The Feit-Thompson conjecture and cyclotomic polynomials
To the memory of professors Kazuo Kishimoto and Yôichi Miyashita

Kaoru Motose

Abstract : We can see that Feit-Thompson conjecture is true using factorizations of cyclotomic polynomials on the finite prime field.

Key Words : cyclotomic polynomials, finite fields, splitting field.

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Feit and Thompson conjectured in [2, p.970, last paragraph]

$$s := \Phi_p(q) = \frac{q^p-1}{q-1} \text{ never divides } t := \Phi_q(p) = \frac{p^q-1}{p-1} \text{ for distinct primes } p \text{ and } q,$$

where $\Phi_m(x)$ is the m-th cyclotomic polynomial (see **R1**). The utility and the cause of this conjecture are also stated in [1, p.1], and [3, B25]. Using computer, Stephens [7] found the unique example: for $p = 17$ and $q = 3313$, the prime $r = 2pq + 1 = \gcd(s, t)$ that shows $s \nmid t$ from $r < s$ (see **R3** or [6, p.82]). We show this conjecture is true.

Reviews. In **R1** and **R2,** let ℓ be a prime with $\ell \nmid m$ for natural number m.

R1. We define cyclotomic polynomials over $\mathbb{Q}$ by $\Phi_m(x) := \prod_k (x - \zeta_m^k)$ where $\zeta_m = e^{\frac{2\pi i}{m}}$ and k runs over $E_m := \{k \mid 1 \le k < m \text{ with } \gcd(k, m) = 1\}$. Euler function $\varphi(m) := |E_m| = \deg \Phi_m(x)$ is defined. All roots of $x^m - 1$ are distinct by its derivation mx^{m-1}. Thus all roots of $x^m - 1$ on $\mathbb{Q}$ or $\mathbb{F}_\ell$ forms the cyclic group $\langle \zeta_m \rangle$ of order m. Hence $x^m - 1 = \prod_{d|m} \Phi_d(x)$ on $\mathbb{Q}$ or on $\mathbb{F}_\ell$ by classifying roots by orders (see [4, p.64, 2.45.Theorem]). $\Phi_m(x)$ is irreducible over $\mathbb{Q}[x]$ since it is invariant and minimal by the automorphisms $\sigma_k : \zeta_m \to \zeta_m^k$ for $k \in E_m$. $\Phi_m(x) \in \mathbb{Z}[x]$ by induction on m.

R2. This review is not so popular but important for our theorem. Let $|a|_m$ be the order of a mod m for natural numbers a and m with $\gcd(a, m) = 1$. $\Phi_m(x)$ on $\mathbb{F}_\ell$ factorizes into irreducible polynomials $u_{k_i}(x) = \prod_{h=0}^{|\ell|_m-1} (x - \zeta_m^{k_i \ell^h})$ of the same degree $|\ell|_m$, where $k_i \ell^h \not\equiv k_j \bmod m$ for all h with $0 \le h \le |\ell|_m - 1$, $k_i \in E_m$ and for some $1 \le i \ne j \le \varphi(m)/|\ell|_m$ since $u_{k_i}(x)$ are invariant and minimal by Frobenius automorphism $\sigma_\ell : \zeta_m^{k_i} \to \zeta_m^{k_i \ell}$ (see also [4, p.65, 2.47.Theorem.(ii)]).

R3. Example of Stephens. Using the programuMPQSX3vattached to the package of language UBASIC designed by professor Yuji Kida, we have the prime factorization $s = r_1 r_2 r_3$ for $p = 17, q = 3313$ where r_1, r_2 and r_3 are primes and $r_k - 1 (k = 1, 2, 3)$ are as the next table. We can see $\gcd(s, t) = r_1$ by $q = 3313 \nmid (r_2 - 1)(r_3 - 1)$ in this table.

$$\begin{aligned} r_1 - 1 &= 2 \times 17 \times 3313, \\ r_2 - 1 &= 2 \times 2 \times 5 \times 17 \times 35081 \times 2007623, \\ r_3 - 1 &= 2 \times 17 \times 1609 \times 763897 \times 1869248598543746584721506723. \end{aligned}$$

R4. The next shows $s = t$ iff $p = q$. Since $\frac{x}{\log x}$ is strictly increasing for $3 \le x < y$,

$$\frac{x}{\log x} < \frac{y}{\log y}, \; y^x < x^y \text{ and } \frac{y^x - 1}{y - 1} < \frac{x^y - 1}{y - 1} < \frac{x^y - 1}{x - 1}.$$

R5. $\underline{r \equiv 1 \bmod 2pq}$ for any prime divisor r of s under the conditions $s \mid t$ and $2 < p < q$. If $r \mid s$ and $q \equiv 1 \bmod r$ then $0 \equiv s = q^{p-1} + \cdots + q + 1 \equiv p \bmod r$ and $r = p$. We have a contradiction $0 \equiv t \equiv 1 \bmod r$ by $r = p$. Thus $|q|_r = p$ by $q^p \equiv 1 \bmod r$ and $q \not\equiv 1 \bmod r$. Similarly $|p|_r = q$. Hence we have $r \equiv 1 \bmod 2pq$ by Fermat little theorem.

Theorem. If s divides t, then p is odd and $p = q$.

PROOF. We shall prove this theorem by reduction to absurdity. Hence we assume $p < q$, namely, $s < t$ by **R4** or [5, p.16, Remark]. If $p = 2$, then $s \nmid t$ since $s = q + 1$ is even and $t = 2^q - 1$ is odd. We also see that s, t are odd and $r \equiv 1 \bmod 2pq$ for any prime divisor r of s by **R5** or [7] or [5, p.16, Lemma. (3)]. We can see $|p|_t = q, |p|_s = q$ and $|q|_s = p$ by $p^q \equiv 1 \bmod t, p^q \equiv 1 \bmod s$ and $q^p \equiv 1 \bmod s$ from $p < q < s$ and $s \mid t$.

p : Both $\Phi_t(x)$ and $\Phi_s(x)$ on $\mathbb{F}_p$ have the minimal splitting field $\mathbb{F}_{p^q}$ from $|p|_t = q = |p|_s$ and **R2**. The isomorphism $\zeta_t \to \zeta_s$ over $\mathbb{F}_p$ is contrary to $s < t$, where $\zeta_m = e^{\frac{2\pi i}{m}}$. □.

q : $\Phi_t(x)$ on $\mathbb{F}_q$ factorizes into $\varphi(t)/|q|_t$ irreducible factors by **R2**, where $\varphi(t) = \deg \Phi_t(x)$. We have $|q|_s = p$ divides $|q|_t$ by $q^{|q|_t} \equiv 1 \bmod s$ and the inequality $\varphi(t)/|q|_t \geq \varphi(t)/|q|_s = \varphi(t)/p$ by $|q|_s = p$ because $\Phi_s(x)$ on $\mathbb{F}_q$ already factorizes into $\varphi(s)/|q|_s = \varphi(s)/p$ irreducible factors and hence $\Phi_t(x)$ on $\mathbb{F}_\ell$ factorizes at least into $\varphi(t)/|q|_s = \frac{\varphi(t)}{\varphi(s)} \cdot \frac{\varphi(s)}{p}$ irreducible factors. Thus $|q|_t = p$. If a prime $\ell \mid \gcd(t, (q-1))$, then $q \equiv 1 \bmod \ell$ then we have $|q|_\ell = 1$ is contrary to $\ell \mid \gcd(t, (q-1))$, by the same method as the above. Thus $\gcd(t, (q-1)) = 1$ and $t \mid s(q-1)$, namely, $|q|_t = p$ imply $s = t$, contrary to $s < t$. □

p and q : $\Phi_t(x)$ and $\Phi_s(x)$ on $\mathbb{F}_p$ (resp. $\mathbb{F}_q$) has the minimal splitting field $\mathbb{F}_{p^q} = \mathbb{F}_p(\zeta_t) = \mathbb{F}_p(\zeta_s)$ (resp. $\mathbb{F}_{q^p} = \mathbb{F}_q(\zeta_t) = \mathbb{F}_q(\zeta_s)$) by $|p|_t = |p|_s = q$ (resp. $|q|_t = |q|_s = p$) (see the above **p, q**). $\Phi_t(x)$ has the only one minimal splitting field $\mathbb{Q}(\zeta_t)$, we obtain a contrary $p^q = |\mathbb{F}_{p^q}| = |\mathbb{F}_{q^p}| = q^p$. □

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EMERITUS PROFESSOR, HIROSAKI UNIVERSITY

Home post address: TORIAGE 5-13-5, HIROSAKI, 036-8171, JAPAN

E-mail address: motose@hirosaki-u.ac.jp