New Whole Numbers Classification

Jean-Yves BOULAY

Abstract. According to new mathematical definitions, the set (ℕ) of whole numbers is subdivided into four subsets (classes of numbers), one of which is the fusion of the sequence of prime numbers and numbers zero and one. This subset, at the first level of complexity, is called the set of ultimate numbers. Three other subsets, of progressive level of complexity, are defined since the initial definition isolating the ultimate numbers and the non-ultimate numbers inside the set ℕ. The interactivity of these four classes of whole numbers generates singular arithmetic arrangements in their initial distribution, including exact 3/2 or 1/1 value ratios.

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1 Introduction

The concept of numbers ultimity has already been introduced in the article *"The ultimate numbers and the 3/2 ratio"* [1] where singular arithmetic phenomena are presented in relation to the different classifications of numbers* deduced from this new concept.

In this previous article, a new classification of whole numbers was therefore proposed and introduced. This new article describes more fully how the set ℕ (of whole numbers) can be organized into subsets with arithmetic properties proper and unique but also, simultaneously interactive.

* In statements, when this is not specified, the term "number" always means "whole number". It is therefore agreed that the number zero (0) is well integrated into the set of whole numbers.

2 The ultimate numbers

The definition of thus called *prime numbers* did not allow the numbers zero (0) and one (1) to be included in this set of primes. Thus, the set of whole numbers was scattered in four entities: prime numbers, non-prime numbers, but also ambiguous numbers *zero* and *one* at exotic arithmetic characteristics. The double definition of ultimate and non-ultimate numbers proposed here makes it possible to properly divide the set of whole numbers into two groups of numbers with well-defined and absolute characteristics: a number is either ultimate or non-ultimate. In addition to its non-triviality, the fact of specifying the numerically lower nature of a divisor to any envisaged number effectively allows that there is no difference in status between the ultimate numbers zero (0) and one (1) and any other number described as ultimate.

2.1 Definition of an ultimate number

Considering the set of whole numbers, these are organized into two sets: ultimate numbers and non-ultimate numbers.

Ultimate numbers definition:

> ***An ultimate number not admits any non-trivial divisor (whole number) being less than it.***

Non-ultimate numbers definition:

> ***A non-ultimate number admits at least one non-trivial divisor (whole number) being less than it.***

Note: a non-trivial divisor of a whole number *n* is a whole number which is a divisor of *n* but distinct from *n* and from 1 (which are its trivial divisors).

2.2 Other definitions

Let *n* be a whole number (belonging to ℕ), this one is ultimate if no divisor (whole number) lower than its value and other than 1 divides it.

Let *n* be a natural whole number (belonging to ℕ), this one is non-ultimate if at least one divisor (whole number) lower than its value and other than 1 divides it.

2.3 Development

Below are listed, to illustration of definition, some of the first ultimate or non-ultimate numbers defined above, especially particular numbers zero (0) and one (1).

- 0 is ultimate: although it admits an infinite number of divisors superior to it, **since it is the first whole number**, the number 0 does not admit any divisor **being inferior to it**.

- 1 is ultimate: since the division by 0 has no defined result, the number 1 does not admit any divisor (whole number) being less than it.

- 2 is ultimate: since the division by 0 has no defined result, the number 2 does not admit any divisor* being less than it.

- 4 is non-ultimate: the number 4 admits the number 2 (number being less than it) as divisor*.

- 6 is non-ultimate: the number 6 admits numbers 2 and 3 (numbers being less than it) as divisors*.

- 7 is ultimate: since the division by 0 has no defined result, the number 7 does not admit any divisor* being less than it. The non-trivial divisors 2, 3, 4, 5 and 6 cannot divide it into whole numbers.

- 12 is non-ultimate: the number 6 admits numbers 2, 3, 4 and 6 (numbers being less than it) as divisors*.

Thus, by these previous definitions, the set of whole numbers is organized into these two entities:

- the set of ultimate numbers, which is the fusion of the prime numbers sequence with the numbers 0 and 1.

- the set of non-ultimate numbers identifying to the non-prime numbers sequence, deduced from the numbers 0 and 1.

* non-trivial divisor.

2.4 Conventional designations

As "primes" designates prime numbers, it is agree that designation "ultimates" designates ultimate numbers. Also it is agree that designation "non-ultimates" designates non-ultimate numbers. Other conventional designations will be applied to the different classes or types of whole numbers later introduced.

2.5 The first ten ultimate numbers and the first ten non-ultimate numbers

Considering the previous double definition, the sequence of ultimate numbers is initialized by these ten numbers:

0 1 2 3 5 7 11 13 17 19

Considering the previous double definition, the sequence of non-ultimate numbers is initialized by these ten numbers:

4 6 8 9 10 12 14 15 16 18

3. The four classes of whole numbers

The segregation of whole numbers into two sets of entities qualified as ultimate and non-ultimate is only a first step in the investigation of this type of numbers. Here is a further exploration of this set of numbers revealing its organization into four subsets of entities with their own but interactive properties.

3.1 Four different types of numbers

From the definition of ultimate numbers introduced above, it is possible to differentiate the set of whole numbers into four final classes, inferred from the three source classes and progressively defined according to these criteria:

Whole numbers are subdivided into these two categories:

- **<u>ultimates</u>**: an ultimate number not admits any non-trivial divisor (whole number) being less than it.

- **<u>non-ultimates</u>**: a non-ultimate number admits at least one non-trivial divisor (whole number) being less than it. Non-ultimate numbers are subdivided into these two categories:

- **<u>raiseds</u>**: a raised number is a non-ultimate number, power of an ultimate number.

- **<u>composites</u>**: a composite number is a non-ultimate and not raised number admitting at least two different divisors.

Composite numbers are subdivided into these two categories:

- **<u>pure composites</u>**: a pure composite number is a non-ultimate and not raised number admitting no raised number as divisor.

- **<u>mixed composites</u>**: a mixed composite number is a non-ultimate and not raised number admitting at least a raised number as divisor.

3.1 Degree of complexity of number classes

The table in Figure 1 summarizes these different definitions. It is more fully developed in Figure 5 Chapter 5.1 where the interactions of the four classes of whole numbers are highlighted.

<table>
<tr><th colspan="4">The whole numbers:</th></tr>
<tr><td rowspan="3">The ultimates:

an ultimate number not admits any non-trivial divisor (whole number) being less than it</td><td colspan="3">The non-ultimates:

A non-ultimate number admits at least one non-trivial divisor (whole number) being less than it</td></tr>
<tr><td rowspan="2">The raiseds:

a raised number is a non-ultimate number, power of an ultimate number</td><td colspan="2">The composites:
a composite number is a non-ultimate and not raised number admitting at least two different divisors</td></tr>
<tr><td>The pure composites:

a pure composite number is a non-ultimate and not raised number admitting no raised number as divisor</td><td>The mixed composites:

a mixed composite number is a non-ultimate and not raised number admitting at least a raised number as divisor</td></tr>
<tr><td>level 1</td><td>level 2</td><td>level 3</td><td>level 4</td></tr>
<tr><td colspan="4">degree of complexity of the final four classes of numbers</td></tr>
</table>

Fig.1 Classification of whole numbers from the definition of ultimate numbers (see Fig. 5 and 7 also).

4. New whole numbers classification

4.1 The four subsets of whole numbers

By the previous definitions and demonstrations, we propose the classification of the set of whole numbers into four subset or classes of numbers:

- the ultimate numbers called ***ultimates*** (***u***),
- the raised numbers called ***raiseds*** (***r***),
- the pure composite numbers called ***composites*** (***c***),
- the mixed composite numbers called ***mixes*** (***m***).

4.1.1 Conventional denominations

So it is agree that designation "ultimates" designates ultimate numbers (as "primes" designates prime numbers). Also it is agree that designation "raiseds" designates raised numbers, designation "composites" designates pure composite numbers and designation "mixes" designates mixed composite numbers. It is also agreed that is called u an ultimate number, r a raised number, c a pure composite and m a mixed composite number.

4.2 Organization charts of whole numbers

This new classification of whole numbers requires some other illustrations of the organization of the $\mathbb{N}$ set.

4.2.1 Hierarchical organizational chart

Thus this set ℕ can be described by a hierarchical organization of its components. At the end of the hierarchy are the four new classes of numbers previously introduced. Figure 2 illustrates this organization.

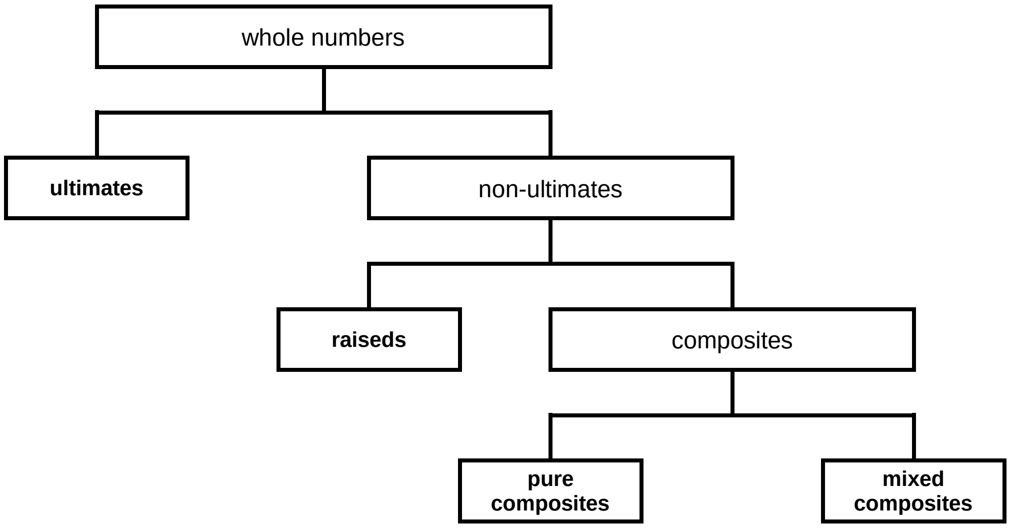


Fig.2 Hierarchical classification of whole numbers since the definition of ultimate numbers.

4.2.2 Inclusive diagram

Also, as illustrated in Figure 3, an inclusive organization is revealed in the organization of the set ℕ.

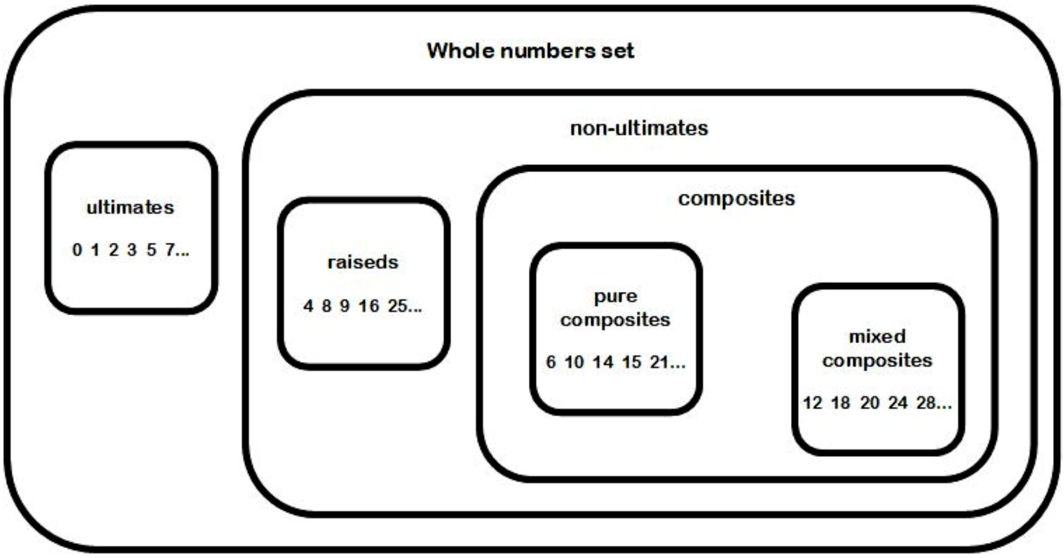


Fig.3 Inclusive (Euler's) diagram of the classification of whole numbers.

Thus the set of whole numbers contains the set of ultimates and that of non-ultimates, the set of non-ultimates contains the set of raiseds and that of composites, this latter set contains the one of pure composites and that of mixed composites.

Conversely, can we conclude that set of the mixed composites is therefore included in that of the composites, this one latter being included in that of the non-ultimates, itself included in set of the whole numbers. Set of the pure composites is found in the same inclusions.

Set of the raiseds is included in that of the non-ultimates, this one latter being included in set of the whole numbers. Finally, set of ultimates is only included in that of whole numbers.

The table in Figure 4 summarizes this inclusive organization of the set of whole numbers.

whole numbers set ($\mathbb{N}$)	$\supseteq$	**ultimates set**	$\supseteq$	**raiseds set**	$\supseteq$	**pure composites set**
		non-ultimates set		**composites set**		**mixed composites set**

Fig. 4 Inclusion of the seven sets of numbers constituting the set of whole numbers.

5 Ultimate divisor

The distinction of whole numbers into different classes deduced from the definition of ultimate numbers allows us to propose the double concept of *ultimate divisor* and *ultimate algebra*.

5.1 Ultimate divisor: definition

An ultimate divisor of a whole number is an ultimate number ***less than this whole number*** *and non-trivial divisor of this whole number.*

For example the number 12 has six divisors, the numbers 1, 2, 3, 4, 6 and 12 but only two ultimate divisors: 2 and 3. Also, the numbers zero (0) and one (1), although definite numbers as ultimate, are never ultimate divisors. As a reminder, the division by zero (0) is not defined and therefore this number is not an ultimate divisor. The number one (1) is a trivial divisor, it does not divide a number into some smaller part.

5.2 Concept of ultimate algebra

The ultimate algebra applies only to the set of whole numbers and is organized, on the one hand, around the definition of ultimate divisor (previously introduced), on the other hand around the definition of ultimate number (previously introduced). This algebra states that any whole number is either an ultimate number having no ultimate divisor, or a non-ultimate number (which can be either a raised, or a pure composite, or a mixed composite) breaking down into several ultimate divisors. In this algebra, no whole number x can be written in the form $x = x \times 1$ but only in the form $x = x$ (ultimate) or in the form $x = y \times y \times \ldots$ (raised) or $x = y \times z \times \ldots$ (composite) or $x = (y \times y \times \ldots) \times z \times \ldots$ (mixed). Also in this algebra, it is not allowed to write for example $0 = 0 \times y \times z \times \ldots$ but only $0 = 0$.

5.2.1 Specific features of the numbers zero and one

By these postulates proposing a concept of ultimate algebra, it is agreed and recalled that although defined as ultimate numbers, the numbers zero (0) and one (1) are neither ultimate divisors, nor composed of ultimate divisors.

5.3 Ultimate divisors and number classes

The table in Figure 5 synthesizes the four interactive definitions of the four classes of whole numbers by incorporating the double concept of ultimate divisor and ultimate algebra.

An ultimate number (***u***) not admits any non-trivial divisor (whole number) being less than it.
Can be written in the form:

$$\boldsymbol{u = u}$$

→ without ultimate divisors

a raised number (***r***) is a non-ultimate number, power of an ultimate number.
Can be written in the form:

$$\boldsymbol{r = u \times u}$$

→ only with identical ultimate divisors

$$\begin{tikzcd} \boldsymbol{u} \arrow[r] \arrow[d] & \boldsymbol{r} \arrow[d] \\ \boldsymbol{c} \arrow[r] & \boldsymbol{m} \end{tikzcd}$$

A pure composite number (***c***) is a non-ultimate and not raised number admitting no raised number as divisor.
Can be written in the form:

$$\boldsymbol{c = u \times u'}$$

→ only at different ultimate divisors

A mixed composite number (***m***) is a non-ultimate and not raised number admitting at least a raised number as divisor
Can be written in the form:

$$\boldsymbol{m = u \times u \times u'}$$
$$\boldsymbol{= r \times u' = c \times u}$$

→ simultaneously at identical ultimate divisors and at different ultimate divisors

Fig. 5 Interactions of the four classes of whole numbers. See Fig. 1 and 7 also.

Crosswise to the hierarchical or inclusive organizations (illustrated in Figures 2 and 3) of the different sets of whole numbers, the four final natures of numbers therefore also have a linear and semi-circular interaction. Thus, illustrated in Fig. 6, is it possible to oppose the two classes of ultimate (*u*) and mixed (*m*) numbers to the two classes of raised (*r*) and composite (*c*) numbers and to qualify these two groups as classes extreme and median.

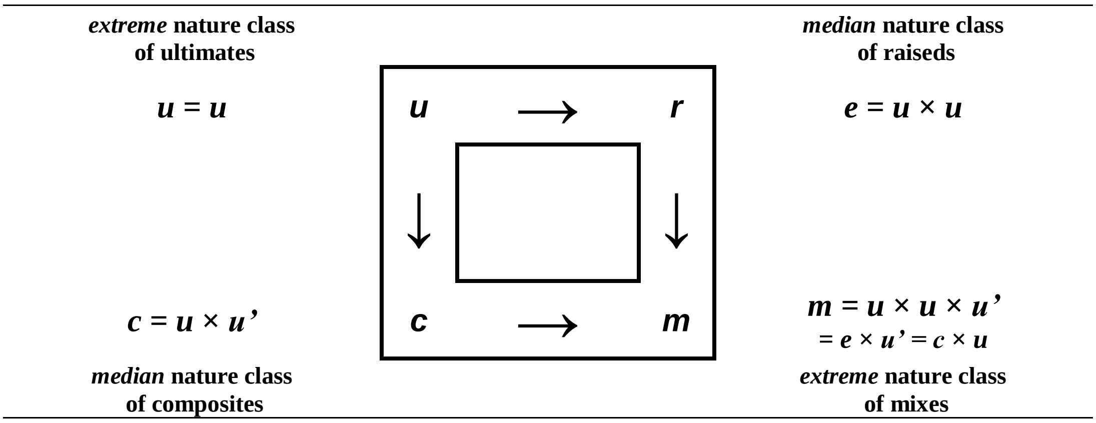


Fig.6 Nature and interactions of the four classes of whole numbers. See Fig. 5 also.

6. New classification and 3/2 ratio

The new classification of whole numbers generates singular arithmetic phenomena in the initial distribution of the different sets of numbers considered. These phenomena result into varied and very often transcendent ratios of exact value 3/2 (or / and reversibly of value 2/3).

6.1 Number classes and 3/2 ratio

The progressive differentiation of source classes and final classes of whole numbers is organized (Figure 7) into a powerful arithmetic arrangement generating transcendent ratios of value 3/2. Thus, the source set of whole numbers includes, among its first ten numbers, 6 ultimate numbers against 4 non-ultimate numbers. The next source set, that of the non-ultimates, includes, among its first ten numbers, 4 raised numbers against 6 composite numbers. Finally, the source set of composites includes, among its first ten numbers, 6 pure composites against 4 mixed composites.

The first 10 whole numbers: 0 1 2 3 4 5 6 7 8 9

6 ultimates: 0 1 2 3 5 7 ← **ratio 3/2** → 4 non-ultimates: 4 6 8 9

The first 10 non-ultimates: 4 6 8 9 10 12 14 15 16 18

4 raiseds: 4 8 9 16 ← **ratio 2/3** → 6 composites: 6 10 12 14 15 18

The first 10 composites: 6 10 12 14 15 18 20 21 22 24

6 pure: ← **ratio 3/2** → 4 mixed:

The first 10 ultimates		The first 10 raiseds		The first 10 pure composites		The first 10 mixed composites
0 1 2 3 5 7	← **ratio 3/2** →	**4 8 9 16**	← **ratio 2/3** →	**6 10 14 15 21 22**	← **ratio 3/2** →	**12 18 20 24**
11 13 17 19	← **ratio 2/3** →	25 27 32 49 64 81	← **ratio 3/2** →	26 30 33 34	← **ratio 2/3** →	28 36 40 44 45 48
ratio 3/2		**ratio 2/3**		**ratio 3/2**		**ratio 2/3**

10 mixed composites

20 composites (pure and mixed)

30 non-ultimates

The 40 primordial numbers

Fig. 7 From the first ten numbers of the three source classes of whole numbers, generation inside 3/2 ratios of the first ten numbers of each of the four final classes of numbers: the 40 primordials. See Fig. 1 and 5 and 7 also.

A very strong entanglement links all these sets of numbers which oppose in multiple ways in ratios of value 3/2 (or reversibly of ratios 2/3). For example, the first 6 ultimates (0-1-2-3-5-7) are simultaneously opposed to the 4 non-ultimates (4-6-8-9) among the first 10 natural numbers, to the 4 raiseds of the first 10 non-ultimates (4-8-9-16) and to the 4 ultimates beyond the first 10 whole numbers (11-13-17-19).

6.1.2 The forty primordial numbers

This entangled classification of whole numbers makes it possible to define (Figure 7) a set of forty primordial numbers. These forty primordial numbers are the set of first ten numbers in each of the four final classes of whole numbers. It is understood that the term "primordials" designates these forty primordial numbers.

6.1.3 Initial numbers and 3/2 ratio

Also, as shown in Figure 7 and in other viewing angle Figure 8, these four sets of ten numbers are all made up of subgroups of always four and six entities according to their respective initial formation and, depending of this initial formation, a value ratio 3/2 (or reversibly of 2/3) always exists between adjacent complexity level subgroups (see Figure 1).

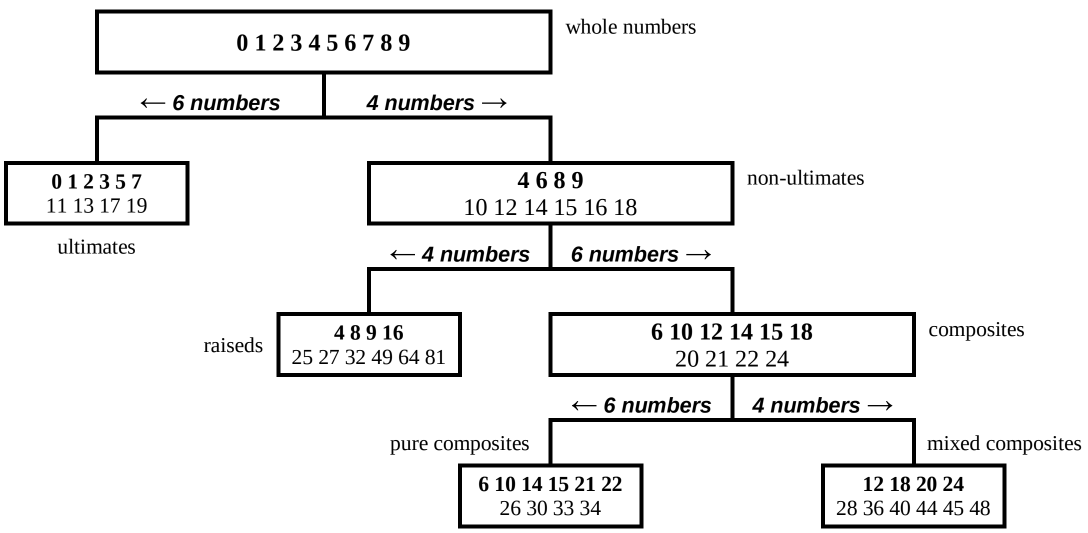


Fig.8 Initial arithmetic arrangements in 3/2 ratios inside hierarchical classification of whole numbers. See Fig. 7 also.

6.2 Matrix of twenty-five entities and 3/2 ratio

The value 25 is the first that can be subdivided into four others (9 + 6 + 6 + 4) generating a triple value ratio to 3/2. As illustrated in figure 9, the value 9 (3 × 3) is opposed in 3/2 ratio to the value 6 (3 × 2) then the value 6 (2 × 3) is opposed to the value 4 (2 × 2). These four values oppose themselves two by two in the 15/10 ratio, extension of the 3/2 ratio. This arithmetic demonstration is a *geometric* variant of the remarkable identity $(a + b)^2 = a^2 + 2ab + b^2$ where a and b have here the values 3 and 2, values opposing in the 3/2 ratio.

It so happens that the matrix of the first 25 numbers (matrix configured from this remarkable identity) generates these phenomena by the opposition of the numbers of extreme classes (ultimates and mixes) to those of median classes (raiseds and composites) of which it is made up.

$$\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2 \rightarrow (3+2)^2 = 3^2 + 2(3\times2) + 2^2 \rightarrow 9 + 6 + 6 + 4 = 25 \text{ entities}}$$

	0u	**1u**	**2u**	**3u**	*4r*	
9 a^2	**5u**	*6c*	**7u**	*8r*	*9r*	**6** *ab*
	10c	**11u**	**12m**	**13u**	*14c*	
6 *ba*	*15c*	*16r*	**17u**	**18m**	**19u**	**4** b^2
	20m	*21c*	*22c*	**23u**	**24m**	

15 extremes (*u* or *m*) ← **3/2 ratio** → 10 medians (*r* or *c*)

Fig. 9 Opposition in 3/2 ratio of the first 15 extremes and the first 10 medians in a matrix of 25 entities deduced from the remarkable identity $(a + b)^2 = a^2 + 2ab + b^2$ where a and b have the values 3 and 2.

It thus appears, Figure 9, that among the first 25 numbers are 15 entities of extreme classes which oppose in a ratio of value 3/2 to 10 entities of median classes.

Also, these two types of classes of numbers are opposed, Figure10, in different transcendent ratios of value 3/2 in and between the sub-matrices whose configuration is deduced from the remarkable identity $(a + b)^2 = a^2 + 2ab + b^2$ where a and b have 3 and 2 to respective value.

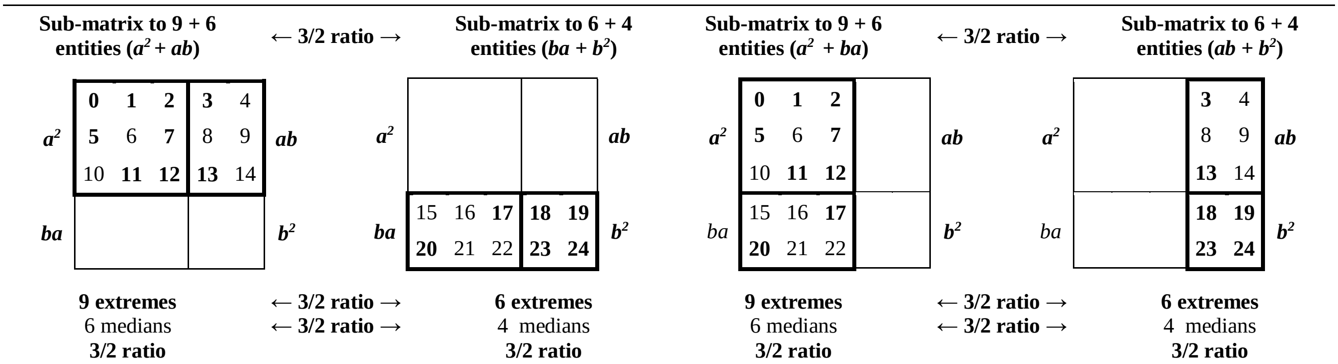


Fig. 10 Distribution of the numbers to extreme and median classes in two double sub-matrices of the first 25 numbers. Configurations inferred from the identity $(a + b)^2 = a^2 + 2ab + b^2$ where a and b have the values 3 and 2. See Fig. 9 also.

6.3 Classes of numbers and pairs of numbers

According to their classification into four classes as defined in Chapter 4 (u = ultimate, r = raised, c = composite and m = mix, see Figure 5 Chapter 5.3 also), whole numbers can be associated two by two in ten different configurations.

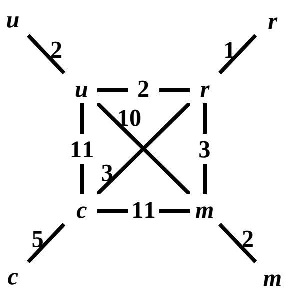
6.3.1 The ten associations of number classes

In the matrix of the first hundred whole numbers classified linearly into ten lines of ten consecutive entities, it is possible to form 50 pairs of consecutive numbers. These couples can be arranged in ten different ways according to the respective class of the two entities constituting them. It turns out Figure 11 that, in this matrix, all the ten possible associations are represented including only one but very ever-present association of two raised class numbers: the couple 8-9 (2^3 and 3^2).

For fun (but maybe not) it's nice to note that these two numbers are the last two of the digit numbers. Also, their respective root values are in a ratio of 2/3 and they are respectively raised by 3 and 2 powers: another ratio of 3/2. Also, (the demonstration will not be done here) it would seem that it is the only pair of consecutive raised class numbers among the set of whole numbers.

30 couples ***r-r*** ***r-m*** ***m-u*** ***u-c*** ***c-c***

0-1	*u-u*	2-3	*u-u*	4-5	*r-u*	**6-7**	***c-u***	**8-9**	***r-r***
10-11	***c-u***	**12-13**	***m-u***	**14-15**	***c-c***	16-17	*r-u*	**18-19**	***m-u***
20-21	*m-c*	**22-23**	***c-u***	**24-25**	***m-r***	26-27	*c-r*	**28-29**	***m-u***
30-31	***c-u***	32-33	*r-c*	**34-35**	***c-c***	**36-37**	***m-u***	**38-39**	***c-c***
40-41	***m-u***	**42-43**	***c-u***	44-45	*m-m*	**46-47**	***c-u***	**48-49**	***m-r***
50-51	*m-c*	**52-53**	***m-u***	54-55	*m-c*	56-57	*m-c*	**58-59**	***c-u***
60-61	***m-u***	62-63	*c-m*	64-65	*r-c*	**66-67**	***c-u***	68-69	*m-c*
70-71	***c-u***	**72-73**	***m-u***	74-75	*c-m*	76-77	*m-c*	**78-79**	***c-u***
80-81	***m-r***	**82-83**	***c-u***	84-85	*m-c*	**86-87**	***c-c***	**88-89**	***m-u***
90-91	*m-c*	92-93	*m-c*	**94-95**	***c-c***	**96-97**	***m-u***	98-99	*m-m*



20 couples *u-u* *u-r* *r-c* *c-m* *m-m*

Fig. 11 Count of the associations of classes of numbers of the pairs of adjacent numbers of the matrix of the first 100 numbers. See Fig. 12 also.

6.3.2 Symmetric associations of number classes

By grouping together five particular associations of pairs of numbers and five others, it turns out that, in a ratio of 3/2, 30 couples are made up of these first five associations considered and 20 couples are made up of the other five possible associations. As shown in Figure 12, these two groups of five associations are not arbitrary but are organized in two sub symmetrical hyper configurations which can be called *configuration N* and *configuration Z*. This, with reference to the image released from these hyper configurations of twice five associations of numbers in the schematization of these configurations.

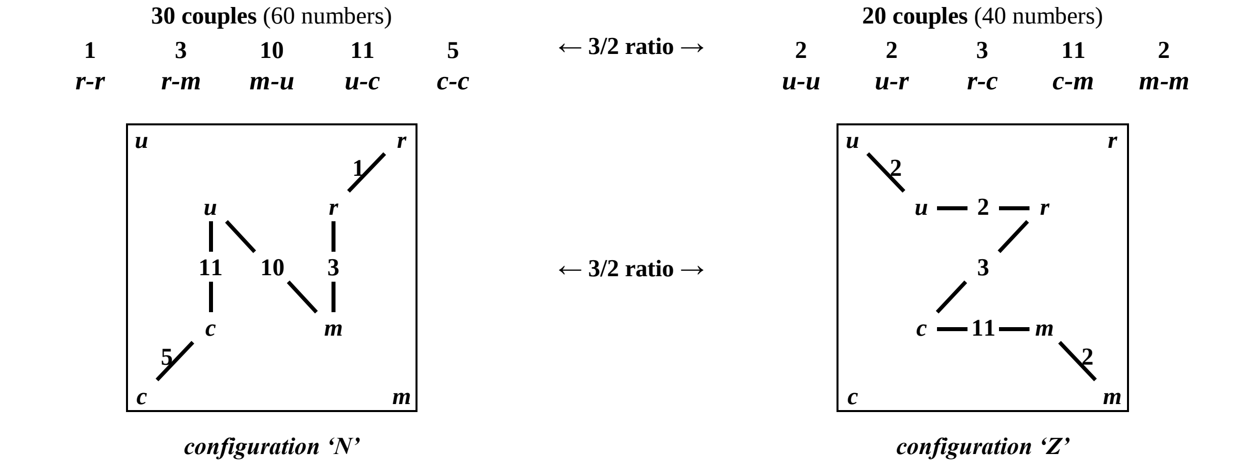


Fig. 12 Classification of the 50 pairs of numbers according to two symmetrical configurations of associations of couples. In a 3/2 value ratio: 30 pairs to N configuration versus 20 pairs to Z configuration. See Fig.11 also.

The N-type configuration has two *protuberances* made up of associations of two raiseds (*r*-*r*) and two composites (*c*-*c*), namely types of numbers of median classes. The Z-type configuration has its two similar and symmetrical protuberances made up of associations of two ultimates (*u*-*u*) and two mixes (*m*-*m*), namely types of numbers of extreme classes.

Also, illustrated in the left part of Figure 13, among the 30 couples of configuration *N*, 6 couples are formed of two numbers of the same classes (5 *c*-*c* couples and 1 *r*-*r* couple) and among the 20 couples of configuration *Z*, 4 couples are formed of two numbers of the same classes (2 *u*-*u* couples and 2 *m*-*m* couples). Again, these sets of couples are in opposition in a 3/2 value ratio.

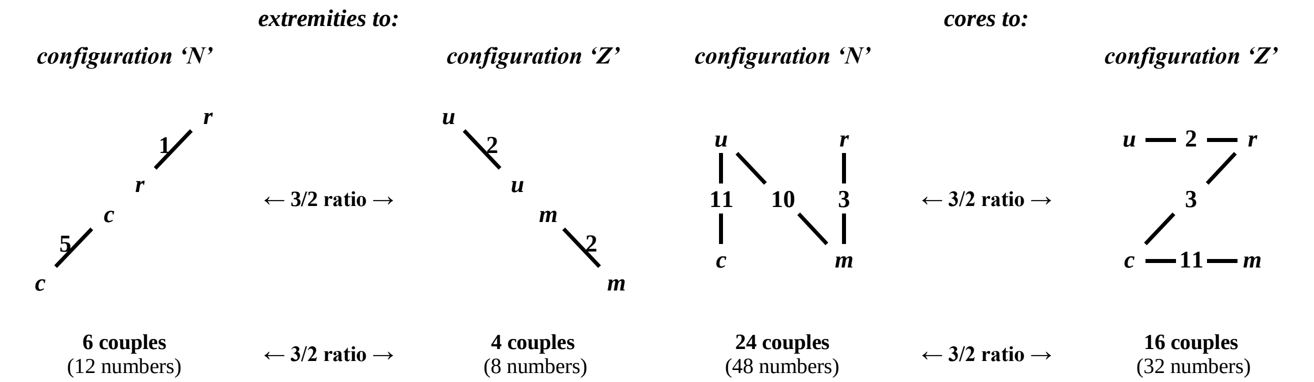


Fig. 13 Maintaining of the 3/2 ratio in the protuberances (associations of entities of the same nature) and the cores (associations of entities of different natures) in the *N* and *Z* configurations of number couples. See Fig. 12 also.

Also, illustrated in the right part of Figure 13, the cores of the two configurations (stripped of their protuberances) therefore also oppose in a 3/2 value ratio with 24 couples of configuration *N* versus 16 couples of configuration *Z*.

Regarding this matrix of the first hundred numbers (Figure 11) and their associations coupled according to their four different natures (ultimates, raiseds, composites or mixes), many arithmetic demonstrations always involving 3/2 value ratios are illustrated in the initial article *"The ultimate numbers and the 3/2 ratio"* [1] and so the reader is strongly invited to consult it.

6.4 Matrix of the twenty fundamentals and number classes

In the initial article *"The ultimate numbers and the 3/2 ratio"* [1], the high importance of the entanglement of the first twenty whole numbers, which are conventionally called the twenty fundamentals, is demonstrated.

The addition matrix of the first ten ultimate with the first ten non-ultimate numbers (which happen to be the first twenty whole numbers), generates, Figure 14, one hundred values which can be distinguished according to the four classes of numbers defined Chapter 1: ultimates (*u*), raiseds (*r*), composites (*c*) and mixes (*m*).

As shown in Figure 14, in this addition matrix of the twenty fundamentals, the classes of numbers oppose two by two in ratios of value 3/2 or value 1/1 depending on the considered configurations.

the first 10 non-ultimates

the first 10 ultimates

+	4	6	8	9	10	12	14	15	16	18
0	*4r*	*6c*	*8r*	*9r*	*10c*	*12m*	*14c*	*15c*	*16r*	*18m*
1	*5u*	*7u*	*9r*	*10c*	*11u*	*13u*	*15c*	*16r*	*17u*	*19u*
2	*6c*	*8r*	*10c*	*11u*	*12m*	*14c*	*16r*	*17u*	*18m*	*20m*
3	*7u*	*9r*	*11u*	*12m*	*13u*	*15c*	*17u*	*18m*	*19u*	*21c*
5	*9r*	*11u*	*13u*	*14c*	*15c*	*17u*	*19u*	*20m*	*21c*	*23u*
7	*11u*	*13u*	*15c*	*16r*	*17u*	*19u*	*21c*	*22c*	*23u*	*25r*
11	*15c*	*17u*	*19u*	*20m*	*21c*	*23u*	*25r*	*26c*	*27r*	*29u*
13	*17u*	*19u*	*21c*	*22c*	*23u*	*25r*	*27r*	*28m*	*29u*	*31u*
17	*21c*	*23u*	*25r*	*26c*	*27r*	*29u*	*31u*	*32r*	*33c*	*35c*
19	*23u*	*25r*	*27r*	*28m*	*29u*	*31u*	*33c*	*34c*	*35c*	*37u*

60 numbers of level 1 and 2 class ← **3/2 ratio** → 40 numbers of level 3 and 4 class

39 ultimates | **21 raiseds** | **29 composites** | **11 mixes**

50 numbers of level 2 and 3 class

↑ **1/1 ratio** ↓

50 numbers of level 1 and 4 class

Fig. 14 Distribution of the 4 classes of numbers generated from the additions matrix of the 20 fundamentals segregated into 10 ultimates versus 10 non-ultimates. See Fig. 1 and 6 also.

Thus, in this matrix of one hundred entities, the opposition of the extreme classes (of level 1 and 4 of complexity) to the middle classes (of level 2 and 3 of complexity) is organized into an exact ratio of value 1/1 and the opposition of the first two classes of 1st and 2nd level of complexity to the last two classes of 3rd and 4th level of complexity is organized into an exact 3/2 value ratio.

6.5 Classes of the first 30 numbers

As illustrated in Figure 15, the first thirty whole numbers, which therefore include the ten digits and the twenty fundamentals, are opposed in various 3/2 value ratios according to their belonging to the different types of classes.

30 first numbers

20 fundamental numbers

10 digit numbers

0u 1u 2u 3u 4r 5u 6c 7u 8r 9r	*10c 11u 12m 13u 14c 15c 16r 17u 18m 19u*	*20m 21c 22c 23u 24m 25r 26c 27r 28m 29u*

6 ultimates / 4 non-ultimates: → 3/2 ratio
(level class 1 / levels class 2, 3 and 4)

12 ultimates or mixes / 8 raiseds or composites: → 3/2 ratio
(extreme complexity level classes / middle level class)

18 ultimates or raiseds / 12 composites or mixes: → 3/2 ratio
(level 1 and 2 complexity classes / level 3 and 4 class)

Fig.15 Opposition in various 3/2 ratios of the first thirty numbers according to their belonging to the types of classes of the whole numbers set. See Fig. 1 and 6 also.

6.6 Inclusion depth

6.6.1 Depth of inclusion of number classes

Any whole number belongs to a subset of the set ℕ. Also, any whole number is positioned at a certain depth of inclusion within ℕ. As illustrated in Figure 16,

- ultimates have an inclusion depth of level 1,
- raiseds have an inclusion depth of level 2,
- pure composites like mixed composites, have an inclusion depth of level 3.

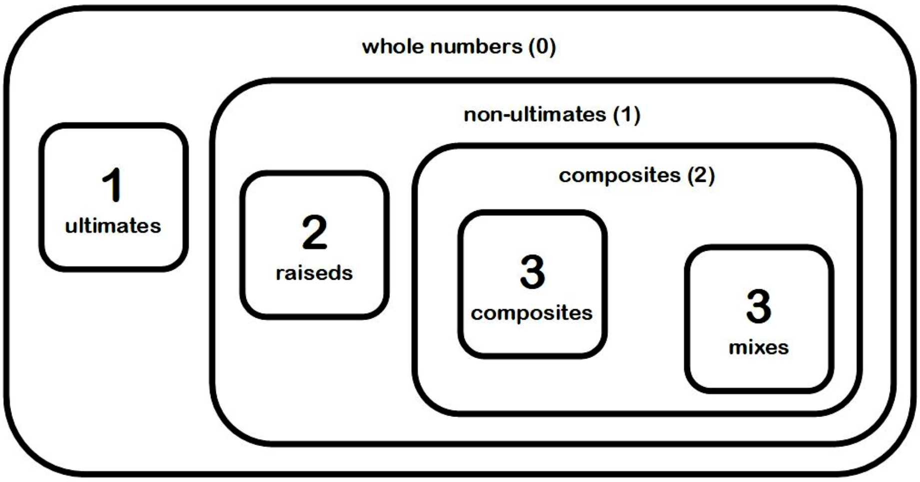


Fig.16 Level of inclusion depth of classes of whole numbers. See Fig. 4 also.

6.6.2 Inclusion depth of the first 30 numbers

As shown in Figure 17, the inclusion levels of the first thirty numbers are not arbitrary. It turns out that exactly 2/5th of these numbers have a level 1 inclusion depth, then 1/5th have a level 2 and again 2/5th have a level 3 inclusion depth.

first 30 numbers inclusion level :

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
1	**1**	**1**	**1**	**2**	**1**	**3**	**1**	**2**	**2**	**3**	**1**	**3**	**1**	**3**	**3**	**2**	**1**	**3**	**1**	**3**	**3**	**3**	**1**	**3**	**2**	**3**	**2**	**3**	**1**

inclusion of level 1:	inclusion of level 2:	inclusion of level 3:
0 1 2 3 5 7 11 13 17 19 23 29	*4 8 9 16 25 27*	*6 10 12 14 15 18 20 21 22 24 26 28*
12 entities	**6 entities**	**12 entities**
2/5	**1/5**	**2/5**

Fig.17 Level of inclusion depth of the first 30 whole numbers.

Thus, Figure 18, can we see that in a 3/2 value ratio, 18 entities of inclusion level 1 or 2 oppose 12 entities of inclusion level 3. Simultaneously with this, the 12 entities of level 1 and the 6 level 2 entities total 24 levels of depth of inclusion and the 12 level 3 entities total 36 levels. These two groups oppose in an inverse ratio of 2/3 value.

Also, following these arithmetic arrangements, the first thirty numbers accumulate 60 levels of inclusion depth within the set ℕ, so an exact level 2 average for these thirty particular numbers. Investigations of the same type do not reveal any similar arithmetic phenomena beyond these thirty numbers, this legitimizing the interest that is taken here in their particularities.

inclusion level:	inclusion of level 1 or 2		inclusion of level 3
entities :	*0 1 2 3 5 7 11 13 17 19 23 29* *4 8 9 16 25 27*		*6 10 14 15 21 22 26* *12 18 20 24 28*
	12 + 6 = **18 entities**	**← 3/2 ratio →**	**12 entities**
cumulative levels:	(12 × 1) + (6 × 2) = **24 levels**	**← 2/3 ratio →**	(12 × 3) = **36 levels**

Fig.18 Level of inclusion depth and cumulated levels of inclusion of the first 30 whole numbers. See Fig. 16 and 17.

Figure 19 heightens the legitimacy of the peculiarities of the first thirty numbers when it comes to their respective but also collective level of inclusion depth. In these two matrices, one of 3 times 10 numbers and the other of 6 times 5 numbers, these thirty numbers are symmetrically opposed in two groups of 18 versus 12 entities cumulating inclusion levels of 36 versus 24. All this is again organized in 3/2 value ratios.

Matrix to 3 by 10 entities

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	1	1	1	2	1	3	1	2	2
10	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
3	1	3	1	3	3	2	1	3	1
20	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
3	3	3	1	3	2	3	2	3	1

0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	1	1	1	2	1	3	1	2	2
10	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*
3	1	3	1	3	3	2	1	3	1
20	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*
3	3	3	1	3	2	3	2	3	1

18 entities **36 levels** ← **3/2 ratio** → *12 entities* **24 levels**

Matrix to 6 by 5 entities

0	*1*	*2*	*3*	*4*
1	1	1	1	2
5	*6*	*7*	*8*	*9*
1	3	1	2	2
10	*11*	*12*	*13*	*14*
3	1	3	1	3
15	*16*	*17*	*18*	*19*
3	2	1	3	1
20	*21*	*22*	*23*	*24*
3	3	3	1	3
25	*26*	*27*	*28*	*29*
2	3	2	3	1

0	*1*	*2*	*3*	*4*
1	1	1	1	2
5	*6*	*7*	*8*	*9*
1	3	1	2	2
10	*11*	*12*	*13*	*14*
3	1	3	1	3
15	*16*	*17*	*18*	*19*
3	2	1	3	1
20	*21*	*22*	*23*	*24*
3	3	3	1	3
25	*26*	*27*	*28*	*29*
2	3	2	3	1

18 entities **36 levels** ← **3/2 ratio** → *12 entities* **24 levels**

Fig.19 Symmetrical oppositions of 18 versus12 entities and 36 levels of inclusion versus 24 in two matrices of the first thirty numbers.

7. Conclusion

The twin concept of ultimity or non-ultimity of whole numbers which is based on a new mathematical definition emphasizing the inferiority of the components of the digital entities considered allows us to propose a new classification of these whole numbers.

Thus, any whole number can only belong to one of the four classes of numbers newly introduced here. These four classes of numbers are conventionally called according to their degree of complexity:

- the class of ultimates (*u*), source class of first level of complexity,
- the class of raiseds (*r*), class of second level of complexity,
- the class of pure composites (*c*), called composites, class of third level of complexity,
- the class of mixed composites (*m*), called mixes, class of fourth and last level of complexity.

These four classes of numbers form four subsets of the set ℕ which is also made up, because of this proposed new classification, of the set of non-ultimates and the global set of composites.

Thus the set ℕ is made up of six sets whose characteristics all depend on the original definition of the ultimate numbers. Within the set ℕ, these six sets have a level of inclusion depth varying from 1 to 3:

- the ultimates set and the one of the non-ultimates have a level 1 of inclusion,
- the raiseds set and the one of the composites have a level 2 of inclusion,
- the pure composites set and the one of the mixed composites have a inclusion level to 3.

Since ℕ conventionally denotes the set of natural whole numbers, It is suggested to represent these six new sets by the same types of designations.

Also, the singular but yet real arithmetic arrangements of the initial organization of these different new sets of numbers, most of which are in 3/2 ratios, confirm the idea of the legitimacy of this new classification of whole numbers.

References :

Jean-Yves Boulay. The ultimate numbers and the 3/2 ratio. 2020. ⟨ hal-02508414v2⟩

Jean-Yves BOULAY independent researcher (without affiliation) – FRANCE - e-mail: jean-yvesboulay@orange.fr