VORTICITY QUANTITY CONSERVATION LAW IN A VISCOUS INCOMPRESSIBLE FLUID.

Andrey Preobrazhenskiy

ABSTRACT. The paper gives the definition of the "vorticity quantity" concept. It is further shown that the system of generalized Helmholtz equations for a viscous incompressible fluid is a mathematical expression of the vorticity quantity conservation law for each of its components. These results are used to analyze the solutions of the Euler and Navier-Stokes equations. The physical mechanism of a blowup scenario for the 3D Euler equations solutions is described. The impossibility of such a scenario for the 3D Navier-Stokes equations solutions is shown.

To date, unfortunately, it has not been possible to fully understand the properties of the 3D Navier-Stokes equations solutions for the viscous incompressible fluid. Moreover, this cannot be done even for Euler equations describing an inviscid fluid. The level of understanding in this case can be illustrated by the situation allowed by the analysis of the solutions properties of these equations, taken from [1]: "…fluid starts from rest at time t = 0, begins to move at time t = 1 with no outside stimulus, and returns to rest at time t = 2 …". The question of how well the Navier-Stokes equations describe the behavior of real viscous fluids also remains open.

In this paper, an attempt is made to understand the Navier-Stokes equations by considering them in a transformed form - generalized Helmholtz equations. This allowed to identify previously unknown patterns.

Below viscous incompressible fluid is considered, the equation

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \qquad (1)$$

expresses a condition of its incompressibility. For this fluid, consider the system of Navier-Stokes equations:

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \Delta V_x$$

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \Delta V_y \tag{2}$$

$$\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \Delta V_z$$

For any of the three couples of equations (2), using the operation of cross-differentiation, one can exclude the derivative $\partial^2 P/\partial x_i \partial x_j$, as a result, one of the equations of the generalized Helmholtz equations system will be obtained:

$$\frac{\partial \Omega_x}{\partial t} + V_x \frac{\partial \Omega_x}{\partial x} + V_y \frac{\partial \Omega_x}{\partial y} + V_z \frac{\partial \Omega_x}{\partial z} = \Omega_x \frac{\partial V_x}{\partial x} + \Omega_y \frac{\partial V_x}{\partial y} + \Omega_z \frac{\partial V_x}{\partial z} + \nu \Delta \Omega_x$$

$$\frac{\partial \Omega_y}{\partial t} + V_x \frac{\partial \Omega_y}{\partial x} + V_y \frac{\partial \Omega_y}{\partial y} + V_z \frac{\partial \Omega_y}{\partial z} = \Omega_x \frac{\partial V_y}{\partial x} + \Omega_y \frac{\partial V_y}{\partial y} + \Omega_z \frac{\partial V_y}{\partial z} + \nu \Delta \Omega_y \tag{3}$$

$$\frac{\partial \Omega_z}{\partial t} + V_x \frac{\partial \Omega_z}{\partial x} + V_y \frac{\partial \Omega_z}{\partial y} + V_z \frac{\partial \Omega_z}{\partial z} = \Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z} + \nu \Delta \Omega_z$$

where $\Omega_x, \Omega_y, \Omega_z$ - are the vorticity components.

$$\Omega_x = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}; \quad \Omega_y = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}; \quad \Omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$$

Equations (1) and (3) form a system of four equations with three unknown functions V_x, V_y, V_z. It can be easily shown that in the system of equations (3) one of the equations is excessive. The fact is that if the velocity field $\boldsymbol{V}$ satisfies any of the two equations of system (3), then it will be automatically satisfying the third equation of this system. This happens because the vorticity components Ω_i are interconnected by the identity

$$\frac{\partial \Omega_x}{\partial x} + \frac{\partial \Omega_y}{\partial y} + \frac{\partial \Omega_z}{\partial z} \equiv 0 \tag{4}$$

Let us define the concept of vorticity quantity (or amount of vorticity) for the i^{-th} component. The vorticity quantity σ_i enclosed in a small volume of fluid $dxdydz$ is equal to the product of the vorticity value Ω_i by the value of this volume

$$\sigma_i = \Omega_i dxdydz$$

As will become apparent further, in 2D and 3D cases of fluid motion fundamental differences are existing. Therefore, one begins the analysis with the case of 2D fluid motion, in this case the velocities V_x and V_y are independent of the Z coordinate, and $V_z = 0$ ($V_z = const$ is permissible). Then $\Omega_x = 0$, $\Omega_y = 0$, and only one equation remains in system (3), which will also be greatly simplified

$$\frac{\partial \Omega_z}{\partial t} + V_x \frac{\partial \Omega_z}{\partial x} + V_y \frac{\partial \Omega_z}{\partial y} = \nu \Delta \Omega_z \tag{5}$$

Further, the index “z” in the letter Ω can be omitted, since there is only one vorticity component, the vector of which is orthogonal to the plane of fluid motion.

All equations of system (3) and equation (5) in particular, can be obtained without knowing about the existence of the Navier-Stokes equations (2). For this, it is enough to make two assumptions about the vorticity properties.

The first assumption relates to the physical mechanisms by which vorticity can propagate in a fluid. It is assumed that there are only two such mechanisms — convection and diffusion; there are no other mechanisms.

The second assumption is the statement that the vorticity quantity does not disappear in a viscous incompressible fluid. In the space of fluid motion, single out a simply connected volume W bounded by a closed surface S. For this volume, the statement about the vorticity quantity conservation can be formulated as follows:

The change of vorticity quantity in the volume W during the time dt is equal to vorticity quantity that fell into this volume during this time through the surface S surrounding it, minus the vorticity quantity that left it through the same surface during the same time.

The physical mechanisms by which vorticity enters the volume W (and leaves it) are different. Let us consider in more detail each of the mechanisms separately.

Convection - is the transfer of vorticity by a flowing fluid. The action of the convection mechanism can be illustrated by the following analogy. During the movement of an unevenly colored fluid, the dye is transferred by the fluid from one point in space to another. An analogue of the vorticity Ω (one of its components) is the concentration of the dye C at a given point. An analogue of the vorticity quantity σ is the quantity of dye contained in the considered volume of fluid.

Write down the condition of the vorticity quantity conservation for a small volume element $dxdydz$.

Along the coordinate axis X, through the surface element $dydz$, a volume of fluid equal to $V_x(x)dydzdt$ flows into the considered volume element during the time dt. This volume of fluid contains the vorticity quantity equal to $\Omega(x)V_x(x)dydzdt$. The volume of the leaked fluid and leaked quantity of vorticity will be equal to $V_x(x+dx)dydzdt$ and $\Omega(x+dx)V_x(x+dx)dydzdt$ respectively. Then the change of vorticity quantity will be equal

$$\partial\sigma = \big(\Omega(x)V_x(x) - \Omega(x+dx)V_x(x+dx)\big)dydzdt = -\frac{\partial}{\partial x}(\Omega V_x)dxdydzdt$$

Similarly, along the Y axis

$$\partial\sigma = -\frac{\partial}{\partial y}\big(\Omega V_y\big)dydxdzdt$$

For the two-dimensional case, given that $\partial\sigma/dxdydz = \partial\Omega$, one obtains

$$\frac{\partial\Omega}{\partial t} = -V_x\frac{\partial\Omega}{\partial x} - V_y\frac{\partial\Omega}{\partial y} - \Omega\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)$$

By virtue of (1), the expression in parentheses is zero, then the equation describing the mechanism of vorticity convection can be written as follows:

$$\frac{\partial\Omega}{\partial t} + V_x\frac{\partial\Omega}{\partial x} + V_y\frac{\partial\Omega}{\partial y} = 0 \qquad (6)$$

From this equation it follows that the vorticity value at some point, Ω, after time dt will be transferred to a neighboring point lying in the direction of the velocity vector $\boldsymbol{V}$ at a distance Vdt.

Let us consider the mechanism of vorticity diffusion. The diffusion process is described by two Fick's laws. In the simplest one-dimensional case, the expression for the first Fick law has the form

$$j = -k\frac{\partial C}{\partial x} \qquad (7)$$

where j is the diffusion flux density, k is the diffusion coefficient, C is the diffusant concentration. If one uses the analogy already considered above, then C is the value of the dye concentration, i.e. the vorticity Ω. The diffusion coefficient k is the fluid viscosity ν. Now equation (7) can be rewritten in terms of the problem under consideration by generalizing it to two-dimensional case

$$j_i = -\nu \frac{\partial \Omega}{\partial x_i} \tag{8}$$

where j_i is the vorticity flux density (in the direction of the i^{-oh} coordinate). Here, a remark should be made, because, strictly speaking, the value j should be called "the vorticity quantity flux density". However, this name is unsound and unnecessarily concrete.

Clarify the dimension of value j_i (hereinafter L and T are the dimensions of length and time, respectively).

$$[j_i] = [\text{vorticity flux density }] = \left[\frac{\text{vorticity quantity}}{\text{time unit } \cdot \text{square unit}}\right] = \frac{L^3/T}{T \cdot L^2} = \frac{L}{T^2}$$

It is easy to verify that the value j_i in formula (8) has precisely this dimension ($[\Omega] = T^{-1}$, $[\nu] = L^2 T^{-1}$, $[\partial \Omega / \partial x_i] = T^{-1} L^{-1}$).

Again, consider in space a small element of volume $dxdydz$ and for this element write down the condition of the vorticity quantity conservation.

Along the coordinate axis X, through the surface element $dydz$, vorticity quantity $j_x(x)dydzdt$ flows into the considered volume element during the time dt, leaked vorticity quantity $j_x(x+dx)dydzdt$. Then the change of the vorticity quantity will be equal to

$$\partial \sigma = (j_x(x) - j_x(x+dx))dydzdt = -\frac{\partial}{\partial x} j_x \, dxdydzdt = \nu \frac{\partial^2 \Omega}{\partial x^2} dxdydzdt$$

Similarly, along the Y axis

$$\partial \sigma = -\frac{\partial}{\partial y} j_y \, dydxdzdt = \nu \frac{\partial^2 \Omega}{\partial y^2} dxdydzdt$$

Then we have

$$\frac{\partial \Omega}{\partial t} = \nu \left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} \right) \equiv \nu \Delta \Omega \tag{9}$$

As is well known, this equation is called “the equation of diffusion” and is the mathematical expression of second Fick's law. An important property of the diffusion mechanism should be noted. It follows from (8) that the diffusion process is always directed against the vorticity gradient; it always tends to equalize the vorticity values at neighboring points.

To write down the condition for the vorticity quantity conservation under the simultaneous action of convection and diffusion mechanisms, two equations (6) and (9) must be combined. As a result, equation (5) will be obtained, as required.

This completes the consideration of the two-dimensional case. However, one important feature of the two-dimensional Cauchy problem for the Navier-Stokes equations should be noted. Suppose that in the initial vorticity distribution there is a point at which the vorticity value is maximum (in absolute value) $|\Omega|_{max}$ in the entire infinite volume of fluid. Is it possible that at some point in time at some point the vorticity value will exceed the value $|\Omega|_{max}$? The answer to this question is negative, it is impossible. As follows from the consideration of vorticity propagation mechanisms, the convection mechanism can transfer its initial value to nearby points, nothing more. The diffusion mechanism will tend to reduce this value, since the vorticity values at neighboring points are smaller. As a result, the vorticity localized at the initial moment of time will spread over time over the entire infinite volume of the fluid. There are no features in this situation. A rigorous proof of the existence and uniqueness of the solution of the two-dimensional Cauchy problem for the Navier-Stokes equations was already carried out quite a long time by O. Ladyzhenskaya [2].

Consider the three-dimensional case of fluid motion, i.e. equations system (3). In this system of equations, all equations have the same structure, so it will be enough to consider one of the equations, consider the third equation.

$$\frac{\partial \Omega_z}{\partial t} + V_x \frac{\partial \Omega_z}{\partial x} + V_y \frac{\partial \Omega_z}{\partial y} + V_z \frac{\partial \Omega_z}{\partial z} = \Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z} + \nu \Delta \Omega_z$$

There is no doubt that part of this equation

$$V_x \frac{\partial \Omega_z}{\partial x} + V_y \frac{\partial \Omega_z}{\partial y} + V_z \frac{\partial \Omega_z}{\partial z}$$

describes 3D convection, and its other part

$$\nu \Delta \Omega_z$$

describes 3D diffusion of the Z component of vorticity, Ω_z. This part of the equation raises questions

$$\Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z} \tag{10}$$

It is obvious that the appearance of this part is connected with the possibility of non-uniform motion of the fluid in the direction of the vorticity vector (Z axis) because this was not possible with plane motion. And here not even the movement itself is important, i.e. velocity V_z, but its gradients $\partial V_z/\partial x$, $\partial V_z/\partial y$, $\partial V_z/\partial z$ are important. Consideration of the following simple task will help to clarify the situation.

The swirling flow of an ideal ($\nu = 0$) incompressible fluid moves in a round conical pipe. The axis of the pipe coincides with the coordinate axis Z. At the entrance of the pipe $z = 0$ are given: r_0 is the input radius of the pipe, V_0 and ω_0 are the linear and angular velocities of the fluid. It is required to estimate the vorticity flux in an arbitrary section of a pipe of radius r.

Obviously, an equal volume of fluid must flow through any pipe section per unit of time (incompressibility property). The angular momentum crossing any pipe section per unit of time should be the same because there is no friction on the walls (angular momentum conservation law). These two conditions make it possible to estimate the velocity and vorticity in an arbitrary pipe section,

$$V = V_0 \left(\frac{r_0}{r}\right)^2, \quad \Omega = 2\omega_0 \left(\frac{r_0}{r}\right)^2$$

hence the vorticity flux density,

$$j = \Omega V = 2\omega_0 V_0 \left(\frac{r_0}{r}\right)^4$$

and the vorticity flux

$$\Phi = \pi r^2 j = 2\pi r_0^2 \omega_0 V_0 \left(\frac{r_0}{r}\right)^2$$

It can be clearly seen from the last formula that the vorticity flux Φ in a pipe depends on its radius r, it increases with a decrease in radius and decreases with its increase. The vorticity (its quantity) can be formed (absorbed) directly in the mass of fluid due to the condensation (rarefaction) of vortex lines. The condensation (rarefaction) of vortex lines is called in the literature - vortex stretching [3]. In the context of this paper, this name does not convey the physical essence of what is happening in terms of vorticity quantity formation. Here another name will be used – “the vorticity divergence”, why, it will become clear a little later. The concept of the vorticity divergence (this is technical slang) should not be confused with the concept of the divergence of the vorticity vector field, this value is always equal to zero (4).

Here the question involuntarily arises: if the vorticity quantity is formed (absorbed) directly in the mass of fluid, what law of its conservation can be discussed? However, it can be clearly seen from the task considered above that the vorticity divergence is a reversible process, it can be either formed or absorbed. If we talk about the Cauchy problem, then in the vicinity of any point a vorticity quantity can be formed, and in the vicinity of a neighboring point it can be absorbed. Moreover, the total balance of the amount of vorticity formed and absorbed as a result of its divergence over the entire infinite volume of fluid will be equal to zero. Below this statement will be rigorously proved.

In a tapering pipe fragment, the vorticity quantity emanating from it exceeds the vorticity quantity flowing into the pipe. Under these conditions, in accordance with the above assumption that the vorticity quantity is conserved, the vorticity in the pipe Ω should decrease with time. However, this does not happen; the problem under consideration is stationary. From this it becomes clear how to modify the assumption about the vorticity quantity conservation:

The change of vorticity quantity in volume W over time dt is equal to the vorticity quantity that fell into this volume during this time through the surface *S*, plus the vorticity quantity that has formed (absorbed) during this time in the volume under consideration, minus the vorticity quantity that has left this volume through this same surface in the same time.

This formulation is universal, it is valid for the flat case of fluid motion, in this case, vorticity divergence will not occur.

How to determine the vorticity quantity formed due to its divergence? First of all, it is necessary to determine the factors on which the process of vorticity formation depends. As mentioned above, the formation of vorticity can occur only in the case of three-dimensional motion of a fluid with its uneven movement in the direction of the vorticity vector. Since now we are talking about the Z component of vorticity, then the Z component of the velocity V_z will be one of such factors. Obviously, the second such factor can only be the vorticity field **Ω** itself. The desired formula can include only the values V_z and **Ω**.

The vorticity forms a vector field **Ω**, and when one speaks of the presence of field sources, one considers its divergence

$$\operatorname{div}\mathbf{\Omega} = \frac{\partial \Omega_x}{\partial x} + \frac{\partial \Omega_y}{\partial y} + \frac{\partial \Omega_z}{\partial z}$$

The field **Ω** has zero divergence (4), but the divergence of the field $V_z\mathbf{\Omega}$ is not equal to zero, it has sources of the field,

$$\operatorname{div}(V_z\mathbf{\Omega}) = \frac{\partial}{\partial x}(V_z\Omega_x) + \frac{\partial}{\partial y}\left(V_z\Omega_y\right) + \frac{\partial}{\partial z}(V_z\Omega_z) = \Omega_x\frac{\partial V_z}{\partial x} + \Omega_y\frac{\partial V_z}{\partial y} + \Omega_z\frac{\partial V_z}{\partial z} +$$

$$+V_z\left(\frac{\partial \Omega_x}{\partial x} + \frac{\partial \Omega_y}{\partial y} + \frac{\partial \Omega_z}{\partial z}\right) = \Omega_x\frac{\partial V_z}{\partial x} + \Omega_y\frac{\partial V_z}{\partial y} + \Omega_z\frac{\partial V_z}{\partial z} \qquad (11)$$

(the expression in parentheses is identically equal to zero (4)). This is the desired part of the equation (10), its physical meaning has become clearer. Let's check the dimension of the value (11), in the physical sense it should be

$$\left[\frac{\text{vorticity quantity}}{\text{time unit } \cdot \text{volume unit}}\right] = \frac{L^3/T}{T \cdot L^3} = \frac{1}{T^2}$$

formula (11) has precisely this dimension ($[\Omega_i] = T^{-1}, [\partial V_z/\partial x_i] = T^{-1}$). Now it is necessary to strictly show that equation part (10) describes the formation of precisely the Z component of the vorticity quantity, and not something else.

For an arbitrary point in the velocity field, consider a rectangular coordinate system ξ, η, ζ such that the axis ξ passing through this point has a direction coinciding with the direction of the vorticity vector $\mathbf{\Omega}$ at this point. Therefore, at the point under consideration, the vector $\mathbf{\Omega}$ has only one nonzero component Ω_ξ. Also consider the velocity vector at this point $\boldsymbol{V}$ denote its projection onto the ξ axis by V_ξ. Now calculate the divergence of the field $V_\xi \mathbf{\Omega}$ at this point, from (11) it is clear

$$\operatorname{div}\left(V_\xi \mathbf{\Omega}\right) = \Omega_\xi \frac{\partial V_\xi}{\partial \xi}$$

Divergence is an invariant to the choice of a coordinate system at a given point. If its value is known in some coordinate system, then this value will remain the same in any other coordinate system. Then we can write

$$\operatorname{div}\left(V_\xi \mathbf{\Omega}\right) = \Omega_\xi \frac{\partial V_\xi}{\partial \xi} = \Omega_x \frac{\partial V_\xi}{\partial x} + \Omega_y \frac{\partial V_\xi}{\partial y} + \Omega_z \frac{\partial V_\xi}{\partial z} \tag{12}$$

Let γ be the angle between the Z axis and the ξ axis, then it follows from geometric considerations that

$$dV_\xi \cos(\gamma) = dV_z$$

Now the formula (12) can be rewritten as

$$\operatorname{div}\left(V_\xi \mathbf{\Omega}\right) = \Omega_\xi \frac{\partial V_\xi}{\partial \xi} = \frac{1}{\cos(\gamma)} \left(\Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z}\right) \tag{13}$$

The physical meaning of $\operatorname{div}(V_\xi \mathbf{\Omega})$ is the formed vorticity quantity σ_ξ (referred to the unit of volume and unit of time). However, if the value of σ_ξ is known, then the value of σ_z can be determined using the obvious relation arising from geometric considerations

$$\sigma_z = \sigma_\xi \cos(\gamma) \tag{14}$$

Then it follows from formulas (13) and (14) that

$$\Omega_\xi \frac{\partial V_\xi}{\partial \xi}\cos(\gamma) = \left(\Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z}\right) \equiv \mathrm{div}\,(V_z \mathbf{\Omega})$$

and the physical meaning of the value div $(V_z \mathbf{\Omega})$ is the quantity of the Z component of the formed vorticity, σ_z (referred to the unit of volume and unit of time), which was to be proved. Summarizing the formula obtained, it can be argued that the values div $(V_x \mathbf{\Omega})$ and div $(V_y \mathbf{\Omega})$ have the same meaning for the X and Y vorticity components. So, the physical meaning of the equation part (10) is finally established. This part of the equation describes the process of vorticity quantity formation (absorption) – the vorticity divergence (of Z component).

Now it is needed to strictly show that in an infinite volume of fluid for the Cauchy problem, the vorticity quantity formed and absorbed per unit of time will be exactly equal. In other words, it is needed to show that the integral

$$\iiint_0^\infty \left(\Omega_x \frac{\partial V_z}{\partial x} + \Omega_y \frac{\partial V_z}{\partial y} + \Omega_z \frac{\partial V_z}{\partial z}\right) dxdydz \qquad (15)$$

at all-time instants will be identically equal to zero.

As applied to the Cauchy problem, we are only interested in a rather rapidly decreasing velocity field $\boldsymbol{V}$. Let the velocity field $\boldsymbol{V}$ decrease at infinity faster than r^{-1}. Then, the decay rate of the field $\mathbf{\Omega}$ will be faster than r^{-2}. Rewrite the integral (15) in the form

$$\iiint_0^\infty \mathrm{div}(V_z \mathbf{\Omega}) dxdydz$$

and apply the Gauss-Ostrogradsky formula (divergence theorem), which expresses the values of the integral over the volume through the values of the integral over the surface covering this volume. As such a surface, consider a sphere of infinitely large radius

$$\iiint_0^\infty \mathrm{div}(V_z \mathbf{\Omega}) dxdydz = \iint_S V_z \Omega_n dS \qquad (16)$$

here S is the surface of the sphere. The values of the normal component of the field $V_z \Omega_n$ on a sphere of radius r will have a decreasing order at infinity faster than r^{-3}. Then it follows that the integral (16), and hence (15), are identically equal to zero at all time instants, as required.

Now one can summarize the preliminary results, which can be formulated as two conclusions:

1. The vorticity field evolution process can be represented as a superposition of three physical processes - convection, diffusion and divergence of the vorticity.

2. In a viscous incompressible fluid there is a specific conservation law expressed by the equations of system (3). For the Cauchy problem, the vorticity quantity of each of its components of the entire infinite fluid volume

$$\iiint_{0}^{\infty} \Omega_i dxdydz = \sigma_{i0} \tag{17}$$

will remain constant in time, i.e. these values are conserved and are determined only by the initial data of the problem.

The persistence of the integrals (17) for the solutions of the Euler equations is known [3], for the solutions of the Navier-Stokes equations this is established only for the 2D case [3]. Whether the fact that these integrals are preserved for solutions of the 3D Navier-Stokes equations was previously established is unknown to the author of this paper. Nothing is also known about attempts to consider equations (3) as formulations of the conservation law.

Above, for the 2D Cauchy problem, it was concluded that the vorticity cannot grow above its initial values $|\Omega|_{max}$. In the 3D case, the situation is different, now there is the possibility of the vorticity divergence. The vorticity divergence can lead to an increase in its local values. In a viscous fluid, the process of local increase of vorticity will always be counteracted by the mechanism of its diffusion. The mechanisms of divergence and diffusion of vorticity are contra directional; they always work against each other. In an inviscid fluid, the vorticity diffusion mechanism does not work, and there are no other mechanisms preventing the unlimited growth of vorticity. In this regard, the following scenario of vorticity growth becomes possible in an inviscid fluid.

Suppose that in an inviscid fluid there is some initial axisymmetric distribution of one of the vorticity components, for example, Ω_z, the other two components Ω_x, Ω_y, are initially absent. This corresponds to the rotation of the entire mass of fluid around the Z axis. Schematically, this situation can be represented as shown in

Figure 1a, here the color density conveys the vorticity level.

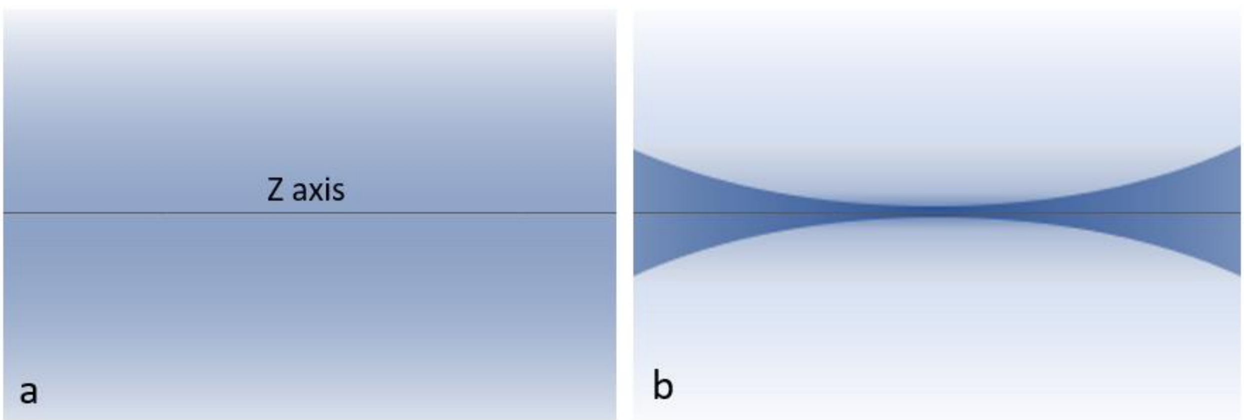

Figure 1. Explanatory drawing.

Now imagine that the vorticity suddenly began to redistribute spontaneously. In this case, an axisymmetric (along the Z axis) center of vorticity concentration began to form in the fluid, surrounded by a region where the vorticity level decreased. This situation is shown schematically in

Figure 1b. In this scheme, the fact of redistribution of vorticity is the key. An increase in the level of vorticity occurs only since somewhere near the level of vorticity falls.

Is this situation acceptable from the point of view of the equations (3)? Yes, it is acceptable; there is no change in the vorticity quantity σ_z, i.e. the conservation law is fulfilled.

It can be argued that in the described process fluid deformations occur, which means that the vorticity components Ω_x and Ω_y should be present, absent initially. However, as can be clearly seen from equations (3), the presence of one of the vorticity components Ω_z makes it possible the emergence and growth of the other two components Ω_x and Ω_y. To do this, it is sufficient to allow an infinitesimal initial perturbation in the fluid so that the derivatives $\partial V_x/\partial z$ and $\partial V_y/\partial z$ become small values.

Here again it can be argued that if the vorticity components Ω_x and Ω_y grow, then the total amount of vorticity of these components σ_x and σ_y will change, but this cannot be so, since they did not exist initially. The total vorticity quantity of these components σ_x and σ_y will not change, it will be zero at all time instants. This condition is ensured by the axial symmetry of the problem under consideration. Under axial symmetry, regions with a positive value of the Ω_x and Ω_y always correspond to regions symmetric to them with negative values Ω_x and Ω_y. Therefore, the values of σ_x and σ_y in this case will always be zero for all the time. It is difficult to disagree that this situation is very much alike the situation described in the staged part of this paper, when an inviscid fluid begins to move without outside stimulus.

Let analyze this situation, now from the point of view of the Euler equations. To do this, consider a vortex that rotates as a solid body with an angular velocity ω. First consider this vortex as isolated, i.e. located not in the mass of fluid but in empty space. The Euler equations give the following picture of the pressure distribution along the radius of the vortex.

$$P = \frac{\rho\omega^2 r^2}{2} - \frac{\rho\omega^2 r_0^2}{2}$$

The constant value in this formula is selected from the condition that the pressure P on the surface of the vortex $r = r_0$, is equal to zero. Then everywhere inside the vortex, $r < r_0$, the pressure will be negative, i.e. tensile (positive) stresses are acting, which corresponds to the physical essence of the problem. Further suppose that a local infinitesimal change in the radius of the vortex occurred and a constriction area has formed. The angular velocity of the fluid in the constriction area will increase and become equal to

$$\omega = \omega_0 (r_0/r_1)^2$$

here ω_0 is the initial angular velocity value, r_1 is the radius of constriction area, $r_1 < r_0$. Pressure in constricted area will increase

$$P = \frac{\rho {\omega_0}^2 {r_0}^2}{2} \left(\frac{r_0}{r_1}\right)^2 - \frac{\rho {\omega_0}^2 r_0^2}{2}$$

The pressure gradient will appear on both sides of the constriction area, directed toward the constriction. Then, in accordance with the Euler equations, on both sides of the constriction, the fluid will start moving against the pressure gradient, i.e. from the constriction area. As a result, the radius of the constriction area will begin to decrease, this process will end with a collapse, the vorticity in the constriction will turn into infinity!

Now, this isolated vortex can be returned in to the fluid and similar reasoning can be carried out, while the qualitative picture will not change. As a result, the conclusion about the instability of a swirling inviscid fluid is inevitable. This conclusion is not in contradiction with existing notions, for example [3], [4]. One may also notice that the idea of self-amplification of vorticity caused by deformation of vortex tubes is not new. This idea in its general features has already been expressed, for example, by R. Feynman in the pages of his lectures on physics.

In paper [5] it was shown that the equation similar in several key properties to three-dimensional Navier-Stokes equation for incompressible fluid, has a smooth solution which blows up in finite time. This circumstance is a serious argument in favor of the possibility of the existence of such solutions for the Navier-Stokes system.

Consider the 3D Cauchy problem for the Navier-Stokes equations with limited initial data. The question must be answered: will the vorticity value remain limited at all time?

As follows from the entire previous analysis, the only possible mechanism for the growth of vorticity above its initial values is the vorticity divergence. In a viscous fluid, the process of local increase of vorticity will always be counteracted by the mechanism of its diffusion. Two contra directional mechanisms will enter the confrontation, and the answer to the question above depends on its outcome. This process can be modeled by running a scenario of the simultaneous action of vorticity divergence and diffusion mechanisms. Moreover, in this scenario, both mechanisms should be started with the greatest possible intensity. This scenario corresponds to the process of radial vortex deformation, consider this process in detail.

From the solution of the problem of a single vortex diffusion (M. Oseen, 1911), the distribution of vorticity in infinite space for any moment of time is known

$$\Omega = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

where Γ is the initial vortex circulation. Now we will only be interested of the vorticity distribution along the radius of the vortex, so this formula can be written as

$$\Omega = \frac{\Gamma}{\pi\alpha} e^{-\frac{r^2}{\alpha}} \tag{18}$$

and consider the value α as a parameter. The value of this parameter controls the degree of vortex deformation, with a decrease in α, the localization of the vortex increases. If calculate the vorticity quantity in infinite space (per unit length along the Z axis), will be obtained

$$\sigma^* = \int_0^\infty 2\pi r \Omega dr = \frac{2\Gamma}{\alpha}\int_0^\infty e^{-\frac{r^2}{\alpha}}\, r dr = -\Gamma e^{-\frac{r^2}{\alpha}} |_0^\infty = \Gamma \tag{19}$$

The sign * means the amount of vorticity per unit length along the Z axis. The vortex described by formula (18) has no sharp boundaries; vorticity is present at arbitrarily large values of radius r. Let us agree to consider the radius r_{out}, as the boundary of the vortex, inside which 95% of the total amount of vorticity of the vortex is contained. Then a condition arises from (19) which determines the external boundary of the vortex.

$$\frac{r_{out}^2}{\alpha} = -\ln(1 - 0.95) \approx 3 \tag{20}$$

Now, speaking of the vortex, with the vorticity quantity σ^*, we assume that all this vorticity quantity is concentrated inside the boundary defined by condition (20). Let the values σ^* and r_{out} be known, then from the formulas (18), (19) and (20) we can determine the maximum value of the vorticity (in the vortex center, $r = 0$)

$$\Omega_m = \frac{3}{\pi}\frac{\sigma^*}{r_{out}^2} \tag{21}$$

From (18), (19) and (20) we can also obtain the formula for the vorticity gradient at the vortex boundary

$$\frac{\partial\Omega}{\partial r}_{r=r_{out}} = -\frac{18}{\pi e^3}\frac{\sigma^*}{r_{out}^3} \tag{22}$$

Consider a vortex, which at time $t = 0$, has radius r_0 and contains vorticity quantity equal to σ_0^*. At time moment $t = 0$, the process of vortex deformation begins at the velocity ε, while the external radius of the vortex begins to change with the time according to the law

$$r_{out} = r_0 - \varepsilon t \tag{23}$$

There is no flow of fluid through the vortex boundary. A gradient of radial and axial velocity appears in the vortex, the vorticity divergence process begins. The velocity ε is the input parameter of the problem; the value of this parameter determines the intensity of the vorticity divergence process, $0 \leq \varepsilon < \infty$.

Upon vortex deformation, the vorticity value Ω_m starts to increase. If the diffusion mechanism did not work in the fluid, then under deformation of the vortex, the amount of vorticity in it would not change (this can be seen from formula (19), the amount of vorticity $\sigma^* = \Gamma$ does not depend on the degree of vortex deformation α). In this case, the value of Ω_m could be determined from formula (21), since the quantity σ^* is the constant. But since the diffusion mechanism works, the outflow of vorticity quantity through the lateral surface of the vortex begins, and the quantity σ^* begins to decrease. One can compose an equation describing the evolution of σ^* value. To do this, one must first calculate the vorticity quantity lost by the vortex during time t, obviously it will be equal to (per unit length along the Z axis)

$$\int_0^t 2\pi r_{out} j dt$$

where j is the vorticity flux density on the side surface of the vortex, is determined by formulas (8) and (22)

$$j = -\nu \frac{\partial \Omega}{\partial r} = \nu \frac{18}{\pi e^3} \frac{\sigma^*}{r_{out}^3} \tag{24}$$

Now one can write an equation that determines the residual vorticity quantity for any moment of time

$$\sigma^* = \sigma_0^* - \int_0^t 2\pi r_{out} j dt \approx \sigma_0^* - 1{,}8\nu \int_0^t \frac{\sigma^*}{r_{out}^2} dt$$

Differentiating this equation with respect to time, obtain

$$\frac{d\sigma^*}{dt} = -1.8\nu \frac{\sigma^*}{r_{out}^2}$$

substituting (23), obtain

$$\frac{d\sigma^*}{dt} = -1{,}8\nu\frac{\sigma^*}{(r_0 - \varepsilon t)^2}$$

The solution to this equation, satisfying the initial conditions, is the function

$$\sigma^* = \sigma_0^* e^{\frac{1{,}8\nu}{\varepsilon r_0}} e^{-\frac{1{,}8\nu}{\varepsilon(r_0 - \varepsilon t)}}$$

From here, using (21), determine the vorticity

$$\Omega_m(t) = \sigma_0^* \ e^{\frac{1{,}8\nu}{\varepsilon r_0}} \ \frac{3}{\pi} \frac{e^{-\frac{1{,}8\nu}{\varepsilon(r_0 - \varepsilon t)}}}{(r_0 - \varepsilon t)^2}$$

The dependence $\Omega_m(t)$ for different values of ε is shown in Figure 2.

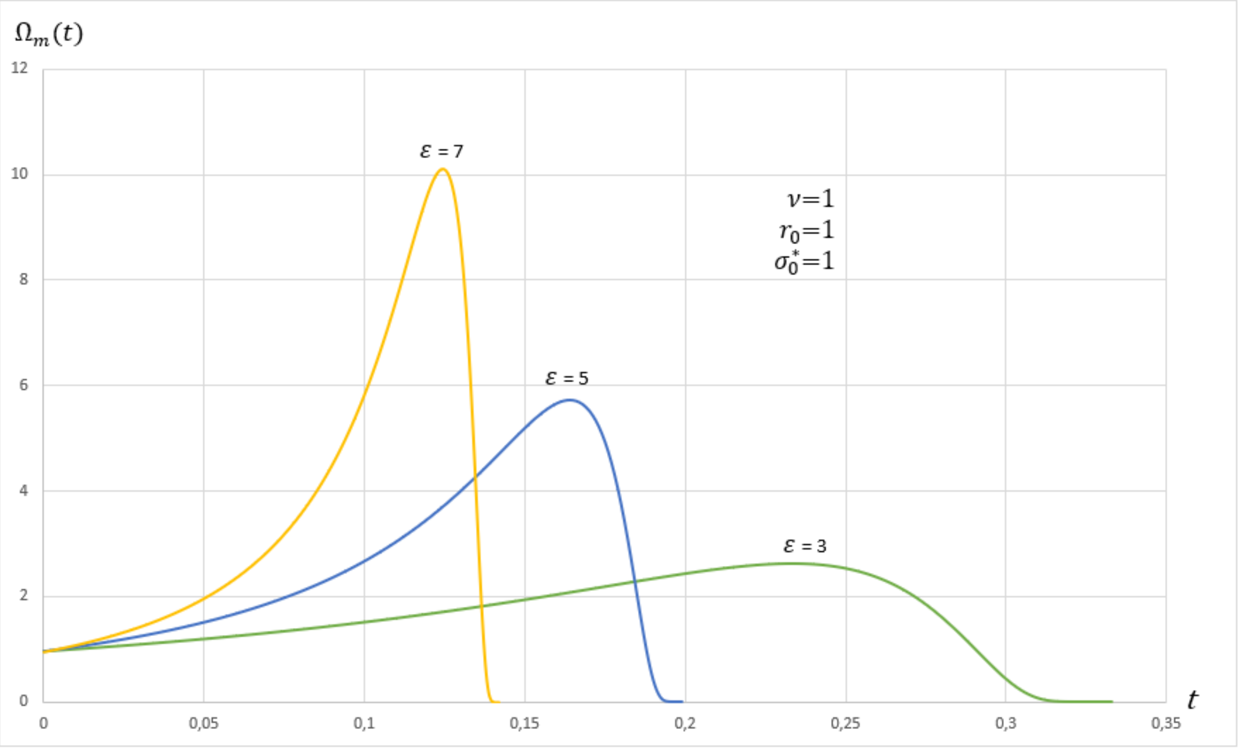

Figure 2. Function $\Omega_{\mathrm{m}}(t)$ diagram.

The function $\Omega_m(t)$ for $t = (r_0/\varepsilon - \ 1{,}8\nu/2\varepsilon^2)$ has a maximum

$$\Omega_{max} \approx 1.179 \ \sigma_0^* \ \frac{e^{((1{,}8\nu/\varepsilon r_0)-2)}}{\nu^2} \varepsilon^2 \qquad (25)$$

when $\varepsilon r_0/\nu \leq 0{,}9$, the maximum of the function $\Omega_m(t)$ disappears. Obviously, the dimensionless number R is an analogue of the Reynolds number Re

$$\mathrm{R} = \varepsilon r_0/\nu$$

One can talk about the vorticity amplification factor λ (for $\mathrm{R} > 0{,}9$ only)

$$\lambda = \frac{\Omega_{max}}{\Omega_{m0}} \approx 1.235 \ e^{((1.8\nu/\varepsilon r_0)-2)} \left(\frac{\varepsilon r_0}{\nu}\right)^2 = 1{,}235 e^{(1.8/\mathrm{R}-2)}\mathrm{R}^2$$

So, for example, the vorticity of a water vortex with a diameter of two centimeters at ε equal to one meter per second (at t = 20°C) will increase by almost seventeen million times!!!

It follows from (25) that the amplified value of the vorticity Ω_{max} is large and overly sensitive to the ε value, $\Omega_{max} \sim \varepsilon^2$. But the value of Ω_{max} remains finite for finite values of ε. However, before making final conclusions, it is necessary to evaluate the conservatism of the model used here in relation to these conclusions themselves.

It will probably be difficult to point out the imperfection of the model with respect to the mechanism of vorticity divergence. From the formula (21) it is seen that the fundamental property of the vortex tubes $\Omega r_{out}^2 = const$ is satisfied. The value specifying the intensity of the vorticity divergence process ε is a parameter that varies as wide as possible $0 \leq \varepsilon < \infty$.

Regarding the mechanism of vorticity diffusion, the model has a clear flaw. The model tacitly assumes that the deformable vortex is infinitely long, i.e. the vorticity has no gradients along the axis of the vortex. Such an assumption is not true and will lead to an underestimation of the diffusion processes intensity. If vorticity diffusion along the axis of the vortex is included in the model, then the diffusion intensity will increase. But in this model, the intensity of the diffusion process is enough to dominate the divergence process. And now one can draw the main and final conclusion:

1. In the 3D Cauchy problem for the Navier-Stokes equations for the incompressible fluid with finite initial data, an unlimited increase in vorticity level is impossible.
2. The model on the basis of which this conclusion is drawn is conservative.

The vorticity field in the solutions of the 3D Navier-Stokes equations can have extremely high and sharp maxima, but it always remains smooth. The derivatives $\partial\Omega_i/\partial x_j$ always exist everywhere and they are continuous functions of coordinates. Then, the velocity field and pressure field will have the same properties.

P.S.

The use of dependence (18) approximating the distribution of vorticity along the radius of the vortex is justified only for small values of velocity ε. In the context of this paper, the results for small values of ε are of no interest. For large values of ε, using approximation (18) is no better than linear approximation ($\Omega = \Omega_m$ for $r = 0$, $\Omega = 0$ for $r = r_{out}$). In the latter case, only the numerical coefficients in the formulas will slightly change without changing the main conclusion.

REFERENCES

1. C. Fefferman. Existence and Smoothness of the Navier-Stokes Equation. http://claymath.org/
2. O. Ladyzhenskaya. The Mathematical Theory of Viscous Incompressible Flows (2nd edition), Gordon and Breach, New York, 1969.
3. A. Bertozzi and A. Majda. Vorticity and Incompressible Flows, Cambridge U. Press, Cambridge, 2002.
4. P. Constantin. Some open problems and research directions in the mathematical study of fluid dynamics, in Mathematics Unlimited–2001 and Beyond, Springer Verlag, Berlin, 2001, 353–360.
5. Terence Tao. Finite Time Blowup for an Averaged Three-Dimensional Navier-Stokes Equation. http://arxiv.org/pdf/1402.0290v2.pdf

andreypr59@gmail.com