Inferences from the Taylor Series

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Abstract

The Taylor series is an important tool in mathematical analysis and it has wide ranging applications .Nevertheless there are inconsistent features related to it. The article intends to demonstrate such features.

Introduction

The Taylor series is well known for its application in mathematics and in physics. The article brings out some anomalous features about the Taylor expansion

Various Inconsistencies

Case 1.

We consider

$$f(x+2h)=f\big((x+h)+h\big) \quad (1)$$

Expanding about $(x+h)$

$$f(x+2h)=f(x+h)+\frac{h}{1!}f'(x+h)+\frac{h^2}{2!}f''(x+h)+\frac{h^3}{3!}f'''(x+h)+\cdots\ldots.(2)$$

Expanding about $x=x$

$$f(x+2h)=f(x)+\frac{2h}{1!}f'(x+h)+\frac{4h^2}{2!}f''(x+h)+\frac{8h^3}{3!}f'''(x+h)+\cdots\ldots.(3)$$

From (2) and (3)

$$f(x+h)+\frac{h}{1!}f'(x+h)+\frac{h^2}{2!}f''(x+h)+\frac{h^3}{3!}f'''(x+h)+\cdots\ldots.$$
$$=f(x)+\frac{2h}{1!}f'(x)+\frac{4h^2}{2!}f''(x)+\frac{8h^3}{3!}f'''(x)+\cdots\ldots$$

$$f(x+h)-f(x)+h[f'(x+h)-2f'(x)]+\frac{1}{2!}h^2[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h^3[f'''(x+h)-8f'''(x)]+\cdots..=0\ (4)$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-2f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)]+\frac{1}{3!}h[f'''(x+h)-8f'''(x)] \\ +\cdots..=0$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}-\frac{f'(x)}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0\ (5)$$

Equation (5) is considered for $h \neq 0$. Even when we go for $h \to 0, h$ does not become equal to zero. It is in the neighborhood of zero without becoming equal to zero]

$$\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+Lim_{h\to 0}\frac{[f'(x+h)-f'(x)]}{h} \\ +\frac{1}{2!}Lim_{h\to 0}[f''(x+h)-4f''(x)]+\frac{1}{3!}Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\dots..] \\ =0\ \ (6)$$

We are considering a function for which

$$Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0$$

Then

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+f''(x)-\frac{3}{2}f''(x)=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{h}=\frac{1}{2}f''(x)\ (7)$$

We apply L' Hospital's rule[1`] to obtain

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\left[-\frac{1}{h^2}\big(f(x+h)-f(x)\big)+\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)$$

$$\left[-Lim_{h\to 0}\frac{1}{h^2}\big(f(x+h)-f(x)\big)+Lim_{h\to 0}\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)(8)$$

$$-\infty+f''(x)=\frac{1}{2}f''(x)$$

$$-\frac{1}{2}f''(x)=-\infty$$

As claimed we have brought out an aspect of inconsistency with Taylor Series.

Case 2.Let us have another situation for our analysis. We write the Taylor series

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9).$$

The increment h may be sufficiently large subject to the fact that the series has to converge.

$$\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)+hf''(x_0)+\frac{h^2}{2!}f'''(x_0)+\cdots(10).$$

$$lim_{h\to 0}\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)\ \ (11)$$

$$lim_{h\to 0}F_h(x_0+h)=f'(x_0)$$

The limit $f'(x_0)$ is independent of h.This is an example of uniform convergence . We may analyze as follows:

$$\frac{\partial f(x)}{\partial h}=\frac{\partial f(x_0+h)}{\partial h}$$

is evaluated for different values of h: $\left[\frac{\partial f(x)}{\partial h}\right]_{h_1},\left[\frac{\partial f(x)}{\partial h}\right]_{h_2},\left[\frac{\partial f(x)}{\partial h}\right]_{h_3}$ ….

The limit $f'(x_0)$ is independent of x

Therefore we can interchange the derivative and the limit[2].

$$\frac{\partial}{\partial h} lim_{h\to 0} \frac{\partial f(x_0+h)}{\partial h} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0 \ (12)$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial^2 f(x_0+h)}{\partial h^2} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)} \frac{\partial (x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial (x_0+h)}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right] \frac{\partial (x_0+h)}{\partial h} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial x}\left[\frac{\partial f(x)}{\partial x}\right] = 0$$

where$x = x_0 + h$

We now have,

$$lim_{h\to 0} \frac{\partial^2 f(x)}{\partial x^2} = 0 \ (13)$$

$$\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_0} = 0$$

But $x = x_0$ could be any arbitrary point.

By differentiating (10) we obtain the expected result

$$\frac{\partial^2 f(x_0+h)}{\partial h^2} = f''(x_0) (14)$$

which contradicts the earlier result given by (13)unless $f''(x_0) = 0$

Direct Calculations

We write the Taylor series

$$f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots\ldots(15).$$

$$\frac{\partial}{\partial x}f(x+h)=f'(x)+\frac{h}{1!}f''(x)+\frac{h^2}{2!}f'''(x)+\frac{h^3}{3!}f''''(x)+\cdots\ldots(16).$$

$$\frac{\partial}{\partial h}f(x+h)=f'^{(x)}+\frac{h}{1!}f''(x)+\frac{h^2}{2!}f'''(x)+\frac{h^3}{3!}f''''(x)+\cdots(17)$$

From (10) and (11) we have,

$$\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)(18)$$

Differentiating (10) with respect to $x+h$[holding x as constant]

$$\frac{\partial}{\partial x}f(x+h)=\frac{d}{d(x+h)}f(x+h)\ (19)$$

$$\left[\frac{\partial}{\partial x}f(y)\right]_{y=x}=\left[\frac{\partial}{\partial h}f(y)\right]_{y=y}\quad(20)$$

$\frac{\partial}{\partial x}f(y)=\frac{df(x)}{dx}$is a constant on $(x,x+h)$. This notion may be considered to show that$\frac{df(x)}{dx}$ is constant everywhere.[we take $(x,x+h)$, $(x+h,x+2h)$, $(x+2h,x+3h)$ ….and consider the proof given over and over again]

$$\frac{\partial}{\partial x}f(x)=const\Rightarrow\frac{\partial^2}{\partial x^2}f(x)=0$$

which we got earlier

Now [treating f as a function of x and h we may write

$$df(x+h)=\frac{\partial}{\partial x}f(x+h)dx+\frac{\partial}{\partial h}f(x+h)dh\ (21)$$

Again

$$df(x+h)=\frac{\partial}{\partial(x+h)}f(x+h)d(x+h)\quad(22)$$

$$\Rightarrow df(x+h)=\frac{\partial}{\partial(x+h)}f(x+h)dx+\frac{\partial}{\partial(x+h)}f(x+h)dh\quad(23)$$

From (21) and (22) we have,

$$\left[\frac{\partial}{\partial(x+h)}f(x+h)-\frac{\partial}{\partial x}f(x+h)\right]dx+\left[\frac{\partial}{\partial(x+h)}f(x+h)-\frac{\partial}{\partial h}f(x+h)\right]dh=0$$

$$\frac{\partial}{\partial(x+h)}f(x+h)=\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)\ \ (24)$$

We clearly see that the function $\frac{df}{dx}$is a constant function that is $\frac{d^2f}{dx^2}=0$

Further Considerations

We recall (9)

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9)$$

We differentiate the above with respect to $x=x_0+h';h'<h$

$$\left[\frac{df(x_0+h)}{dh}\right]_{h=h'}=f'(x_0)+h'f''(x_0)+\frac{h'^2}{2!}f'''(x_0)+\cdots..=f'(x_0+h')\ (25)$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\frac{dh}{d(x_0+h)}=\frac{df(x_0+h)}{dh}$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\ \ (26)$$

We obtain an indication of constancy of $\frac{df(x_0+h)}{dh}$ from (26) and keeping in mind equation (18) we have

$$\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)=\frac{df(x+h)}{d(x+h)}$$

Next we consider a truncated Taylor series which has been approximated with ‘n’ terms. Now we have an equation and not an identity and there are discrete solutions for h. Since we have taken an approximation to the Taylor series it is least likely the corresponding roots will cause a divergence of the infinite series in the Taylor expansion. It would be better to take a truncation which is not an approximation but the infinite Taylor series is convergent for it. These solutions for ‘h’ will not satisfy the entire Taylor series with an infinite number of terms. Suppose one solution of ‘h’ from approximated equation[equation with finite number of terms] satisfied the infinite Taylor series, we will have (9) as well as a truncated (9)[approximated up to ‘n’ terms. The situation has been delineated below

We now cconsider the Maclaurin expansion for e^x

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots..$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\epsilon_n(x)\ \ (26)$$

$\epsilon_n(x)$: Remainder after the nth term count starting from zero: n=0,1,2…….

$$\epsilon_0(x) = e^x - 1$$

Differentiating (26) with respect to 'x' for a fixed 'n'we obtain

$$\frac{de^x}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (27)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (28)$$

$$\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x) = \frac{x^n}{n!} (29)$$

$$\frac{d^2e^x}{dx^2} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-2}}{(n-1)!} + \frac{d^2\epsilon_n(x)}{dx^2}$$

We consider a positive interval $(x_1, x_2)$and make $n \to \infty$.For such an interval

$$lim_{n\to\infty} \frac{x^n}{n!} = 0$$

$$lim_{n\to\infty} \left[\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right] = lim_{n\to\infty} \frac{x^n}{n!}$$

$$lim_{n\to\infty} \left[\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right] = 0$$

For sufficiently large ,n$\left|\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right|$ can be made arbitrarily close to zero

For the concerned interval we have in the limit n tending to infinity [for the interval (x_1, x_2)]the following[rigorous]equation

$$\frac{d\epsilon_\infty(x)}{dx} - \epsilon_\infty(x) = 0 \ (30.1)$$

$$ln\epsilon_\infty(x) = \mathrm{x} + C' \ (30.2)$$

If C'=0

$$\epsilon_\infty(x) = e^x \ (31.1)$$

If $C' \neq 0, C' = lnC$

$$\epsilon_\infty(x) = Ce^x (31.2)$$

Again

$$C = 0 \Rightarrow C' = -\infty \ (32)$$

That means we used $-\infty$ as the constant of integration in equation (30.2).Suppose we take $|C'| \gg 0; C' < 0$ so that C is a very small fraction:

$$Ce^{x_1} < \epsilon_\infty(x) < Ce^{x_2}$$

But the point is that once we decide on the value of C[C cannot be minus infinityby itself if a value is considered] we cannot vary it. Though $\epsilon_\infty(x)$ will be very small we cannot take it arbitrarily close to zero

Next we consider a much larger the interval (x_1, x_2') which contains the interval $(x_1, x_2); x_2' \gg x_2[x_2'$ remaining finite. We have the same equation as given by (30.1) and the same solution $\epsilon_\infty(x) = Ce^x$.This time we cannot change the value of the constant. If we changed it to C_{new}then we have untenable results li8ke $\epsilon_\infty(x_1) = C_{new}e^{x_1}$ and$\epsilon_\infty(x_2) = C_{new}e^{x_2}$. But with the old constant $\epsilon_\infty(x_2') = Ce^{x_2'} \gg 0$ since $x_2' \gg x_2$

It is not possible to cover the entire x axis or the semi x axis:$(0, \infty)$ by a single constant having a numerical value.

The discrepancy we have found should not surprise us in view of the earlier discrepancies , for example those notified through case1 and case2.

Further Investigation

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \epsilon_n(x)$$

We define $f(x, n)$ as follows:

$$f(x, n) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \ (33)$$

$$f(x, n) - f(x, n-1) = \frac{x^n}{n!}$$

We make r $f(x, n)$ a smooth[obviously continuous) by interpolation with a suitable curve where n isposotive everywhere .

$$\frac{\partial f(x, n)}{\partial n} - \frac{\partial f(x, n-1)}{\partial n} = \frac{x^n lnx}{n!} + x^n \frac{d}{dn}\left(\frac{1}{n!}\right) \ (34)$$

$$\frac{\partial f(x, n)}{\partial n} - \frac{\partial f(x, n-1)}{\partial n} = \frac{x^n lnx}{n!} - x^n \frac{1}{n!}\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right] \ (35)$$

In the above $n! = n(n-1) \ldots .up\ to\ |n|$ *terms . The equation considers right handed derivatives exclusively.*

For large ‘n’ the left side of (35) is zero: $f(x, n) \approx f(x, n-1) \approx e^x$

Therefore,

$$lim_{n\to\infty}\left[\frac{x^n lnx}{n!}-x^n\frac{1}{n!}\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]\right]=0$$

That is to say that in the limit irrespective of the value of x---x very large or very small

$$\frac{lnx}{n!}\approx\frac{1}{n!}\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]$$

$$\ln x\approx\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]$$

We have a strange result

$$x\approx e^{\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]}; n: very\ large\ but\ constant\ in\ value\ (36)$$

NB: For testing ,in the above, we have taken a very large but fixed value of x so as to ensure that we have an approximation with (35)..

Next we move on to the following case.

We expand $f(x)$about three distinct points x_1, x_2 and x_3 with the same increment h assuming convergence for all three cases

$$f(x_1+h)=f(x_1)+\frac{h}{1!}f'(x_1)+\frac{h^2}{2!}f''(x_1)+\frac{h^3}{3!}f'''(x_1)+\cdots\dots.(38.1)$$

$$f(x_2+h)=f(x_2)+\frac{h}{1!}f'(x_2)+\frac{h^2}{2!}f''(x_2)+\frac{h^3}{3!}f'''(x_2)+\cdots\dots.(38.2)$$

$$f(x_3+h)=f(x_3)+\frac{h}{1!}f'(x_3)+\frac{h^2}{2!}f''(x_3)+\frac{h^3}{3!}f'''(x_3)+\cdots\dots(38.3).$$

The increment 'h' in the above three equations may be azrbitrary to the extent it does not upset convergence.

Adding the last three equations we have

$$f(x_1+h)+f(x_2+h)+f(x_2+h)$$
$$=f(x_1)+f(x_2)+f(x_3)+\frac{h}{1!}[f(x_1)+f(x_2)+f(x_3)]+\frac{h^2}{2!}[f(x_1)+f(x_2)+f(x_3)]$$
$$+\frac{h^2}{3!}[f(x_1)+f(x_2)+f(x_3)]\dots(38)$$

Let x satisfy the following equation

$$f(x)=f(x_1)+f(x_2)+f(x_3)\ (39)$$

Equation (38) does not involve 'h'

$$\Rightarrow f(x_1+h)+f(x_2+h)+f(x_2+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots\ldots$$

$$f(x_1+h)+f(x_2+h)+f(x_2+h)=f(x+h)(40)$$

But h could be arbitrary. In fact we may very it continuously over a small interval without upsetting convergence. On top of that there are many functions for which the corresponding Taylor expansion is convergent for any arbitrary increment for example $Sin\ x, Cos\ x, \exp(x)$

For the same x_1, x_2, x_3 and corresponding x as given by (39) we do have an infinite number of equations of the type (40) for the various 'h' that comply with the convergence issue. . But (40) does not represent an identity. We do have a situation of a gross violation.

If

$$Sin(x_1)+Sin(x_2)+Sin(x_3)=Sin(x)\ (41)$$

By our choice the absolute value of the left side of the above should be less than or equal to unity

Then do we have the following for an arbitrary 'h', [arbitrary to the extent the absolute value of the left side of the following should be less than or equal to unity]?

$$Sin(x_1+h)+Sin(x_2+h)+Sin(x_3+h)=Sin(x+h)(42)$$

Any arbitrary value of 'h' has to satisfy (42) subject to the convergence issue and that the absolute value of the left side[of (42)] should be less than or equal to unity.

The Convergence Condition:

If we consider the Taylor expansion of $Sin(x+h)$ for

$$\frac{t_{n+1}}{t_n}=\frac{h^2}{(n+2)(n+1)}\frac{f^{(n+2)}(x)}{f^n(x)}\quad(43)$$

We apply D'Alembert Test considering the absolute value of each term in the expansion. If this series converges then the original series also converges.

$$lim_{n\to\infty}\frac{|t_{n+1}|}{|t_n|}=lm_{n\to\infty}\left|\frac{f^{(n+2)}(x)}{f^n(x)}\frac{h^2}{(n+2)(n+1)}\right|=\left|\frac{f^{(n+2)}(x)}{f^n(x)}h^2\right|\ lim_{n\to\infty}\frac{1}{(n+2)(n+1)}=0$$
$$<1$$

The modified series consisting of positive terms converges. Hence the Taylor expansion also converges for any arbitrary 'h'[considering finite $\frac{f^{(n+1)}(x)}{f^n(x)}$ for all n and n+1:choosing x in that way and this can be achieved for the sin function]. Those terms for which $f^n(x) = 0$ simply drop out of the series. We may apply D'Alembert ratio test to the remaining terms.

We may alternatively think of Cauchy condition: if $lim_{n\to\infty} {u_n}^{1/n}$<1, then the series converges.

Against the Taylor series we form a series comprising the absolute values of the corresponding terms of the Taylor series.

$$\frac{|x|^n}{n!}\left|f^{(n)}(x)\right|$$

$$lim_{n\to\infty}\frac{|x|^n}{n!} = 0 \Rightarrow lim_{n\to\infty}\left(\frac{|x|^n}{n!}\right)^{1/n} = 0 \Rightarrow lim_{n\to\infty}\frac{|x|}{(n!)^{1/n}} = 0$$

Indeed $lim_{n\to\infty}\frac{|x|^n}{n!} = 0$ implies that for every preassigned $\epsilon > 0$, no matter how small, we have $N > 0$ such that for $n > N$ we have, $\frac{|x|^n}{n!} < \epsilon \Rightarrow \frac{|x|}{(n!)^{1/n}} < \epsilon^{1/n} = \epsilon'$

For any arbitrary $\epsilon' > 0$ no matter how small we can arrange for an $\epsilon = \epsilon'^{1/n}$ so that $\frac{|x|^n}{n!} < \epsilon \Rightarrow \frac{|x|}{(n!)^{1/n}} < \epsilon^{1/n} = \epsilon'$

If $\left|f^{(n)}(x)\right|$ is bounded for all 'n' , $lim_{n\to\infty}\frac{|x|^n}{n!}\left|f^{(n)}(x)\right| = 0 \Rightarrow \frac{|x|}{(n!)^{1/n}}\left|f^{(n)}(x)\right|^{1/n} = 0$

$$lim_{n\to\infty} {u_n}^{1/n}|u_n|^{1/n} = 0$$

Relevant Demonstrations

We may also consider the following

$$f(x_0 + h) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots \frac{h^n}{n!}f^{(n)}(x_0 + h\theta) \ (44)$$

We write $x = x_0 + h \Rightarrow h = x - x_0$

$$f(x) = f(x_0) + \frac{x - x_0}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \cdots + \frac{(x - x_0)^{n-1}}{(n-1)!}f^{(n-1)}(x_0) + \frac{(x - x_0)^n}{n!}f^{(n)}\big(x_0 + \theta(n,x)(x - x_0)\big) \ (45)$$

$$f(x) = f(x_0) + \frac{x - x_0}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \cdots + \frac{(x - x_0)^{n-1}}{(n-1)!}f^{(n-1)}(x_0) + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0 + \theta(n,x)(x - x_0)\big) \ (46)$$

Subtracting (45) from (46) we have,

$$\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$
$$=\frac{(x-x_0)^n}{n!}f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)(47)$$

$$\frac{(x-x_0)^n}{n!}\big[f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)\big]=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$

$$\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{x-x_0}=\frac{1}{n+1}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)\ (48)$$

$$lim_{x\to x_0}\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\theta(n,x)(x-x_0)}\theta(n,x)=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$

$$lim_{x\to x_0}\frac{f^{(n)}\big(x_0+\theta(n,x)(x-x_0)\big)-f^{(n)}(x_0)}{\theta(n,x)(x-x_0)}\times lim_{x\to x_0}\theta(n,x)$$
$$=\frac{1}{n+1}lim_{x\to x_0}f^{(n+1)}\big(x_0+\theta(n,x)(x-x_0)\big)$$

Now

$$lim_{x\to x_0}\theta(n,x)=1\ (49)$$

Let

$$x_0+\theta(n,x)(x-x_0)=x'\Rightarrow x'-x=\theta(n,x)(x-x_0)$$

$$lim_{x'\to x_0}\frac{f^{(n)}(x')-f^{(n)}(x_0)}{x'-x=}=\frac{1}{n+1}f^{(n+1)}(x_0)$$

Therefore

$$f^{(n+1)}(x_0)=\frac{1}{n+1}f^{(n+1)}(x_0)\ (50)$$

The above relation is not possible for $f^{(n+1)}(x_0)\neq 0$

Now we go in for the next demonstration

We consider the equations:

$$Sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}-\frac{x^9}{9!}+\cdots\ldots(51)$$

$$Cos\ x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots . (52)$$

Solutions to

$$Sin\ x = 0 \ \ (53)$$

and

$$Cos\ x = 1 \Rightarrow 1 - Cos\ x = 0 \ \ (54)$$

are given by $x = 2n\pi; k \in I$

Type equation here.By way of approximation we truncate (51) and (52) after a very large number of terms so that the following two equations hold on some interval.

$$Sinx \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots \ldots (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} (55)$$

$$Cos\ x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots (-1)^{n+1} \frac{x^{2n}}{(2n)!} . (56)$$

$$\frac{1 - Cos\ x}{x} \approx \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} - \frac{x^9}{10!} + \cdots \ldots (-1)^{n+1} \frac{x^{2n-1}}{(2n1)!} \ \ (56')$$

Approximate solutions for (2) and (3) are given by

$$x \approx 2k\pi; k \in I$$

We rewrite (55) and (56) as

$$Sinx \approx Ax(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots . . (x - \alpha_n) \ (57)$$

$$\frac{1 - Cos\ x}{x} \approx Bx(x - \beta_1)(x - \beta_2)(x - \beta_3) \ldots . . (x - \beta_n) \ (58)$$

But $\{\alpha_i; i = 1,2..n\} = \{\beta_i; i = 1,2..n\}; \alpha_n = \beta_n \approx 2n\pi$

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots . . (x - \alpha_n) \approx x(x - \beta_1)(x - \beta_2)(x - \beta_3) \ldots . . (x - \beta_n)$$

$$\Rightarrow \frac{1}{A} Sinx \approx \frac{1}{B} \frac{1 - Cos\ x}{x}$$

$$Sinx \approx \frac{A}{B} \frac{1 - Cos\ x}{x}$$

$$Sinx \approx K \frac{1 - Cos\ x}{x} \ \ (59)$$

We do not have a valid expression with (9)

Aswe increase the number of terms [included] in the truncation we have

$$Sinx = K' \frac{1 - Cos\ x}{x} \ (60)$$

Equation (10) is not a valid one.

Point to Observe

1.

$$Lim_{x \to 0} \frac{1 - Cos\ x}{x} = Lim_{x \to 0} \frac{2Sin^2 x}{x} = 2Lim_{x \to 0} \frac{Sin^2 x}{x^2} \times x = 2 \times 1^2 \times 0 = 0$$

Right side of (6') tends to zero with x tending to zero. Therefore (6') is satisfied for $x \to 0$asides $\beta_k = 2k\pi$

2. Instead of (59) we may consider

$$Sinx = K_n(x) \frac{1 - Cos\ x}{x}$$

The roots for the two approximations will not be integers.The corresp0onding roots may not b e exactly equal Nevertheless the above equation will hold with . $K_n(x)$ asa slowly vrying constant

With n tending to infinity [as we take more and more terms into consideration] we have $K_n(x)$ tending to a constant K' that is $lim_{n \to \infty} K_n(x) = K'$[K' being a constant]

Differences between the two sets of roots,$\{\alpha_i\}$ and $\{\beta_k\}$,get flushed out as n tends to infinity.

3. If A+iB is a complex root for (51) then Sin(A+iB)=0 implies Cos (A+iB)=1. These relationswill hold approximately for the approximate equations thus confirming $\{\alpha_i\} \approx \{\beta_k\}$

Examples for Numerical Testing

$$Sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ....$$

$$Sin\ 2\pi n_1 = 2\pi n_1 - \frac{(2\pi n_1)^3}{3!} + \frac{(2\pi n_1)^5}{5!} - \frac{(2\pi n_1)^7}{7!} + \cdots ....$$

$$Sin\ 2\pi n_2 = 2\pi n_1 - \frac{(2\pi n_1)^3}{3!} + \frac{(2\pi n_1)^5}{5!} - \frac{(2\pi n_1)^7}{7!} + \cdots ....$$

If n_1 and n_2 are integers

$$2\pi n_1 - \frac{(2\pi n_1)^3}{3!} + \frac{(2\pi n_1)^5}{5!} - \frac{(2\pi n_1)^7}{7!} + \cdots \ldots . = 0$$

$$2\pi n_2 - \frac{(2\pi n_2)^3}{3!} + \frac{(2\pi n_3)^5}{5!} - \frac{(2\pi n_4)^7}{7!} + \cdots \ldots . = 0$$

By subtraction we obtain

$$2\pi(n_1 - n_2) - \frac{(2\pi)^3}{3!}({n_1}^3 - {n_2}^3) + \frac{(2\pi)^5}{5!}({n_1}^5 - {n_2}^5) - \frac{(2\pi)^7}{7!}({n_1}^7 - {n_2}^7) + \cdots \ldots . = 0 \ (61)$$

$$Sin 2\pi(n_1 - n_2) = 0$$

Therefore

$$Sin2\pi(n_1 - n_2) = 2\pi(n_1 - n_2) - \frac{(2\pi)^3}{3!}({n_1}^3 - {n_2}^3) + \frac{(2\pi)^5}{5!}({n_1}^5 - {n_2}^5) - \frac{(2\pi)^7}{7!}({n_1}^7 - {n_2}^7) + \cdots \ldots . == 0 (62)$$

Now n_1 and n_2 being integers their difference $n_1 - n_2$ is also an integer. Therefore

$$Sin\ 2\pi(n_1 - n_2) = 0$$

$$Sin2\pi(n_1 - n_2) = 2\pi(n_1 - n_2) - \frac{(2\pi)^3(n_1 - n_2)^3}{3!} + \frac{(2\pi)^5(n_1 - n_2)^5}{5!} - \frac{(2\pi)^7}{7!} \ldots . = 0 \ \ (63)$$

One may think of testing (53) numerically.

We should not expect

$${n_1}^k - {n_2}^k = (n_1 - n_2)^k$$

or

$${n_1}^k - {n_2}^k = (n_1 - n_2)^k + 2\pi p; p \in I$$

$${n_1}^k - {n_2}^k - (n_1 - n_2)^k = 2\pi p; p\ const$$

for all $n_1, n_2 \in I$

Precisely e consider m integers $a_1, a_2, a_3 \ldots\ldots a_m$

$$0 = Sin2\pi a_i = 2\pi a_i - \frac{(2\pi)^3 {a_i}^3}{3!} + \frac{(2\pi)^5 {a_i}^5}{5!} - \frac{(2\pi)^7 {a_i}^7}{7!} \ldots \ldots$$

Summing on i we have

$$2\pi\sum_i a_i - \frac{(2\pi)^3}{3!}\sum_i {a_i}^3 + \frac{(2\pi)^5}{5!}\sum_i {a_i}^5 - \frac{(2\pi)^7}{7!}\sum_i {a_i}^7 \ldots\ldots. = 0 \ (64)$$

But $\sum_i a_i$ is an integer

Therefore

$$Sin2\pi\sum_i a_i = 0$$

$$2\pi\sum_i a_i - \frac{(2\pi)^3}{3!}\left(\sum_i a_i\right)^2 + \frac{(2\pi)^5}{5!}\left(\sum_i a_i\right)^3 - \frac{(2\pi)^7}{7!}\left(\sum_i a_i\right)^7 \ldots\ldots. = 0 \ (65)$$

$$2\pi\sum_i a_i - \frac{(2\pi)^3}{3!}\sum_i {a_i}^3 + \frac{(2\pi)^5}{5!}\sum_i {a_i}^5 - \frac{(2\pi)^7}{7!}\sum_i {a_i}^7 \ldots\ldots.$$
$$\equiv 2\pi\sum_i a_i - \frac{(2\pi)^3}{3!}\left(\sum_i a_i\right)^2 + \frac{(2\pi)^5}{5!}\left(\sum_i a_i\right)^3 - \frac{(2\pi)^7}{7!}\left(\sum_i a_i\right)^7 = 0 \ (66)$$

Equation (66) may be put to test on the computer.

Now ,

$$Sin\frac{\pi}{4} = Sin\left(\frac{\pi}{4} + 6\pi\right) = \frac{1}{\sqrt{2}} \ (67)$$

$$Sin0.7855 = Sin19.6375$$

$$Sin\left(\frac{\pi}{4}\right) = \frac{\left(\frac{\pi}{4}\right)}{1!} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \frac{\left(\frac{\pi}{4}\right)^5}{5!} + \cdots..(68)$$

$$Sin\left(\frac{\pi}{4} + 6\pi\right) = \frac{\left(\frac{\pi}{4} + 6\pi\right)}{1!} - \frac{\left(\frac{\pi}{4} + 6\pi\right)^3}{3!} + \frac{\left(\frac{\pi}{4} + 6\pi\right)^5}{5!} - \cdots.(69)$$

$$\frac{\left(\frac{\pi}{4}\right)}{1!} - \frac{\left(\frac{\pi}{4}\right)^3}{3!} + \frac{\left(\frac{\pi}{4}\right)^5}{5!} + \cdots.. = \frac{\left(\frac{\pi}{4} + 6\pi\right)}{1!} - \frac{\left(\frac{\pi}{4} + 6\pi\right)^3}{3!} + \frac{\left(\frac{\pi}{4} + 6\pi\right)^5}{5!} - \cdots.(70)$$

Equation (66) may be put to test on the computer, considering a large number of terms.

On the Convergence of the Exponential Series

Assume that the formula for $Sin(x)$ holds simultaneously over the entire real axis $x \in (-\infty, \infty)$. We4 partition he real axis into accountably infinite number of closed intervals indexed by the natural numbers. The remainder term[Cauchy form],$R_n = \frac{x^n}{n!} Sin(x + \theta h)$ should tend to zero for all intervals we have in the partition. For any preassigned $\epsilon > 0$ no matter how small we have for

Interval 1: $N_1 > 0$ such that for all $n > N_1$ we have $|R_n| < \epsilon$

Interval 2: $N_2 > 0$ such that for all $n > N_2$ we have $|R_n| < \epsilon$

Interval 3: $N_3 > 0$ such that for all $n > N_3$ we have $|R_n| < \epsilon$

Interval k: $N_k > 0$ such that for all $n > N_k$ we have $|R_n| < \epsilon$

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Fort the largest N_k, $|R_n| < \epsilon$ for all intervals. There is no such largest N_k[we cannot denote it numerically]

Therefore our usual formula for e^x will not hold for the entire real axis at one stroke..

Considering any finite interval on the x axis or a finite union of finite intervals the exponential series is uniformly convergent. But it is not uniformly convergent when the entire x axis is taken into consideration.

Analogous conclusions may follow from all non terminating instances of the Taylor series.

Conclusions

As claimed we have arrived at some inconsistent aspects of the Taylor expansion

References

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