριζικές Τριγωνομετρικές Συναρτήσεις" (Radical Trigonometric Functions).

Several mathematicians worked on the same problem. I mention here those who have achieved substantial results, with respectfully to all of those who I do not mention.

Pierluigi Vellucci came up with a similar solution based on an observation linking the nested radicals to the Gray code. Apparently, he matched the signs in the binary digits and sorted the resulting values according to the size of the values of the radical representation. He then examined the relevant sequence in OEIS, finding that this is the sequence of the Gray code. Pierluigi provides proofs of his findings.

Jayantha Senadheera has also come up with a solution, but his method is difficult and complicated.

plousos2505@gmail.com

## References

Pierluigi Vellucci and Alberto Maria Bersani, "Ordering of nested square roots of 2 according to Gray code" (2016).

Pierluigi Vellucci and Alberto Maria Bersani, "New formulas for  $\pi$  involving infinite nested square roots and Gray code" (2018).

Jayantha Senadheera, "On the periodic continued radicals of 2 and generalization for Vieta's product" (2013).

Seth Zimmerman & Chungwu Ho, "On Infinitely Nested Radicals" (2008).

George Plousos, "Nested Radicals and Trigonometry"