Inferences from the Taylor Series

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Abstract

The Taylor series is an important tool in mathematical analysis and it has wide ranging applications .Nevertheless there are inconsistent features related to it. The article intends to demonstrate such features.

Introduction

The Taylor series is well known for its application in mathematics and in physics. The article brings out some anomalous features about the Taylor expansion

Various Inconsistencies

Derivation 1

We expand $f(x)$ in Taylor series[1] about three distinct points x_1, x_2 and x_3 with the same increment h assuming convergence for all three cases

$$f(x_1+h)=f(x_1)+\frac{h}{1!}f'(x_1)+\frac{h^2}{2!}f''(x_1)+\frac{h^3}{3!}f'''(x_1)+\cdots\ldots.(1)$$

$$f(x_2+h)=f(x_2)+\frac{h}{1!}f'(x_2)+\frac{h^2}{2!}f''(x_2)+\frac{h^3}{3!}f'''(x_2)+\cdots\ldots.(2)$$

$$f(x_3+h)=f(x_3)+\frac{h}{1!}f'(x_3)+\frac{h^2}{2!}f''(x_3)+\frac{h^3}{3!}f'''(x_3)+\cdots\ldots(3).$$

The increment ‘h’ in the above three equations may be arbitrary to the extent it does not upset convergence.

Adding the last three equations we have

$$\begin{aligned}f(x_1+h)+f(x_2+h)+f(x_3+h)&\\ =f(x_1)+f(x_2)+f(x_3)+\frac{h}{1!}[f'(x_1)+f'(x_2)+f'(x_3)]&\\ +\frac{h^2}{2!}[f''(x_1)+f''(x_2)+f''(x_3)]+\frac{h^3}{3!}[f'''(x_1)+f'''(x_2)+f'''(x_3)]\ldots(4)\end{aligned}$$

Let x satisfy the following equation

$$f(x) = f(x_1) + f(x_2) + f(x_3)\ (39)(5)$$

Equation (39) does not involve 'h'explicitly

$$\Rightarrow f(x_1 + h) + f(x_2 + h) + f(x_2 + h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \ldots$$

$$f(x_1 + h) + f(x_2 + h) + f(x_2 + h) = f(x + h)(40)(6)$$

But h could be arbitrary. In fact we may very it continuously over a small interval without upsetting convergence. On top of that there are many functions for which the corresponding Taylor expansion is convergent for any arbitrary increment for example $Sin\ x, Cos\ x, \exp(x)$

For the same x_1, x_2, x_3 and corresponding x as given by (39) we do have an infinite number of equations of the type (40) for the various 'h' that comply with the convergence issue. . But (40) (6)does not represent an identity. We do have a situation of a gross violation.

It follows from the Taylor's expansion(from what we have discussed in the preceding part):

If

$$f(x_1) + f(x_2) + f(x_2) = f(x)$$

then

$$f(x_1 + h) + f(x_2 + h) + f(x_2 + h) = f(x + h)$$

subject to the issue f convergence that the Taylor expansion has to converge for the function f for $x_1 + h, x_2 + h$ and $x_3\ + h$

The above inference ensuing from Taylor series does not hold in general[with the exception of the exponential function:$f(x) = e^x$]

If

$$Sin(x_1) + Sin(x_2) + Sin(x_3) = Sin(x)\ (7)$$

By our choice the absolute value of the left side of the above should be less than or equal to unity

Then do we have the following for an arbitrary 'h', [arbitrary to the extent the absolute value of the left side of the following should be less than or equal to unity]?

$$Sin(x_1 + h) + Sin(x_2 + h) + Sin(x_3 + h) = Sin(x + h)\ (8)$$

Any arbitrary value of 'h' has to satisfy (42) subject to the convergence issue and that the absolute value of the left side[of (42)] should be less than or equal to unity.

The Convergence Condition:

If we consider the Taylor expansion of $Sin(x+h)$ for

$$\frac{t_{n+1}}{t_n}=\frac{h^2}{(n+2)(n+1)}\frac{f^{(n+2)}(x)}{f^n(x)}$$

We apply D'Alembert Test considering the absolute value of each term in the expansion. If this series converges then the original series also converges.

$$lim_{n\to\infty}\frac{|t_{n+1}|}{|t_n|}=lm_{n\to\infty}\left|\frac{f^{(n+2)}(x)}{f^n(x)}\frac{h^2}{(n+2)(n+1)}\right|=\left|\frac{f^{(n+2)}(x)}{f^n(x)}h^2\right|\,lim_{n\to\infty}\frac{1}{(n+2)(n+1)}=0<1$$

The modified series consisting of positive terms converges. Hence the Taylor expansion also converges for any arbitrary 'h'[considering finite $\frac{f^{(n+1)}(x)}{f^n(x)}$ for all n and n+1:choosing x in that way and this can be achieved for the sin function]. Those terms for which $f^n(x)=0$ simply drop out of the series. We may apply D'Alembert ratio test to the remaining terms.

We may alternatively think of Cauchy condition: if $lim_{n\to\infty}{u_n}^{1/n}$<1, then the series converges.

Against the Taylor series we form a series comprising the absolute values of the corresponding terms of the Taylor series.

$$\frac{|x|^n}{n!}\left|f^{(n)}(x)\right|$$

$$lim_{n\to\infty}\frac{|x|^n}{n!}=0\Rightarrow lim_{n\to\infty}\left(\frac{|x|^n}{n!}\right)^{1/n}=0\Rightarrow lim_{n\to\infty}\frac{|x|}{(n!)^{1/n}}=0$$

Indeed $lim_{n\to\infty}\frac{|x|^n}{n!}=0$ implies that for every preassigned $\epsilon>0$, no matter how small, we have $N>0$ such that for $n>N$ we have, $\frac{|x|^n}{n!}<\epsilon\Rightarrow\frac{|x|}{(n!)^{1/n}}<\epsilon^{1/n}=\epsilon'$

For any arbitrary $\epsilon'>0$ no matter how small we can arrange for an $\epsilon=\epsilon'^{1/n}$ so that $\frac{|x|^n}{n!}<\epsilon\Rightarrow$ $\frac{|x|}{(n!)^{1/n}}<\epsilon^{1/n}=\epsilon'$

If $\left|f^{(n)}(x)\right|$ is bounded for all 'n', $lim_{n\to\infty}\frac{|x|^n}{n!}\left|f^{(n)}(x)\right|=0\Rightarrow\frac{|x|}{(n!)^{1/n}}\left|f^{(n)}(x)\right|^{1/n}=0$

$$lim_{n\to\infty}{u_n}^{1/n}|u_n|^{1/n}=0$$

Derivation2

We consider

$$f(x+2h) = f\big((x+h)+h\big) \ (9)$$

Expanding about $(x+h)$

$$f(x+2h) = f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots (10)$$

Expanding about $x = x$

$$f(x+2h) = f(x) + \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) \ (11)$$

From (10) and (11)

$$f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots = f(x) + \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \cdots \ldots$$

$$f(x+h) - f(x) + h[f'(x+h) - 2f'(x)] + \frac{1}{2!}h^2[f''(x+h) - 4f''(x)] + \frac{1}{3!}h^3[f'''(x+h) - 8f'''(x)] + \cdots .. = 0 \ (12)$$

$$\frac{f(x+h) - f(x)}{h}\frac{1}{h} + \frac{[f'(x+h) - 2f'(x)]}{h} + \frac{1}{2!}[f''(x+h) - 4f''(x)] + \frac{1}{3!}h[f'''(x+h) - 8f'''(x)] + \cdots .. = 0$$

$$\frac{f(x+h) - f(x)}{h}\frac{1}{h} + \frac{[f'(x+h) - f'(x)]}{h} - \frac{f'(x)}{h} + \frac{1}{2!}[f''(x+h) - 4f''(x)] + \frac{1}{3!}h[f'''(x+h) - 8f'''(x)] + h[\ldots..] = 0 \ (13)$$

Equation (5) is considered for $h \neq 0$. Even when we go for $h \to 0, h$ does not become equal to zero. It is in the neighborhood of zero without becoming equal to zero]

$$\left[\frac{f(x+h) - f(x)}{h} - f'(x)\right]\frac{1}{h} + \frac{[f'(x+h) - f'(x)]}{h} + \frac{1}{2!}[f''(x+h) - 4f''(x)] + \frac{1}{3!}h[f'''(x+h) - 8f'''(x)] + h[\ldots..] = 0$$

$$Lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} - f'(x)\right]\frac{1}{h} + Lim_{h\to 0}\frac{[f'(x+h) - f'(x)]}{h} + \frac{1}{2!}Lim_{h\to 0}[f''(x+h) - 4f''(x)] + \frac{1}{3!}Lim_{h\to 0}h[f'''(x+h) - 8f'''(x)] + h[\ldots..] = 0 \ (14)$$

We are considering a function for which

$$Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\ldots..]=0$$

Then

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+f''(x)-\frac{3}{2}f''(x)=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{h}=\frac{1}{2}f''(x)\ (15)$$

We apply L' Hospital's rule[1`] to obtain

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\frac{d}{dh}[f'(x)-f'(x)]}{1}=\frac{1}{2}f''(x)$$

$$\frac{1}{2}f''(x)=0(16)$$

As claimed we have brought out an aspect of inconsistency with Taylor Series.

Derivation 3

We may also consider the following

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\frac{h^n}{n!}f^{(n)}(x_0+h\theta)\ \ (17)$$

We write $x=x_0+h\Rightarrow h=x-x_0$

We consider the Taylor expansion of a function up to n terms[inclusive of the remainder term]

$$f(x)=f(x_0)+\frac{x-x_0}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots+\frac{(x-x_0)^{n-1}}{(n-1)!}f^{(n-1)}(x_0)+\frac{(x-x_0)^n}{n!}f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)\ (18)$$

Next We consider the Taylor expansion of the same function up to n+1 terms[inclusive of the remainder term]

$$f(x)=f(x_0)+\frac{x-x_0}{1!}f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+\cdots+\frac{(x-x_0)^{n-1}}{(n-1)!}f^{(n-1)}(x_0)$$
$$+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)\ (19)$$

Subtracting (18) from (19) we have,

$$\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$
$$=\frac{(x-x_0)^n}{n!}f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)(20)$$

$$\frac{(x-x_0)^n}{n!}\left[f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)\right]=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$

$$f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)=\frac{x-x_0}{n+1}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$

For $x=x_0$ the above equation is satisfied. It is also satisfied for other values of x inclusive of those arbitrarily close to x_0.

$$\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{x-x_0}=\frac{1}{n+1}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)\ (21)$$

Equation (21) is undefined for $x=x_0$

$$lim_{x\to x_0}\left[\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{x_0+\theta(n,x)(x-x_0)-x_0}\times\frac{x_0+\theta(n,x)(x-x_0)-x_0}{x-x_0}\right]$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$

$$lim_{x\to x_0}\left[\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{x_0+\theta(n,x)(x-x_0)-x_0}\times\theta(n,x)\right]$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$

$$lim_{x\to x_0}\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\theta(n,x)(x-x_0)}\times lim_{x\to x_0}\theta(n,x)$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)(22)$$

Now

$$lim_{x\to x_0}[x_0+\theta(n,x)(x-x_0)]=x_0\ (23)$$

Also

$$\theta(n,x)=\frac{x'-x_0}{x-x_0}$$

Indeed $x_0+\theta(n,x)(x-x_0)=x'$ is a point between x_0 and $x_0+h; h=x-x_0; x=x_0+h$

$$lim_{x\to x_0}\theta(n,x)=lim_{x\to x_0}\frac{x'-x_0}{x-x_0}$$

As $x\to x_0$ we have $x'\dashrightarrow x_0$

$$lim_{x\to x_0}\theta(n,x)=lim_{x\to x_0}\frac{x'-x_0}{x-x_0}=1$$

$$lim_{x\to x_0}\theta(n,x)=1\ (24)$$

$$lim_{x\to x_0}\theta(n+1,x)=1\ (25)$$

W e recall (22) and proceed as delineated below

$$lim_{x\to x_0}\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\theta(n,x)(x-x_0)}\times lim_{x\to x_0}\theta(n,x)$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n+1,x)(x-x_0)\big)$$

$$lim_{x\to x_0}\left[\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\big(x_0+\theta(n,x)(x-x_0)\big)-x_0}\times\frac{\big(x_0+\theta(n,x)(x-x_0)\big)-x_0}{x-x_0}\right]$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$

$$lim_{x\to x_0}\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\big(x_0+\theta(n,x)(x-x_0)\big)-x_0}\times lim_{x\to x_0}\frac{\big(x_0+\theta(n,x)(x-x_0)\big)-x_0}{x-x_0}$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$

$$f^{(n+1)}(x_0)lim_{x\to x_0}\left[\frac{x_0-x_0}{x-x_0}+\frac{\theta(n,x)(x-x_0)}{x-x_0}\right]=\frac{1}{n+1}f^{(n+1)}(x_0)$$

$$f^{(n+1)}(x_0)lim_{x\to x_0}[0+\theta(n,x)]=\frac{1}{n+1}f^{(n+1)}(x_0)$$

$$f^{(n+1)}(x_0)lim_{x\to x_0}\theta(n,x)=\frac{1}{n+1}f^{(n+1)}(x_0)$$

Recalling (50) we obtain

$$f^{(n+1)}(x_0).1=\frac{1}{n+1}f^{(n+1)}(x_0)\Rightarrow f^{(n+1)}(x_0)=0(26)$$

Derivation 4

We consider the equations:

$$Sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}-\frac{x^9}{9!}+\cdots\dots(26)$$

$$Cos\ x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots.(27)$$

Solutions to

$$Sin\ x=0\ \ (28)$$

or

$$Cos\ x=1\Rightarrow 1-Cos\ x=0\ \ (29)$$

are given by $x=2n\pi;k\in I$

By way of approximation we truncate (26) and (27) after a very large number of terms so that the following two equations hold on some interval reduce to

$$Sinx\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}-\frac{x^9}{9!}+\cdots\dots(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}(30)$$

$$Cos\ x\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots(-1)^{n+1}\frac{x^{2n}}{(2n)!}.(31)$$

$$\frac{1-Cos\ x}{x}\approx\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\frac{x^7}{8!}-\frac{x^9}{10!}+\cdots\dots(-1)^{n+1}\frac{x^{2n-1}}{(2n1)!}\ (32)$$

Solutions for

$$Sinx = \frac{1 - Cos\,x}{x}$$

are given by

$$x = 2k\pi; k \in I; k \neq 0$$

$$Sinx = \frac{1 - Cos\,x}{x}$$

implies in an approximate sense

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots \ldots (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} - \frac{x^9}{10!} + \cdots \ldots (-1)^{n+1}\frac{x^{2n-1}}{(2n!)!}$$

$$Ax(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_n) = Bx(x-\beta_1)(x-\beta_2)(x-\beta_3)\ldots\ldots(x-\beta_n) \quad (33)$$

But $\{\alpha_i; i = 1,2..n\} = \{\beta_i; i = 1,2..n\}; \alpha_i = \beta_i \approx 2i\pi$

$$\Rightarrow A \approx B$$

When we consider the two truncated series for $Sinx$ and for $\frac{1-Cos\,x}{x}$ we look out for n+k roots[out of an infinitude of them] that satisfy it sufficiently well. When we factorize each series we consider 'n' terms for both the series for $\alpha_i = \beta_i$. For 'x' we use some other root from the 'k' roots we have set aside in order to prove $A \approx B$

The values A and B are independent of the choice of x and hence their ratio is also independent of the choice of x when the sine or cosine series is considered for a finite number of terms.

The values of A and B are approximately equal irrespective of the value of x. We investigate this further by considering the polynomials to the right of (30)and (32)for arbitrary x when equation (33) may not hold.

Coefficient for the lowest degree for sin x=$A\prod_{i=1}^{n}\alpha_i = 1$

Coefficient for the lowest degree for$\frac{1-Cos\,x}{x}$=$B\prod_{i=1}^{n}\beta_i = \frac{1}{2}$

But $A \approx B \Rightarrow \prod_{i=1}^{n}\alpha_i = 2\prod_{i=1}^{n}\beta_i$ irrespective of the values of x

Coefficient for the highest degree term for sin x=$A = (-1)^{n+1}\frac{1}{(2n-1)!}$

Coefficient for the highest degree for $\frac{1-Cos\ x}{x}$=$B = (-1)^{n+1}\frac{1}{(2n)!} = (-1)^{n+1}\frac{1}{2n(2n-1)!}$

$$A \approx 2nB$$

which contradicts our earlier conclusion that $A \approx B$

[One has to keep in mind that since we have considered a truncated series; 'n' has a fixed value and each of the sine and the cosine series has been approximated to a polynomial]

Derivation 5

We may try out testing he identity $Sin^2x + Cos^2x = 1$ using the Taylor expansion

$$\left(\boldsymbol{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.}\right)^2 + \left(\boldsymbol{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.}\right)^2 = \boldsymbol{1}$$

Cancelling out unity from either side we obtain an in finite power series in the even powers of x. The coefficients are not identically zero. We do not have an identity as such

Derivation 6

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \epsilon_n(x)$$

We define $f(x,n)$ as follows:

$$f(x,n) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

$$f(x,n) - f(x,n-1) = \frac{x^n}{n!}$$

We make $f(x,n)$ a smooth[obviously continuous) with respect to 'n' by interpolation with a suitable curve where n is positive everywhere .

$$\frac{\partial f(x,n)}{\partial n} - \frac{\partial f(x,n-1)}{\partial n} = \frac{x^n lnx}{n!} + x^n\frac{d}{dn}\left(\frac{1}{n!}\right) \quad (34)$$

$$\frac{\partial f(x,n)}{\partial n} - \frac{\partial f(x,n-1)}{\partial n} = \frac{x^n lnx}{n!} - x^n\frac{1}{n!}\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right] \quad (35)$$

In the above $n! = n(n-1)\ldots.up\ to\ |n|$ *terms . The equation considers right handed derivatives exclusively.*

For large 'n' the left side of (35) is zero: $f(x,n) \approx f(x,n-1) \approx e^x$

Therefore,

$$lim_{n\to\infty}\left[\frac{x^n lnx}{n!} - x^n \frac{1}{n!}\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right]\right] = 0$$

That is to say that in the limit ,n tending to infinity,irrespective of the value of x,x very large or very small

$$\frac{lnx}{n!} \approx \frac{1}{n!}\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right]$$

$$\ln x \approx \left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right]$$

For n tending to infinity the right side blows up while the left side is a variable depending on the value of 'x'

We have a strange result

$$x \approx e^{\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right]}; n: very\ large\ but\ constant\ in\ value\ (36)$$

NB: For testing ,in the above, we have taken a very large but fixed value of x so as to ensure that we have an approximation with (35)..[$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \ldots \ldots = \infty$]

Derivation 7

$$Sin2^c = Sin(2\pi + 2) = Sin(4\pi + 2) = Sin(6\pi + 2) = -Sin(2\pi - 2) = -Sin(4\pi - 2) = -Sin(6\pi - 2)\ (37)$$

Testing the above fails. I tried it with a Blue Java program[Blue Java, University of Kent, Version 3.1.5.]Disagreement starts from the thirteenth place of decimal while using the double variable[Blue compiler consider 16 significant places for the double variable]

[200000 terms of sine series considered in the program initially for each case and later 2000000 terms of the sine series were considered;there was no change in the values of sine;all angles in radians as required of the sine series]

For $|Sin(2\pi + 2)| = Sin(2\pi - 2)$ differences start from the thirteenth place

For $|Sin(2\pi + 4)| = Sin(2\pi - 4)$ differences start from the thirteenth place

For $|Sin(2\pi + 6)| = Sin(2\pi - 6)$ differences start from the tenth place

Program[Blue Java, version 3.1.5,University of Kent]

```
import java.lang.*;

import java.util.Scanner;

class Sin_Verif2

{

   public static void main(String [] args)

   {

      double pi;

      pi=3.14159265358979323846;


      System.out.println("pi="+pi);

      double x;

           x=2*pi-2;

           double arr[]={x,x+4,x-2,x+6,x-4,x+8};


      double i;

      double j;

      double sum;

      double t;


      for(int k=0;k<=5;k++)

      {


       t=arr[k];


      sum=0;
```

```
        for(i=1;i<=2000000;i++)

        {

           sum=sum+t;

                      j=2*i+1;

           t=((-1)*t*arr[k]*arr[k])/(j*(j-1));

           //in the above line  we have effectively used the recursion t(n+1)=(-1)t(n)x^2/n(n+1)


        }

        System.out.println("\n sum: "+sum);


       }


}}
```

Another program to verify that it considers 16 signifcant digits[sometimes seventeen??]

```
import java.lang.*;

import java.util.Scanner;

class November1

{

   public static void main(String [] args)

   {

        double x;

        x=0.000000000000000123456789786523456544;

        System.out.println(x);
```

```
  }
}
```

Conclusion

As claimed at the outset the article brings out some anomalous features about the Taylor expansion

References

1. Wikipedia, Taylor Series, Link:

https://en.wikipedia.org/wiki/Taylor_series