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RESEARCH PAPER

Exponential Factorization of Multivectors in $Cl(p, q)$, $p + q < 3$

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Summary

In this paper we consider general multivector elements of Clifford algebras $Cl(p, q)$, $p+q < 3$, and study multivector factorization into products of exponentials and idempotents, where the exponents are blades of grades zero (scalar) to n (pseudoscalar).

KEYWORDS:
Clifford algebra, factorization, hyperbolic numbers, idempotents

1 | INTRODUCTION

The polar representation of complex numbers and of quaternions are widely known. They employ exponential functions with imaginary and pure vector arguments, respectively. We therefore generalize this approach here and look for analogous factorizations of real Clifford algebra multivectors in $Cl(p, q)$, $p + q < 3$, where whenever possible each factor is given by an exponential function with scalar argument (amplitude of the multivector), vector argument (the vector may have zero, negative or positive square), or with a bivector argument (the bivector may have negative or positive square). We find that in some cases idempotent factors (which are not invertible) occur, and in the final result up to eight case distinctions may be required, that are inherited from hyperbolic planes. In each case we specify whether a multivector is invertible or not. If it is invertible we also specify the factorization of the inverse as a form of corollary to the original factorization of the multivector itself. We use Clifford geometric algebras as defined in[2,5,14,8]. Factorization in terms of exponential functions is naturally related to logarithms in geometric algebra[1]. An alternative treatment of multivector elements in terms of tensor products of quaternions can be found in[4].

As summarized in the conclusions (Section 8) the computational approach taken in this work and the results obtained may be relevant in special relativity (one-dimensional space, one-dimensional time), for the construction and geometric understanding of kernels of integral transformations, in the study of versors, pinors and spinors, polar decomposition of multivectors, roots of multivectors, in physics, signal- and image processing, neural network computations, computer algebra, encryption, robotics, computer vision, etc. The results that will be obtained will also be of immediate use in higher dimensional Clifford algebras, whenever the low-dimensional algebras we consider here appear as subalgebras. A promising route to further extend our results may be the quaternion type classification scheme of Clifford algebras developed in[18].

The paper is structured as follows. Because of its importance in the main part of the paper, we first treat hyperbolic numbers in hyperbolic planes in Section 2. This directly leads to the factorization of multivectors in $Cl(1, 0)$ of Section 3, followed by the factorization of multivectors in $Cl(0, 1)$ of Section 4, isomorphic to complex numbers. Then we treat the multivector factorization in $Cl(2, 0)$ in Section 5, the most interesting case of $Cl(1, 1)$ in Section 6, and factorization in $Cl(0, 2)$ in Section 7, the latter being isomorphic to quaternions. The paper ends with conclusions (including a table summarizing the results and remarks on possible applications) in Section 8, followed by references.

[0]Soli Deo Gloria. Dedicated to Rev. Ralph A. Smith (Mitaka, Tokyo, Japan) on the occasion of his 70th birthday and to Prof. K. Gürlebeck (Weimar, Germany) on the occasion of his 60th birthday.

2 | HYPERBOLIC PLANES

An element $u \neq 1$ that squares to $u^2 = +1$ generates a hyperbolic plane $\{b + au\}$, $a, b \in \mathbb{R}$ with basis $\{1, u\}$. An alternative basis $\{id_-, id_+\}$ is given by two idempotents

$$id_+ = \frac{1+u}{2}, \quad id_- = \frac{1-u}{2}, \quad id_+ + id_- = 1, \quad id_+ - id_- = u,$$
$$id_+^2 = id_+, \quad id_-^2 = id_-, \quad id_+ id_- = id_- id_+ = 0. \tag{2.1}$$

Adopting the definitions

$$x^0 = 1, \qquad 0! = 1, \qquad e^x = \sum_{k=0}^{k=\infty} \frac{x^k}{k!}, \tag{2.2}$$

for powers of a general element x and its exponential, we obtain for $a \in \mathbb{R}$

$$e^{a\, id_\pm} = 1 + (e^a - 1) id_\pm, \qquad e^{au} = \cosh a + u \sinh a. \tag{2.3}$$

Remark 2.1. In this paper we do not make further use of $e^{a\, id_\pm}$. But we note that even though $id_\pm$ is not invertible, $e^{a\, id_\pm}$ has inverse $e^{-a\, id_\pm}$, similar to null-vectors not being invertible, but their exponential functions have a multiplicative inverse.

General *nonzero* elements $m = b + au$ of the hyperbolic plane can be classified by whether $|a| = |b|$, or $|a| \neq |b|$. For $|a| = |b|$ we have the four subcases

$$\begin{aligned}
&b = a > 0, \quad m = 2b\, id_+, \\
&b = a < 0, \quad m = 2b\, id_+ = -2|b|\, id_+, \\
&b = -a > 0, \; m = 2b\, id_-, \\
&b = -a < 0, \; m = 2b\, id_- = -2|b|\, id_-.
\end{aligned} \tag{2.4}$$

Examples are for each line of (2.4): $1 + u = 2(1+u)/2 = 2id_+, -2 - 2u = -4(1+u)/2 = 4(-id_+), 3 - 3u = 6(1-u)/2 = 6id_-, -4 + 4u = -8(1-u)/2 = 8(-id_-)$. Thus for $|a| = |b| \neq 0$ we can always represent m as

$$m = 2|b| h^{id}(u), \qquad h^{id}(u) = \pm id_+ \;\text{ or }\; h^{id}(u) = \pm id_-, \tag{2.5}$$

and therefore as

$$m = e^{\alpha_0} h^{id}(u), \qquad \alpha_0 = \ln(2|b|). \tag{2.6}$$

Note that $h^{id}(u)^2 = id_\pm$. The four values of $h^{id}(u)$ specify four bisector directions, one in each quadrant of the hyperbolic plane. Because idempotents $id_\pm$ are not invertible, hyperbolic numbers with $|a| = |b|$ cannot be inverted.

For general (evidently *nonzero*) elements $m = b + au$ with $|a| \neq |b|$ we also have four subcases

$$\begin{aligned}
&b > |a| \geq 0, \quad m = b + au, \\
&a > |b| \geq 0, \quad m = (a + bu)u, \\
&b < -|a| \leq 0, \; m = -(b + au), \\
&a < -|b| \leq 0, \; m = -(a + bu)u.
\end{aligned} \tag{2.7}$$

Examples for (2.7) are line by line: $4 \pm u, \pm 1 + 4u = (4 \pm u)u, -4 \mp u = -(4 \pm u), \mp 1 - 4u = -(4 \pm u)u$. Thus for $|a| \neq |b|$ we can always represent any m as

$$m = (\beta + \alpha u) h(u), \qquad h(u) = \pm 1, \;\text{ or }\; h(u) = \pm u, \tag{2.8}$$

such that $\beta > |\alpha| \geq 0$, and therefore m can be factored as

$$m = e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_u u} h(u), \quad \alpha_0 = \tfrac{1}{2} \ln(\beta^2 - \alpha^2), \quad \alpha_u = \operatorname{atanh}(\alpha/\beta). \tag{2.9}$$

In the examples for (2.7) we have $\alpha = \pm 1, \beta = 4, \alpha_0 \approx 1.35, \alpha_u \approx \pm 0.255$. Note that $h(u)^2 = 1$ and therefore $h(u)^{-1} = h(u)$. The four possible values of $h(u)$ specify the four quadrants in the hyperbolic plane, delimited by two straight lines (bisectors) with directions $id_\pm$. The inverse of hyperbolic numbers with $|a| \neq |b|$ can always be computed as

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_u u} h(u). \tag{2.10}$$

In summary, any $m = b + au \neq 0$ in the hyperbolic plane can be factorized as

$$m = E(m) = E(a, b, u) = e^{\alpha_0} \begin{cases} h^{id}(u) & \text{for} \quad |a| = |b|, \\ e^{\alpha_u u} h(u) & \text{for} \quad |a| \neq |b|. \end{cases} \tag{2.11}$$

Note that we introduce the new notation $E(m) = E(a, b, u)$ to indicate the factorization (2.11) in terms of one or two exponential functions and eight possible values. The computation of the factorization (2.11) is based on (2.4) to (2.6) for the first four cases involving idempotents, i.e. $h^{id}(u) \in \{+id_+, -id_+, +id_-, -id_i\}$, and on (2.7) to (2.9) for the remaining four cases involving the hyperbolic exponential factor and $h(u) \in \{+1, -1, +u, -u\}$. The hyperbolic number m is only invertible for $|a| \neq |b|$.

3 | FACTORIZATION IN $CL(1,0)$

The algebra $Cl(1,0)$ is isomorphic to hyperbolic numbers. A general element $m \in Cl(1,0)$ is given by

$$m = m_0 + m_1 e_1, \quad e_1^2 = 1, \quad m_0, m_1 \in \mathbb{R}. \tag{3.1}$$

We simply use the representation of the hyperbolic plane discussed previously setting $u = e_1$, $b = m_0$, $a = m_1$, and $\alpha_1 = \alpha_u$ to obtain from (2.11) the factorization

$$m = E(m) = E(m_1, m_0, e_1). \tag{3.2}$$

The element $m \in Cl(1,0)$ can only be inverted for $|m_1| \neq |m_0|$, where m is obviously non-zero for $|m_1| \neq |m_0|$.

4 | FACTORIZATION IN $CL(0,1)$

The algebra $Cl(0,1)$ is isomorphic to complex numbers $\mathbb{C}$. A general element $m \in Cl(0,1)$ is given by

$$m = m_0 + m_1 e_1, \quad e_1^2 = -1, \quad m_0, m_1 \in \mathbb{R}. \tag{4.1}$$

If m is non-zero it can therefore be represented in the polar form of complex numbers

$$m = e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_1 e_1}, \quad \alpha = \tfrac{1}{2}\ln(m_0^2 + m_1^2), \quad \alpha_1 = \mathrm{atan2}(m_1, m_0), \tag{4.2}$$

where atan2 selects an angle between 0 and 2π taking the signs of m_1 and m_0 into account. In particular for positive m_1 and m_0 we simply have $\mathrm{atan2}(m_1, m_0) = \mathrm{atan}(m_1/m_0)$, etc. The inverse of any non-zero $m \in Cl(0,1)$ is given by

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_1 e_1}. \tag{4.3}$$

5 | FACTORIZATION IN $CL(2,0)$

A general nonzero element $m \in Cl(2,0)$ can be represented as

$$m = m_0 + m_1 e_1 + m_2 e_2 + m_{12} e_{12}, \qquad m_0, m_1, m_2, m_{12} \in \mathbb{R}, \tag{5.1}$$

with

$$e_1^2 = e_2^2 = -e_{12}^2 = 1, \qquad e_1 e_{12} = -e_{12} e_1, \qquad e_2 e_{12} = -e_{12} e_2. \tag{5.2}$$

We can rewrite m as

$$\begin{aligned} m &= m_1 e_1 + m_2 e_2 + m_0 + m_{12} e_{12} = a u' + b R, \\ a &= \sqrt{m_1^2 + m_2^2}, \quad b = \sqrt{m_0^2 + m_{12}^2}, \\ u' &= (m_1 e_1 + m_2 e_2)/a, \quad R = (m_0 + m_{12} e_{12})/b, \quad u'^2 = R\widetilde{R} = 1. \end{aligned} \tag{5.3}$$

If $m_1 e_1 + m_2 e_2 = 0$ or $m_0 + m_{12} e_{12} = 0$, then the factorization is already complete in the form

$$\begin{aligned} m &= bR = e^{\alpha_0} e^{\alpha_2 e_{12}}, \qquad \alpha_0 = \ln(b), \qquad \alpha_2 = \mathrm{atan2}(m_{12}, m_0), \\ R &= e^{\alpha_2 e_{12}}, \quad R^{-1} = e^{-\alpha_2 e_{12}} = \widetilde{R}, \qquad m^{-1} = e^{-\alpha_0} e^{-\alpha_2 e_{12}}, \end{aligned} \tag{5.4}$$

or

$$m = a u' = e^{\alpha_0'} u', \quad \alpha_0' = \ln(a), \quad \widetilde{m} = m, \quad m^{-1} = e^{-\alpha_0'} u', \tag{5.5}$$

respectively. The operation $R \to \widetilde{R}$ specifies the reversion anti-involution of Clifford geometric algebra[5]. We therefore assume from now on that both $m_1e_1 + m_2e_2 \neq 0$ and $m_0e_2 + m_{12}e_1 \neq 0$, and compute

$$m = au' + bR = (au'R^{-1} + b)R = (au + b)R,$$
$$u = u'R^{-1} = Ru', \qquad u^2 = uu = \widetilde{u}u = u'\widetilde{R}Ru' = 1. \tag{5.6}$$

We can therefore always rewrite $m \in Cl(2,0)$ as

$$m = E(b + au)R = E(a,b,u)R, \tag{5.7}$$

where $E(a,b,u)$ is given by (2.11). This leads to the general factorization (with R from (5.3))

$$m = E(a,b,u)e^{\alpha_2 e_{12}}, \qquad \alpha_2 = \mathrm{atan2}(m_{12}, m_0), \qquad R = e^{\alpha_2 e_{12}}. \tag{5.8}$$

The inverse of $m \in Cl(2,0)$ exists only iff $a \neq b$, where a, b are defined in (5.3)

$$m^{-1} = R^{-1}\big(E(a,b,u)\big)^{-1} = \widetilde{R}\big(E(a,b,u)\big)^{-1}, \tag{5.9}$$

where $\big(E(a,b,u)\big)^{-1}$ is defined as m^{-1} in (2.10). We further note that the factor $h(u)$ in $E(a,b,u)$ of (5.8) can now be restricted to four (instead of eight) values $h(u) \in \{1, e_{12}, id_-, id_+\}$, because a negative sign can always be absorbed in the factor $e^{\alpha_2 e_{12}}$ by changing the angle $\alpha_2 \to \alpha_2 + \pi$.

6 | FACTORIZING $CL(1,1)$

A general (non-zero) multivector in $Cl(1,1)$ is given as

$$m = m_0 + m_1e_1 + m_2e_2 + m_{12}e_{12}, \tag{6.1}$$

with

$$e_1^2 = -e_2^2 = e_{12}^2 = 1, \quad e_1e_{12} = -e_{12}e_1, \quad e_2e_{12} = -e_{12}e_2, \tag{6.2}$$

where at least one of the four real coefficients m_0, m_1, m_2, m_{12} is taken to be different from zero. In two-dimensional special relativity e_1 can specify the space direction and e_2 the time direction (or vice versa).

If the vector part should equal zero, then the factorization simply becomes

$$m = E(m_{12}, m_0, e_{12}), \tag{6.3}$$

applying equation (2.11). In this case m is invertible using (2.10), iff $|m_0| \neq |m_{12}|$.

If the even part $m_0 + m_{12}e_{12}$ should be zero, then the factorization only applies to the vector part $m_1e_1 + m_2e_2$ and its two cases discussed below in detail.

We consider in the next two subsections the case that the vector part of m is not zero, but squares to zero (light like, no inverse exists):

$$(m_1e_1 + m_2e_2)^2 = m_1^2 - m_2^2 = 0 \Leftrightarrow |m_1| = |m_2| \neq 0, \tag{6.4}$$

or the non-zero square case (implying that the vector part of m is not zero and is invertible) that has either positive square (space like)

$$(m_1e_1 + m_2e_2)^2 > 0 \Leftrightarrow |m_1| > |m_2|, \tag{6.5}$$

or negative square (time like)

$$(m_1e_1 + m_2e_2)^2 < 0 \Leftrightarrow |m_1| < |m_2|. \tag{6.6}$$

6.1 | Light like vector part in $Cl(1,1)$

In this subsection we assume that the vector part $m_1e_1 + m_2e_2$ of m is not zero, but squares to zero, and therefore $|m_1| = |m_2| \neq 0$. We define two more zero vectors N, N' by rescaling and component sign change as

$$m_1e_1 + m_2e_2 = AN, \qquad A = \sqrt{2}|m_1|, \tag{6.7}$$

$$N = \tfrac{1}{\sqrt{2}}\Big(\frac{m_1}{|m_1|}e_1 + \frac{m_2}{|m_1|}e_2\Big), \qquad N' = \tfrac{1}{\sqrt{2}}\Big(\frac{m_1}{|m_1|}e_1 - \frac{m_2}{|m_1|}e_2\Big), \tag{6.8}$$

such that

$$N \cdot N' = \frac{1}{2}(\frac{m_1}{|m_1|}e_1 + \frac{m_2}{|m_1|}e_2) \cdot (\frac{m_1}{|m_1|}e_1 - \frac{m_2}{|m_1|}e_2) = \frac{1}{2}(\frac{m_1^2}{|m_1|^2}e_1^2 - \frac{m_2^2}{|m_1|^2}e_2) \overset{|m_1|=|m_2|}{=} \frac{1}{2}(1-(-1)) = 1. \tag{6.9}$$

Remark 6.1. The pair of null-vectors N, N' also famously appears in conformal geometric algebra $Cl(4,1)$ or its extensions to quadrics and higher order curves and surfaces, as representations for the origin and the point at infinity, compare e.g. equ. (1) in [13].

If the even part $m_0 + m_{12}e_{12}$ should equal zero, then the final factorization is

$$m = e^{\alpha_0}N, \qquad \alpha_0 = \ln(A) = \ln(\sqrt{2}|m_1|). \tag{6.10}$$

In this case m cannot be inverted, because N has no inverse.

If the even part $m_0 + m_{12}e_{12}$ does not equal zero, we distinguish under the light like vector part assumption between the subcases $|m_0| \neq |m_{12}|$ and $|m_0| = |m_{12}|$.

6.1.1 | Assuming $|m_0| \neq |m_{12}|$

Now the even part $m_0 + m_{12}e_{12}$ is a hyperbolic number with $b = m_0$, $a = m_{12}$ and $u = e_{12}$. We can therefore apply (2.7) to (2.9) for the factorization of a hyperbolic number, that is not proportional to an idempotent:

$$m_0 + m_{12}e_{12} = e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}). \tag{6.11}$$

And therefore

$$m = AN + e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}), \tag{6.12}$$

which we rewrite as

$$\begin{aligned} m &= [ANe^{-\alpha_0}e^{-\alpha_{12}e_{12}}h(e_{12}) + 1]e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}) \\ &= [Z+1]e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}), \end{aligned} \tag{6.13}$$

with vector

$$Z = ANe^{-\alpha_0}e^{-\alpha_{12}e_{12}}h(e_{12}). \tag{6.14}$$

We remind ourselves that

$$h(e_{12})^2 = 1, \qquad h(e_{12})^{-1} = h(e_{12}). \tag{6.15}$$

That Z is indeed a vector follows immediately from the multiplication table of $Cl(1,1)$, where a vector times a scalar gives a rescaled vector and a vector times a bivector gives a rotated vector, because $e^{-\alpha_{12}e_{12}}h(e_{12})$ is a sum of scalar plus bivector parts. Furthermore, the vector Z is also light like:

$$\begin{aligned} Z^2 &= ANe^{-\alpha_0}e^{-\alpha_{12}e_{12}}h(e_{12})ANe^{-\alpha_0}e^{-\alpha_{12}e_{12}}h(e_{12}) \\ &= A^2e^{-2\alpha_0}Ne^{-\alpha_{12}e_{12}}h(e_{12})Ne^{-\alpha_{12}e_{12}}h(e_{12}) \\ &= A^2e^{-2\alpha_0}Ne^{-\alpha_{12}e_{12}}e^{+\alpha_{12}e_{12}}h(e_{12})Nh(e_{12}) \\ &= A^2e^{-2\alpha_0}Nh(e_{12})Nh(e_{12}) = \pm A^2e^{-2\alpha_0}NN = 0, \end{aligned} \tag{6.16}$$

where we used that vectors and e_{12} anticommute, that vectors and $h(e_{12})$ either commute or anticommute depending on whether $h(e_{12}) = \pm 1$ or $h(e_{12}) = \pm e_{12}$, and that $N^2 = 0$.

Because $Z = Z_1e_1 + Z_2e_2$ squares to zero, we must have $|Z_1| = |Z_2|$. This motivates us to rescale Z to

$$Z = \sqrt{2}|Z_1|z = \alpha_1 z, \quad \alpha_1 = \sqrt{2}|Z_1|, \quad z = \frac{1}{\alpha_1}Z, \quad z^2 = 0. \tag{6.17}$$

With this in mind, we can rewrite m as

$$\begin{aligned} m &= (1+Z)e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}) = (1+\alpha_1 z)e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}) \\ &= e^{\alpha_1 z}e^{\alpha_0}e^{\alpha_{12}e_{12}}h(e_{12}) = e^{\alpha_0}e^{\alpha_1 z}e^{\alpha_{12}e_{12}}h(e_{12}), \end{aligned} \tag{6.18}$$

the final factorization with the help of three exponential functions, where the exponents are a scalar, a zero vector and a bivector, respectively. This result also shows that in this case m is invertible with

$$m^{-1} = e^{-\alpha_{12}e_{12}}h(e_{12})e^{-\alpha_1 z}e^{-\alpha_0} = e^{-\alpha_0}e^{-\alpha_1 z'}e^{-\alpha_{12}e_{12}}h(e_{12}),$$
$$z' = e^{-\alpha_{12}e_{12}}h(e_{12})\, z\, e^{\alpha_{12}e_{12}}h(e_{12}), \qquad z'^2 = 0. \tag{6.19}$$

6.1.2 Assuming $|m_0| = |m_{12}| \neq 0$

Now we have $|m_1| = |m_2| \neq 0$, and $|m_0| = |m_{12}| \neq 0$, and therefore we rewrite m as

$$\begin{aligned} m &= m_1e_1 + m_2e_2 + m_0 + m_{12}e_{12} \\ &= 2m_1\tfrac{1}{2}(e_1 + \frac{m_2}{m_1}e_2) + 2m_0\tfrac{1}{2}(1 + \frac{m_{12}}{m_0}e_{12}) \\ &= 2m_1e_1\tfrac{1}{2}(1 + \frac{m_2}{m_1}e_{12}) + 2m_0\tfrac{1}{2}(1 + \frac{m_{12}}{m_0}e_{12}). \end{aligned} \tag{6.20}$$

Because $\frac{m_2}{m_1} = \pm 1$ and $\frac{m_{12}}{m_0} = \pm 1$ we know that the two expressions in round brackets will equal the idempotents $id_\pm = \frac{1}{2}(1 \pm e_{12})$. That gives us four combinations, depending on the relative signs of the coefficients $\{m_1, m_2, m_0, m_{12}\}$:

$$m = \begin{cases} 2m_1e_1id_+ + 2m_0\,id_+, & \text{for} \quad m_1 = m_2, \quad m_0 = m_{12}, \\ 2m_1e_1id_- + 2m_0\,id_+, & \text{for} \quad m_1 = -m_2, \; m_0 = m_{12}, \\ 2m_1e_1id_+ + 2m_0\,id_-, & \text{for} \quad m_1 = m_2, \quad m_0 = -m_{12}, \\ 2m_1e_1id_- + 2m_0\,id_-, & \text{for} \quad m_1 = -m_2, \; m_0 = -m_{12}. \end{cases} \tag{6.21}$$

Examples for multivectors with this property (line by line in (6.21)) are: $2e_1 + 2e_2 + 3 + 3e_{12} = 4e_1(1 + e_{12})/2 + 6(1 + e_{12})/2 = 4e_1id_+ + 6id_+, 2e_1 - 2e_2 + 3 + 3e_{12} = 4e_1id_- + 6id_+, 2e_1 + 2e_2 + 3 - 3e_{12} = 4e_1id_+ + 6id_-, 2e_1 - 2e_2 + 3 - 3e_{12} = 4e_1id_- + 6id_-$. Note that a linear combination of idempotents in the hyperbolic plane normally is not an idempotent, since the idempotents in the hyperbolic plane are an alternative basis, such that all elements of the hyperbolic plane can be represented in this way, compare (2.1) with $u = e_{12}$. But the first coefficient $2m_1e_1$ is not a scalar, and has the following commutation property

$$e_1e_{12} = -e_{12}e_1 \quad \Leftrightarrow \quad e_1id_\pm = id_\mp e_1, \tag{6.22}$$

we will therefore compute the square of m in each case. We indicate the specific combination of idempotents with two sign indexes

$$\begin{aligned} m_{++}^2 &= (2m_1e_1id_+ + 2m_0\,id_+)^2 = 4(m_1e_1id_+ + m_0\,id_+)(m_1e_1id_+ + m_0\,id_+) \\ &= 4(m_1^2e_1id_+e_1id_+ + m_1m_0(e_1id_+\,id_+ + \,id_+e_1id_+) + m_0^2\,id_+\,id_+) \\ &= 4(m_1^2e_1^2id_-id_+ + m_1m_0e_1(id_+\,id_+ + id_-\,id_+) + m_0^2\,id_+) \\ &= 4(0 + m_1m_0e_1(id_+ + 0) + m_0^2\,id_+) = 4m_0(m_1e_1id_+ + m_0id_+) \\ &= 2m_0m_{++}. \end{aligned} \tag{6.23}$$

Exchanging in (6.23) all $id_+ \leftrightarrow id_-$ we similarly obtain

$$m_{--}^2 = (2m_1e_1id_- + 2m_0\,id_-)^2 = 2m_0m_{--}. \tag{6.24}$$

Next we compute

$$\begin{aligned} m_{+-}^2 &= (2m_1e_1id_+ + 2m_0\,id_-)^2 = 4(m_1e_1id_+ + m_0\,id_-)(m_1e_1id_+ + m_0\,id_-) \\ &= 4(m_1^2e_1id_+e_1id_+ + m_1m_0(e_1id_+\,id_- + \,id_-e_1id_+) + m_0^2\,id_-\,id_-) \\ &= 4(m_1^2e_1^2id_-id_+ + m_1m_0e_1(id_+\,id_- + id_+\,id_+) + m_0^2\,id_-) \\ &= 4(0 + m_1m_0e_1(0 + id_+) + m_0^2\,id_-) = 4m_0(m_1e_1id_+ + m_0id_-) \\ &= 2m_0m_{+-}. \end{aligned} \tag{6.25}$$

Exchanging in (6.25) all $id_+ \leftrightarrow id_-$ we similarly obtain

$$m_{-+}^2 = (2m_1e_1id_- + 2m_0\,id_+)^2 = 2m_0m_{-+}. \tag{6.26}$$

TABLE 1 Multiplication table of idempotents m'_{++}, m'_{+-}, m'_{-+}, and m'_{--}. First column: left factors, first row: right factors.

	$m'_{++}(\boldsymbol{\mu}_1)$	$m'_{+-}(\boldsymbol{\mu}_2)$	$m'_{-+}(\boldsymbol{\mu}_3)$	$m'_{--}(\boldsymbol{\mu}_4)$
$m'_{++}(\lambda_1)$	$m'_{++}(\lambda_1)$	0	$m'_{++}(\lambda_1)$ $+\boldsymbol{\mu}_3(\lambda_1+e_1)id_-$ $=\lambda_1 id_-(\boldsymbol{\mu}_3+e_1)$ $+m'_{-+}(\boldsymbol{\mu}_3)$	$\boldsymbol{\mu}_4(\lambda_1+e_1)id_-$
$m'_{+-}(\lambda_2)$	$(\boldsymbol{\mu}_1+\lambda_2)e_1 id_+$	$m'_{+-}(\boldsymbol{\mu}_2)$	$\lambda_2 id_-(\boldsymbol{\mu}_3+e_1)$	$(1+\boldsymbol{\mu}_4\lambda_2)id_-$
$m'_{-+}(\lambda_3)$	$(1+\boldsymbol{\mu}_1\lambda_3)id_+$	$\lambda_3 id_+(\boldsymbol{\mu}_2+e_1)$	$m'_{-+}(\boldsymbol{\mu}_3)$	$(\boldsymbol{\mu}_4+\lambda_3)e_1 id_-$
$m'_{--}(\lambda_4)$	$\boldsymbol{\mu}_1(\lambda_4+e_1)id_+$	$m'_{--}(\lambda_4)$ $+\boldsymbol{\mu}_2(\lambda_4+e_1)id_+$ $=\lambda_4 id_+(\boldsymbol{\mu}_2+e_1)$ $+m'_{+-}(\boldsymbol{\mu}_2)$	0	$m'_{--}(\lambda_4)$

We can therefore summarize the uniform result of m^2 for $|m_1| = |m_2| \neq 0$, and $|m_0| = |m_{12}| \neq 0$ as

$$m^2 = 2m_0 m, \tag{6.27}$$

which means that

$$m' = \frac{m}{2m_0} = \frac{m_1}{m_0} e_1 id_\pm + id_\pm, \tag{6.28}$$

where all four sign combinations are meant to be included, is also an idempotent in $Cl(1,1)$

$$m'm' = \frac{m^2}{4m_0^2} = \frac{2m_0 m}{4m_0^2} = \frac{m}{2m_0} = m'. \tag{6.29}$$

Depending on the positive or negative sign of m_0 we have the following factorization

$$m = 2m_0 m' = \pm e^{\alpha_0} m', \qquad \alpha_0 = \ln(2|m_0|), \tag{6.30}$$

in terms of a sign, a scalar exponential and an idempotent. Because idempotents cannot be inverted, multivectors $m \in Cl(1,1)$ with $|m_1| = |m_2| \neq 0$, and $|m_0| = |m_{12}| \neq 0$ can in full generality not be inverted.

Equation (6.28) in principle represents four one-parameter ($\lambda = \frac{m_1}{m_0}$) families of idempotents:

$$m'_{++} = \lambda e_1 id_+ + id_+ = (1+\lambda e_1) id_+, \tag{6.31}$$

$$m'_{+-} = \lambda e_1 id_+ + id_- = id_- \lambda e_1 + id_- = id_-(1+\lambda e_1), \tag{6.32}$$

$$m'_{-+} = \lambda e_1 id_- + id_+ = id_+ \lambda e_1 + id_+ = id_+(1+\lambda e_1), \tag{6.33}$$

$$m'_{--} = \lambda e_1 id_- + id_- = (1+\lambda e_1) id_-. \tag{6.34}$$

In the examples for (6.21) above we simply have $2m_0 = 6, \alpha_0 \approx 1.79, \lambda = m_1/m_0 = 2/3$. It may be of interest to know more about these idempotents. They all are the sum of $id_\pm$ plus a null vector $e_1 id_\pm$ or $id_\pm e_1$, since

$$(e_1 id_\pm)^2 = e_1 id_\pm e_1 id_\pm = e_1 e_1 id_\mp id_\pm = 0, \tag{6.35}$$

and similarly

$$(id_\pm e_1)^2 = 0. \tag{6.36}$$

To learn more about them, we computed the full multiplication table Table 1 using instead of a single parameter $\lambda \in \mathbb{R} \setminus \{0\}$ the parameters $\boldsymbol{\mu}_k$ and λ_k, $k = 1,2,3,4$. All computations can be done easily based on (2.1) and (6.22). Note that with the choice $\boldsymbol{\mu}_1 = 1/\boldsymbol{\mu}_4 = -1/\lambda_3 = -\lambda_2$ (assuming $\boldsymbol{\mu}_4 \neq 0$, $\lambda_3 \neq 0$) we would obtain four more zeros in the table, see Table 2 .

6.2 | Space like or time like vector part in $Cl(1,1)$, $|m_1| \neq |m_2|$

We now study for $m \in Cl(1,1)$ the case that the vector part $m_1 e_1 + m_2 e_2$ does not square to zero. We subdivide this case into the even grade part $m_0 + m_{12} e_{12}$ not proportional to an idempotent ($|m_0| \neq |m_{12}|$), and the even part being proportional to an idempotent ($|m_0| = |m_{12}|$).

TABLE 2 Multiplication table of idempotents m'_{++}, m'_{+-}, m'_{-+}, and m'_{--}, choosing $\boldsymbol{\mu}_1 = 1/\boldsymbol{\mu}_4 = -1/\lambda_3 = -\lambda_2$. First column: left factors, first row: right factors.

	$m'_{++}(\boldsymbol{\mu}_1)$	$m'_{+-}(\boldsymbol{\mu}_2)$	$m'_{-+}(\boldsymbol{\mu}_3)$	$m'_{--}(\boldsymbol{\mu}_4)$
$m'_{++}(\lambda_1)$	$m'_{++}(\lambda_1)$	0	$m'_{++}(\lambda_1)$ $+\boldsymbol{\mu}_3(\lambda_1 + e_1)id_-$ $= \lambda_1 id_-(\boldsymbol{\mu}_3 + e_1)$ $+m'_{-+}(\boldsymbol{\mu}_3)$	$\boldsymbol{\mu}_4(\lambda_1 + e_1)id_-$
$m'_{+-}(\lambda_2)$	0	$m'_{+-}(\boldsymbol{\mu}_2)$	$\lambda_2 id_-(\boldsymbol{\mu}_3 + e_1)$	0
$m'_{-+}(\lambda_3)$	0	$\lambda_3 id_+(\boldsymbol{\mu}_2 + e_1)$	$m'_{-+}(\boldsymbol{\mu}_3)$	0
$m'_{--}(\lambda_4)$	$\boldsymbol{\mu}_1(\lambda_4 + e_1)id_+$	$m'_{--}(\lambda_4)$ $+\boldsymbol{\mu}_2(\lambda_4 + e_1)id_+$ $= \lambda_4 id_+(\boldsymbol{\mu}_2 + e_1)$ $+m'_{+-}(\boldsymbol{\mu}_2)$	0	$m'_{--}(\lambda_4)$

6.2.1 Space like or time like vector part in $Cl(1,1)$ and the even part of m not proportional to an idempotent

Now we start with $|m_1| \neq |m_2|$ and $|m_0| \neq |m_{12}|$. That means that both the vector part and the even grade part cannot be zero. Following equations (2.7) to (2.9) and setting $b = m_0$, $a = m_{12}$, $u = e_{12}$ and $\alpha_{12} = \alpha_u$ we can represent the (invertible) even part $m_0 + m_{12}e_{12}$ as

$$m_0 + m_{12}e_{12} = e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}), \tag{6.37}$$

and insert this in m as

$$\begin{aligned} m &= m_1 e_1 + m_2 e_2 + e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}) \\ &= [(m_1 e_1 + m_2 e_2) e^{-\alpha'_0} e^{-\alpha_{12}e_{12}} h(e_{12}) + 1] e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}) \\ &= [V + 1] e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}), \end{aligned} \tag{6.38}$$

where the vector V will have a positive or negative square. We compute the square of V as

$$\begin{aligned} V^2 &= (m_1 e_1 + m_2 e_2) e^{-\alpha'_0} e^{-\alpha_{12}e_{12}} h(e_{12})(m_1 e_1 + m_2 e_2) e^{-\alpha'_0} e^{-\alpha_1 2} h(e_{12}) \\ &= (m_1 e_1 + m_2 e_2) e^{-2\alpha'_0} e^{-\alpha_{12}e_{12}} e^{\alpha_{12}e_{12}} h(e_{12})(m_1 e_1 + m_2 e_2) h(e_{12}) \\ &= (m_1 e_1 + m_2 e_2) h(e_{12})(m_1 e_1 + m_2 e_2) h(e_{12}) \\ &= e^{-2\alpha'_0} \begin{cases} (m_1 e_1 + m_2 e_2)^2 & \text{for } h(e_{12}) = \pm 1, \\ -(m_1 e_1 + m_2 e_2)^2 & \text{for } h(e_{12}) = \pm e_{12} \end{cases} \\ &= e^{-2\alpha'_0} \begin{cases} m_1^2 - m_2^2 & \text{for } h(e_{12}) = \pm 1, \\ m_2^2 - m_1^2 & \text{for } h(e_{12}) = \pm e_{12} \end{cases}, \end{aligned} \tag{6.39}$$

because e_{12} anticommutes with all vectors. We now divide the vector V by its magnitude

$$m = [1 + V] e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}) = [1 + av] e^{\alpha'_0} e^{\alpha_{12}e_{12}} h(e_{12}), \quad a = \sqrt{|V^2|}. \tag{6.40}$$

Depending on the sign of the square of V in (6.39), the square of v will be ± 1.

For $v^2 = -1$, which occurs for $|m_1| < |m_2|$ and $h(e_{12}) = \pm 1$ or for $|m_1| > |m_2|$ and $h(e_{12}) = \pm e_{12}$, we can factorize m as

$$\begin{aligned} & m = e^{\alpha_0} e^{\alpha_1 v} e^{\alpha_{12}e_{12}} h(e_{12}), \\ & e^{\alpha''_0} = \sqrt{1 + a^2}, \ \ \alpha''_0 = \tfrac{1}{2}\ln(1 + a^2), \ \ \alpha_0 = \alpha'_0 + \alpha''_0, \ \ \alpha_1 = \operatorname{atan}(a). \end{aligned} \tag{6.41}$$

This result is invertible

$$\begin{aligned} & m^{-1} = e^{-\alpha_{12}e_{12}} h(e_{12}) e^{-\alpha_1 v} e^{-\alpha_0} = e^{-\alpha_0} e^{-\alpha_1 v'} e^{-\alpha_{12}e_{12}} h(e_{12}), \\ & v' = e^{-\alpha_{12}e_{12}} h(e_{12})\, v\, e^{\alpha_{12}e_{12}} h(e_{12}), \qquad v'^2 = v^2 = -1. \end{aligned} \tag{6.42}$$

Next we look at $v^2 = +1$, which occurs for $|m_1| > |m_2|$ and $h(e_{12}) = \pm 1$ or for $|m_1| < |m_2|$ and $h(e_{12}) = \pm e_{12}$. We distinguish between $a \neq 1$ and $a = 1$.

Assuming $a \neq 1$ we obtain, with the help of (2.7) to (2.9) and setting $b = 1$, $u = v$, the factorization

$$1 + av = e^{\alpha_0''} e^{\alpha_1 v} h(v), \quad \alpha_0'' = \frac{1}{2}\ln(\beta^2 - \alpha^2), \quad \alpha_1 = \operatorname{atan}(\alpha/\beta), \tag{6.43}$$

and hence in this case

$$\begin{aligned} m &= [1 + av] e^{\alpha_0'} e^{\alpha_{12} e_{12}} h(e_{12}) = e^{\alpha_0} e^{\alpha_1 v} h(v) e^{\alpha_{12} e_{12}} h(e_{12}), \\ \alpha_0 &= \alpha_0' + \alpha_0'', \end{aligned} \tag{6.44}$$

where m is invertible

$$\begin{aligned} m^{-1} &= e^{-\alpha_{12} e_{12}} h(e_{12}) e^{-\alpha_1 v} h(v) e^{-\alpha_0} = e^{-\alpha_0} e^{-\alpha_1 v''} h(v'') e^{-\alpha_{12} e_{12}} h(e_{12}), \\ v'' &= e^{-\alpha_{12} e_{12}} h(e_{12})\, v\, e^{\alpha_{12} e_{12}} h(e_{12}), \qquad v''^2 = v^2 = -1. \end{aligned} \tag{6.45}$$

Assuming instead $a = 1$ we obtain

$$1 + av = 1 + v = 2\frac{1}{2}(1 + v) = e^{\ln(2)}\frac{1}{2}(1 + v) \tag{6.46}$$

with idempotent $\frac{1}{2}(1 + v)$. Hence, in this case we get

$$\begin{aligned} m &= [1 + av] e^{\alpha_0'} e^{\alpha_{12} e_{12}} h(e_{12}) = e^{\alpha_0} \frac{1}{2}(1 + v) e^{\alpha_{12} e_{12}} h(e_{12}), \\ \alpha_0 &= \ln(2) + \alpha_0', \end{aligned} \tag{6.47}$$

and because of the idempotent factor $\frac{1}{2}(1 + v)$ the multivector m will in this case not be invertible.

Finally, we summarize the result for $|m_1| \neq |m_2|$ and $|m_0| \neq |m_{12}|$ as

$$m = \begin{cases} e^{\alpha_0} e^{\alpha_1 v} e^{\alpha_{12} e_{12}} h(e_{12}) & \text{for} \quad v^2 = -1, \\ e^{\alpha_0} e^{\alpha_1 v} h(v) e^{\alpha_{12} e_{12}} h(e_{12}) & \text{for} \quad v^2 = 1, \quad a \neq 1, \\ e^{\alpha_0} \frac{1}{2}(1 + v) e^{\alpha_{12} e_{12}} h(e_{12}) & \text{for} \quad v^2 = 1, \quad a = 1. \end{cases} \tag{6.48}$$

The inverse (in two of three cases) can be summarized as

$$m^{-1} = \begin{cases} e^{-\alpha_0} e^{-\alpha_{12} e_{12}} h(e_{12}) e^{-\alpha_1 v} & \text{for} \quad v^2 = -1, \\ e^{-\alpha_0} e^{-\alpha_{12} e_{12}} h(e_{12}) e^{-\alpha_1 v} h(v) & \text{for} \quad v^2 = 1, \quad a \neq 1, \\ \text{none} & \text{for} \quad v^2 = 1, \quad a = 1. \end{cases} \tag{6.49}$$

Line by line examples for (6.48) are:

- $6 - 4e_1 + 6e_2 + 4e_{12} = 2(1 + e_2)(3 + 2e_{12}) \approx e^{1.84} e^{\frac{\pi}{4} e_2} e^{0.805 e_{12}}$, with $\alpha_0 = \ln(2\sqrt{2}\sqrt{3^2 - 2^2}) \approx 1.84, \alpha_1 = \pi/4, v = e_2, h(v) = 1, \alpha_{12} = \operatorname{atanh}(2/3) \approx 0.805, h(e_{12}) = 1$,
- $24 + 30e_1 + 20e_2 + 16e_{12} = 2(5 + 4e_1)e_1(3 + 2e_{12}) \approx e^{2.60} e^{1.10 e_1} e_1 e^{0.805 e_{12}}$, with $\alpha_0 = \ln(2\sqrt{5^2 - 4^2}\sqrt{3^2 - 2^2}) = \ln(2 * 3\sqrt{5}) \approx 2.60, \alpha_1 = \operatorname{atanh}(4/5) \approx 1.10, v = e_1, h(v) = e_1, \alpha_{12} = \operatorname{atanh}(2/3) \approx 0.805, h(e_{12}) = 1$,
- $4 + 4e_1 - 6e_2 - 6e_{12} = 2(1 + e_1)(2 - 3e_{12}) = 2 * 2\frac{1+e_1}{2}(3 - 2e_{12})(-e_{12}) \approx e^{2.19}\frac{1+e_1}{2} e^{-0.805 e_{12}}(-e_{12})$, with $\alpha_0 = \ln(2 * 2\sqrt{3^2 - 2^2}) \approx 2.19, v = e_1, a = 1, \alpha_{12} = \operatorname{atanh}(-2/3) \approx -0.805, h(e_{12}) = -e_{12}$.

We note that for $v^2 = 1$, $a = 1$ the multivector m is not invertible, otherwise it is invertible, compare (6.49). For the first line of (6.48) with $v^2 = -1$ we note that $h(e_{12})$ can now be restricted to two values $h(e_{12}) \in \{1, e_{12}\}$, instead of four values, because a minus sign can always be absorbed in the factor $e^{\alpha_1 v}$ by changing the angle $\alpha_1 \to \alpha_1 + \pi$. For the second line of (6.48) we further note that $h(e_{12})$ can now also be restricted to two values $h(e_{12}) \in \{1, e_{12}\}$, instead of four values, because a minus sign can always absorbed by the proper choice of sign for $h(v) \in \{\pm 1, \pm v\}$. Vice versa, we could also restrict $h(v) \in \{1, v\}$ and accommodate a possible minus sign in the choice of $h(e_{12}) \in \{\pm 1, \pm e_{12}\}$.

6.2.2 | Space like or time like vector part in $Cl(1,1)$ and the even part of m zero or proportional to an idempotent

Now we start with $|m_1| \neq |m_2|$ and $|m_0| = |m_{12}|$. If $|m_0| = |m_{12}| = 0$, then we can simply factorize the non-zero vector part as

$$m = m_1 e_1 + m_2 e_2 = e^{\alpha_0} v, \quad \alpha_0 = \tfrac{1}{2}\ln(|m_1^2 - m_2^2|),$$
$$v = m e^{-\alpha_0}, \quad v^2 = \pm 1, \tag{6.50}$$

where u is time like (space like), iff m is time like (space like). Obviously, m with $|m_1| \neq |m_2|$ and $|m_0| = |m_{12}| = 0$ is invertible

$$m^{-1} = e^{-\alpha_0} v^{-1}, \qquad v^{-1} = \frac{v}{v^2}. \tag{6.51}$$

If $|m_0| = |m_{12}| \neq 0$, we can write m as

$$m = e^{\alpha_0} v + 2m_0 \tfrac{1}{2}(1 + \frac{m_{12}}{m_0} e_{12}), \tag{6.52}$$

with $m_{12}/m_0 = \pm 1$. Therefore

$$\tfrac{1}{2}(1 + \frac{m_{12}}{m_0} e_{12}) = \tfrac{1}{2}(1 \pm e_{12}) = id_{\pm}, \tag{6.53}$$

simply setting $u = e_{12}$ in the idempotent definition (2.1). Thus

$$m = e^{\alpha_0} v + 2m_0 id_{\pm} = e^{\alpha_0} v(1 + e^{-\alpha_0} v^{-1} 2m_0 id_{\pm}). \tag{6.54}$$

The vector $N = v^{-1} id_{\pm}$ must be a null-vector, because the vector v^{-1} anticommutes with e_{12}

$$N^2 = (v^{-1} id_{\pm})^2 = v^{-1} id_{\pm} v^{-1} id_{\pm} = v^{-1} v^{-1} id_{\mp} id_{\pm} = 0. \tag{6.55}$$

We define n parallel to N as

$$n = \sqrt{2} N, \tag{6.56}$$

and finally factorize m as

$$m = e^{\alpha_0} v(1 + e^{-\alpha_0} m_0 \sqrt{2}\sqrt{2} v^{-1} id_{\pm}) = e^{\alpha_0} v(1 + e^{-\alpha_0} m_0 \sqrt{2}\, n)$$
$$= e^{\alpha_0} v(1 + \alpha_1 n) = e^{\alpha_0} v e^{\alpha_1 n} = e^{\alpha_0} e^{\alpha_1 n'} v \tag{6.57}$$

with

$$\alpha_1 = \sqrt{2} e^{-\alpha_0} m_0, \qquad n' = v n v^{-1}. \tag{6.58}$$

Obviously, for $|m_1| \neq |m_2|$ and $|m_0| = |m_{12}| \neq 0$ we can always invert $m \in Cl(1,1)$ using

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_1 n} v^{-1} = e^{-\alpha_0} v^{-1} e^{-\alpha_1 n'}. \tag{6.59}$$

Note that (6.57) includes (6.50) as special case setting $\alpha_1 = 0$ in (6.57). Similarly, for $\alpha_1 = 0$ (6.59) includes (6.51) as special case.

6.3 | Alternative factorization for $Cl(1,1)$ using $Cl(2,0) \cong Cl(1,1)$

The isomorphism $Cl(2,0) \cong Cl(1,1)$ with

$$1 = 1, \quad E_1 = e_1, \quad E_2 = e_{12}, \quad E_{12} = e_2, \tag{6.60}$$

where $\{e_1, e_2\}$ is the orthonormal basis of $\mathbb{R}^2$, and $\{E_1, E_2\}$ is the orthonormal basis of $\mathbb{R}^{1,1}$, allows to factorize $m \in Cl(1,1)$ by first isomorphically mapping it to $Cl(2,0)$, factorizing it there (as shown above in Section 5), and map the factorized result back to $Cl(1,1)$.

We get

$$m = m_0 + m_1 E_1 + m_2 E_2 + m_{12} E_{12} \overset{(6.60)}{=} m_0 + m_1 e_1 + m_2 e_{12} + m_{12} e_2. \tag{6.61}$$

To factorize this multivector $m_0 + m_1 e_1 + m_2 e_{12} + m_{12} e_2$ in $Cl(2,0)$, we simply exchange the places of m_2 and m_{12} in (5.1) to (5.8). And finally we map the factorization obtained back to $Cl(1,1)$ with (6.60). The inverse will be with $m = e^{\alpha_0} m' : m^{-1} = e^{\alpha_0} \overline{m'}$. Viewed strictly in $Cl(1,1)$, the exponential corresponding to $e^{\alpha_1 u}$ will no longer have a single grade one vector as argument, but a sum of vector plus bivector.

7 | FACTORIZATION OF $CL(0,2)$

We now have

$$e_1^2 = e_2^2 = e_{12}^2 = -1, \qquad e_1 e_{12} = -e_{12} e_1, \qquad e_2 e_{12} = -e_{12} e_2. \tag{7.1}$$

We assume a general $m \neq 0$. Because of the isomorphism to quaternions $Cl(0,2) \cong \mathbb{H}$ one factorization result is straight forward (see e.g. the introduction of[11])

$$\begin{aligned} m &= m_0 + m_1 e_1 + m_2 e_2 + m_{12} e_{12} = |m| e^{\alpha_2' i'} = e^{\alpha_0} e^{\alpha_2' i'}, \\ |m|^2 &= m\overline{m} = m_0^2 + m_1^2 + m_2^2 + m_{12}^2, \quad \alpha_0 = \ln(|m|), \\ i' &= \frac{m_1 e_1 + m_2 e_2 + m_{12} e_{12}}{\sqrt{m_1^2 + m_2^2 + m_{12}^2}}, \\ \alpha_2' &= \operatorname{atan2}(\sqrt{m_1^2 + m_2^2 + m_{12}^2}/|m|,\ m_0/|m|), \\ m^{-1} &= \overline{m}/|m| = e^{-\alpha_0} e^{-\alpha_2' i'}. \end{aligned} \tag{7.2}$$

Note that the overbar in $\overline{m}$ indicates Clifford conjugation with $\overline{1} = 1, \overline{e_1} = -e_1, \overline{e_2} = -e_2, \overline{e_{12}} = -e_{12}$, which is the analogue of quaternion conjugation. Indeed any factorization known for quaternions $\mathbb{H}$ can be realized via the isomorphism $Cl(0,2) \cong \mathbb{H}$ in $Cl(0,2)$ as well.

Furthermore, we can factorize $m \in Cl(0,2)$ into exponentials specified by grade. We observe that

$$\begin{aligned} m &= m_1 e_1 + m_2 e_2 + m_0 + m_{12} e_{12} = \vec{m} + m_{even}, \\ \vec{m} &= m_1 e_1 + m_2 e_2, \qquad m_{even} = m_0 + m_{12} e_{12}, \end{aligned} \tag{7.3}$$

can be expressed as the sum of its vector part $\vec{m}$ and its even grade part m_{even}.

The vector part $\vec{m}$ can always be factorized as

$$\begin{aligned} \vec{m} &= m_v v = e^{\alpha_0} v, \qquad \vec{m}^2 = -m_1^2 - m_2^2, \qquad m_v = \sqrt{|\vec{m}^2|}, \\ \alpha_0 &= \ln(m_v) = \tfrac{1}{2} \ln(m_1^2 + m_2^2), \qquad v = \frac{\vec{m}}{m_v}, \qquad v^2 = \frac{\vec{m}^2}{m_v^2} = -1. \end{aligned} \tag{7.4}$$

If the even grade part should be zero, this would already be the final result, and its inverse is

$$\vec{m}^{-1} = -e^{-\alpha_0} v. \tag{7.5}$$

The even grade part m_{even} can always be factorized as

$$\begin{aligned} m_{even} &= m_e e^{\alpha_{12} e_{12}} = e^{\alpha_0} e^{\alpha_{12} e_{12}}, \qquad m_e^2 = \widetilde{m_{even}} m_{even} = m_0^2 + m_{12}^2, \\ m_e &= \sqrt{m_0^2 + m_{12}^2}, \qquad \alpha_0 = \ln(m_e), \qquad \alpha_{12} = \operatorname{atan2}(m_{12}, m_0). \end{aligned} \tag{7.6}$$

If the vector part should be zero, this would already be the final result, and its inverse is

$$m_{even}^{-1} = e^{-\alpha_0} e^{-\alpha_{12} e_{12}}. \tag{7.7}$$

We can therefore now treat the remaining case, that both vector part $\vec{m}$ and even grade part m_{even} do not equal zero. Using the individual factorizations (7.4) and (7.6) we can write a general multivector with non-zero vector and even grade parts as

$$\begin{aligned} m &= \vec{m} + m_{even} = m_v v + m_e e^{\alpha_{12} e_{12}} = (m_v v e^{-\alpha_{12} e_{12}} + m_e) e^{\alpha_{12} e_{12}} \\ &= (m_v x + m_e) e^{\alpha_{12} e_{12}}, \qquad x = v e^{-\alpha_{12} e_{12}}, \end{aligned} \tag{7.8}$$

where the square of vector x computes to

$$x^2 = v e^{-\alpha_{12} e_{12}} v e^{-\alpha_{12} e_{12}} = v e^{-\alpha_{12} e_{12}} e^{\alpha_{12} e_{12}} v = v^2 = -1. \tag{7.9}$$

We can therefore complete the factorization of m with

$$m = e^{\alpha_0} e^{\alpha_1 x} e^{\alpha_{12} e_{12}}, \qquad \alpha_0 = \tfrac{1}{2} \ln(m_e^2 + m_v^2), \qquad \alpha_1 = \operatorname{atan}(\frac{m_v}{m_e}), \tag{7.10}$$

which factorizes m into a product of three exponential factors with scalar, vector and bivector exponents, respectively. This result subsumes the factorizations of vectors (7.4) or $\alpha_{12} = 0$, and of pure even grade parts (7.6) for $\alpha_1 = 0$, respectively. We know that $Cl(0, 2)$ is isomorphic to quaternions, a division algebra. We can therefore always represent the inverse of m simply by

$$m^{-1} = e^{-\alpha_{12}e_{12}} e^{-\alpha_1 x} e^{-\alpha_0} \tag{7.11}$$

or by

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_1 x'} e^{-\alpha_{12}e_{12}}, \quad x' = e^{-\alpha_{12}e_{12}} x e^{\alpha_{12}e_{12}}, \quad x'^2 = -1. \tag{7.12}$$

8 | CONCLUSION

In this work we studied the factorization of multivectors in Clifford geometric algebras $Cl(1, 0)$ (isomorphic to hyperbolic numbers), $Cl(0, 1)$ (isomorphic to complex numbers), $Cl(2, 0)$ (geometric algebra of the two-dimensional Euclidean plane), $Cl(1, 1)$ and $Cl(0, 2)$ (isomorphic to quaternions). The case of mixed signature $Cl(1, 1)$ turned out to be most interesting, and is relevant to applications of Clifford algebra in special relativity of two dimensions (one time and one space dimension). The invertibility of multivectors [12] was also specified, and if it existed, the factorization of the inverse was also given in an analogous form using exponential factors. We summarize our results in Table 3 , specifying the algebra and conditions on the multivector coefficients, the section where to find the respective treatment in this paper, the equation number for the factorization result and equation for the inverse (replaced by none, if the multivector is not invertible). It may be possible in the future to extend this approach to even higher dimensional Clifford algebras.

The polar form of complex numbers and quaternions are widely applied in many sciences. One important application being the kernels of complex and quaternionic Fourier transforms. Possibly the factorizations that we present here may also find future applications in a number of sciences, including many branches of physics, signal- and image processing, neural network computations, computer algebra, encryption, robotics, computer vision, etc. The present work can further be applied in the study of Lipschitz versors, see e.g. E.4.2 in [19] and [14], pinor and spinor groups, etc. The factors we expose are expected to be related to the factors of polar decomposition of multivectors [17], but we will study this in a later work. Our results may also help with the computation of roots of multivectors [15,1,7,10,9]. All computations in this paper can be tested with computer algebra software, e.g. with the Clifford Multivector Toolbox (for MATLAB) [16].

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TABLE 3 Summary of results. For hyperbolic plane: $m = b + au$. For $p + q = 1$: $m = m_0 + m_1 e_1$. For $p + q = 2$: $m = m_0 + m_1 e_1 + m_2 e_2 + m_{12} m_{12}$.

Algebra and conditions	Sect.	Result	Inverse
Hyperbolic plane			
$\lvert a\rvert = \lvert b\rvert$	2	(2.11)	none
$\lvert a\rvert \neq \lvert b\rvert$	2	(2.11)	(2.10)
$Cl(1,0)$			
$\lvert m_0\rvert = \lvert m_1\rvert$	3	(3.2)	none
$\lvert m_0\rvert \neq \lvert m_1\rvert$	3	(3.2)	(2.10), $a = m_1, b = m_0$
$Cl(0,1)$	4	(4.2)	(4.3)
$Cl(2,0)$			
$m_1 = m_2 = 0$	5	(5.4)	(5.4)
$m_0 = m_{12} = 0$	5	(5.5)	(5.5)
otherwise	5	(5.8)	(5.9)
$Cl(1,1)$			
$\lvert m_1\rvert = \lvert m_2\rvert = 0$,	6	(6.3)	Rem. below (6.3)
$\lvert m_1\rvert = \lvert m_2\rvert \neq 0$, $\lvert m_0\rvert = \lvert m_{12}\rvert = 0$	6.1	(6.10)	none
$\lvert m_1\rvert = \lvert m_2\rvert \neq 0$, $\lvert m_0\rvert \neq \lvert m_{12}\rvert$	6.1.1	(6.18)	(6.19)
$\lvert m_1\rvert = \lvert m_2\rvert \neq 0$, $\lvert m_0\rvert = \lvert m_{12}\rvert \neq 0$	6.1.2	(6.30)	none
$\lvert m_1\rvert \neq \lvert m_2\rvert$, $\lvert m_0\rvert \neq \lvert m_{12}\rvert$	6.2.1	(6.48)	(6.49)
$\lvert m_1\rvert \neq \lvert m_2\rvert$, $\lvert m_0\rvert = \lvert m_{12}\rvert$	6.2.2	(6.57)	(6.59)
$Cl(0,2)$	7	(7.10)	(7.11) or (7.12)

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