Author

Giuliano Bettini

Title

The EVOs Exotic Vacuum Objects

Abstract

This document is specifically dedicated to finding an exact solution of a charged cylindrical wave in a vacuum. It is a charged electromagnetic field that exactly obeys the conditions of Cauchy Riemann. It is a "waveguide" field, but without the waveguide.
Mathematically the solution carries mass, charge and angular momentum and also magnetic charge. It must be better understood, whether it has a physical meaning or not and whether it has to do with the mysterious EVOs of Ken Shoulders. It is certainly exotic and it is certainly in a vacuum.

1 - FOREWORD

I write only the final results of the argument , skipping all the speculations that led me right here. I only say , as far as my notations are concerned , that a brief but exhaustive explanation can be found in a few, few pages, of [1]. Other interesting readings are in Bibliography, [2]to [6]. Of other avant-garde and more or less contested topics, EVOs of Ken Shoulders , Condensed Plasmoids , Ball Lightning etc I have found no trace in the official scientific literature, therefore I have not put them in the Bibliography. However, they can be found on the Internet.

2 – CAUCHY-RIEMANN EQUATIONS

I'll go straight to the topic.
I'm looking for a solution of $\partial^* F = 0$.

With

$$e^{-i(\omega\tau - kz)}$$

I place

$$F = F_1(x, y)e^{-i(\omega\tau - kz)}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F + \left(j\frac{\partial}{\partial z} + T\frac{\partial}{\partial \tau}\right)F = 0$$

$$\left[\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F_1(x, y)\right]e^{-i(\omega\tau - kz)} + (jF_1 ik - TF_1 i\omega)e^{-i(\omega\tau - kz)} = 0$$

Simplifying the exponential I get

$$\partial^*{}_{xy}F_1(x, y) + (jF_1 ik - TF_1 i\omega) = 0$$

I can bring $F_1 i$ to the right and collect it to common factor

$$\partial^*{}_{xy}F_1(x, y)F_1 + (jk - T\omega)F_1 i = 0$$

I place

$$F_1 = (E + TH)$$

With this separation of indices in F1 both what I called E and what I called H contain the indices $1, i, j, ji$

Note: I could put $F_1 = (E + TjiH)$ but I don't know what is more convenient.

$$\partial^*{}_{xy}(E + TH) + (jk - T\omega)(E + TH)i = 0$$

Indexes in motion

$$1, Tj, i, Tji, Ti, ji, T, j$$

and are thus separated , between the part without the T (i.e. E) and the part in front of the T (i.e. H)

$$1, i, j, ji, Tj, Tji, Ti, T$$

$$\partial^*{}_{xy}(E + TH) + (jk - T\omega)(E + TH)i = 0$$

And both E and H contain the indices $1, i, j, ji$

$$\partial^*{}_{xy}E + \partial^*{}_{xy}TH + jkEi + jTkHi - T\omega Ei - \omega Hi = 0$$

I obtain these two equations among (E, H) **with indices** $1, i, j, ji$

$$\begin{cases} \partial^*{}_{xy}E + jkEi - \omega Hi = 0 \\ \partial^*{}_{xy}TH + jTkHi - T\omega Ei = 0 \end{cases}$$

I observe that

$$\partial^*{}_{xy}T = T\partial_{xy}$$

so I bring T on the left and simplify

$$\begin{cases} \partial^*{}_{xy}E + jkEi - \omega Hi = 0 \\ \partial_{xy}H - jkHi - \omega Ei = 0 \end{cases}$$

Now I replace the coordinates from cartesian to cylindrical

$$\partial_{xy} = e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)$$

$$\partial^*{}_{xy} = e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)$$

$$\begin{cases} e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right)E + jkEi - \omega Hi = 0 \\ e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right)H - jkHi - \omega Ei = 0 \end{cases}$$

Arrived at this point

-both E and H contain the indices $1, i, j, ji$ and therefore parts with and without the j $1, i$ and j, ji
- both E and H contain parts 'r' and parts 'phi'.

Suppose immediately to separate the terms E, H in "part 1, i" and "part j, ji" .
Here

$$\begin{cases} e^{i\varphi}\frac{\partial E}{\partial r}+\frac{i}{r}e^{i\varphi}\frac{\partial E}{\partial \varphi}+jkEi-\omega Hi=0 \\ e^{-i\varphi}\frac{\partial H}{\partial r}-\frac{i}{r}e^{-i\varphi}\frac{\partial H}{\partial \varphi}-jkHi-\omega Ei=0 \end{cases}$$

I make the following changes :

$$E \Longrightarrow (E+jE_j)$$
$$H \Longrightarrow (H+jH_j)$$

i.e. I separate in E, H the "parts 1, i" and the "parts j, ji".

$$F_1=(E+jE_j)+T(H+jH_j)$$

With this we get:

$$\begin{cases} e^{i\varphi}\frac{\partial(E+jE_j)}{\partial r}+\frac{i}{r}e^{i\varphi}\frac{\partial(E+jE_j)}{\partial \varphi}+jk(E+jE_j)i-\omega(H+jH_j)i=0 \\ e^{-i\varphi}\frac{\partial(H+jH_j)}{\partial r}-\frac{i}{r}e^{-i\varphi}\frac{\partial(H+jH_j)}{\partial \varphi}-jk(H+jH_j)i-\omega(E+jE_j)i=0 \end{cases}$$

At this point the terms with and without j give rise to separate equations and 4 equations are obtained

$$\begin{cases} e^{i\varphi}\frac{\partial(E)}{\partial r}+\frac{i}{r}e^{i\varphi}\frac{\partial(E)}{\partial \varphi}+jk(jE_j)i-\omega(H)i=0 \\ e^{i\varphi}\frac{\partial(jE_j)}{\partial r}+\frac{i}{r}e^{i\varphi}\frac{\partial(jE_j)}{\partial \varphi}+jk(E)i-\omega(jH_j)i=0 \end{cases}$$

$$\begin{cases} e^{-i\varphi}\frac{\partial(H)}{\partial r}-\frac{i}{r}e^{-i\varphi}\frac{\partial(H)}{\partial \varphi}-jk(jH_j)i-\omega(E)i=0 \\ e^{-i\varphi}\frac{\partial(jH_j)}{\partial r}-\frac{i}{r}e^{-i\varphi}\frac{\partial(jH_j)}{\partial \varphi}-jk(H)i-\omega(jE_j)i=0 \end{cases}$$

I group them differently ie in the following way
(1)

$$\begin{cases} e^{i\varphi}\dfrac{\partial(E)}{\partial r}+\dfrac{i}{r}e^{i\varphi}\dfrac{\partial(E)}{\partial\varphi}+jk(jE_j)i-\omega(H)i=0 \\ e^{-i\varphi}\dfrac{\partial(H)}{\partial r}-\dfrac{i}{r}e^{-i\varphi}\dfrac{\partial(H)}{\partial\varphi}-jk(jH_j)i-\omega(E)i=0 \end{cases}$$

(2)

$$\begin{cases} e^{i\varphi}\dfrac{\partial(jE_j)}{\partial r}+\dfrac{i}{r}e^{i\varphi}\dfrac{\partial(jE_j)}{\partial\varphi}+jk(E)i-\omega(jH_j)i=0 \\ e^{-i\varphi}\dfrac{\partial(jH_j)}{\partial r}-\dfrac{i}{r}e^{-i\varphi}\dfrac{\partial(jH_j)}{\partial\varphi}-jk(H)i-\omega(jE_j)i=0 \end{cases}$$

I try to separate dependencies from r and phi:

$$E+jE_j=\left(R_E\Phi_E+jR_{Ej}\Phi_{E_j}\right)$$
$$H+jH_j=\left(R_H\Phi_H+jR_{Hj}\Phi_{H_j}\right)$$

So that they R_{H_j} and R_{Ej} are with indices 1, i, and as well R_E and R_H -
So it is:

$$\begin{cases} E=R_E\Phi_E \\ H=R_H\Phi_H \end{cases}$$

$$\begin{cases} jE_j=jR_{Ej}\Phi_{E_j} \\ jH_j=jR_{Hj}\Phi_{H_j} \end{cases}$$

I replace in (1).

$$\begin{cases} e^{i\varphi}\dfrac{\partial(R_E\Phi_E)}{\partial r}+\dfrac{i}{r}e^{i\varphi}\dfrac{\partial(R_E\Phi_E)}{\partial\varphi}+jk\left(jR_{Ej}\Phi_{E_j}\right)i-\omega(R_H\Phi_H)i=0 \\ e^{-i\varphi}\dfrac{\partial(R_H\Phi_H)}{\partial r}-\dfrac{i}{r}e^{-i\varphi}\dfrac{\partial(R_H\Phi_H)}{\partial\varphi}-jk\left(jR_{Hj}\Phi_{H_j}\right)i-\omega(R_E\Phi_E)i=0 \end{cases}$$

$$\begin{cases} e^{i\varphi}\dfrac{\partial(R_E)}{\partial r}\Phi_E + \dfrac{i}{r}e^{i\varphi}R_E\dfrac{\partial(\Phi_E)}{\partial\varphi} + jk\left(jR_{Ej}\Phi_{E_j}\right)i - \omega(R_H\Phi_H)i = 0 \\ e^{-i\varphi}\dfrac{\partial(R_H)}{\partial r}\Phi_H - \dfrac{i}{r}e^{-i\varphi}R_H\dfrac{\partial(\Phi_H)}{\partial\varphi} - jk\left(jR_{Hj}\Phi_{H_j}\right)i - \omega(R_E\Phi_E)i = 0 \end{cases}$$

To simplify (i.e. to attempt to simplify) an exponential from the right we admit it is

$$\begin{cases} \Phi_E = e^{in\varphi} \\ \Phi_H = e^{i(n+1)\varphi} \end{cases}$$

$$\begin{cases} \Phi_{E_j} = e^{i(n+1)\varphi} \\ \Phi_{H_j} = e^{in\varphi} \end{cases}$$

Substituting

$$\begin{cases} e^{i\varphi}\dfrac{\partial(R_E)}{\partial r}e^{in\varphi} + \dfrac{i}{r}e^{i\varphi}R_E\dfrac{\partial(e^{in\varphi})}{\partial\varphi} + jk\left(jR_{Ej}e^{i(n+1)\varphi}\right)i - \omega\left(R_He^{i(n+1)\varphi}\right)i = 0 \\ e^{-i\varphi}\dfrac{\partial(R_H)}{\partial r}e^{i(n+1)\varphi} - \dfrac{i}{r}e^{-i\varphi}R_H\dfrac{\partial\left(e^{i(n+1)\varphi}\right)}{\partial\varphi} - jk\left(jR_{Hj}e^{in\varphi}\right)i - \omega\left(R_Ee^{in\varphi}\right)i = 0 \end{cases}$$

and then executing the derivatives with respect to phi

$$\begin{cases} e^{i\varphi}\dfrac{\partial(R_E)}{\partial r}e^{in\varphi} + \dfrac{i}{r}e^{i\varphi}R_Eine^{in\varphi} + jk\left(jR_{Ej}e^{i(n+1)\varphi}\right)i - \omega\left(R_He^{i(n+1)\varphi}\right)i = 0 \\ e^{-i\varphi}\dfrac{\partial(R_H)}{\partial r}e^{i(n+1)\varphi} - \dfrac{i}{r}e^{-i\varphi}R_Hi(n+1)\left(e^{i(n+1)\varphi}\right) - jk\left(jR_{Hj}e^{in\varphi}\right)i - \omega\left(R_Ee^{in\varphi}\right)i = 0 \end{cases}$$

you can simplify the exponential from the right and you get to this megasimplification

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} + \dfrac{i}{r}R_Ein + jk\left(jR_{Ej}\right)i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} - \dfrac{i}{r}R_Hi(n+1) - jk\left(jR_{Hj}\right)i - \omega(R_E)i = 0 \end{cases}$$

(1a)

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r}R_En - k\left(R_{Ej}\right)i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r}R_H(n+1) + k\left(R_{Hj}\right)i - \omega(R_E)i = 0 \end{cases}$$

Now I replace in (2).

$$\begin{cases} E = R_E \Phi_E \\ H = R_H \Phi_H \end{cases}$$

$$\begin{cases} jE_j = jR_{Ej} \Phi_{E_j} \\ jH_j = jR_{Hj} \Phi_{H_j} \end{cases}$$

$$\begin{cases} e^{i\varphi} \dfrac{\partial(jE_j)}{\partial r} + \dfrac{i}{r} e^{i\varphi} \dfrac{\partial(jE_j)}{\partial \varphi} + jk(E)i - \omega(jH_j)i = 0 \\ e^{-i\varphi} \dfrac{\partial(jH_j)}{\partial r} - \dfrac{i}{r} e^{-i\varphi} \dfrac{\partial(jH_j)}{\partial \varphi} - jk(H)i - \omega(jE_j)i = 0 \end{cases}$$

$$\begin{cases} e^{i\varphi} \dfrac{\partial(jR_{Ej})}{\partial r} \Phi_{E_j} + \dfrac{i}{r} e^{i\varphi} jR_{Ej} \dfrac{\partial\left(\Phi_{E_j}\right)}{\partial \varphi} + jk(R_E \Phi_E)i - \omega\left(jR_{Hj} \Phi_{H_j}\right) i = 0 \\ e^{-i\varphi} \dfrac{\partial(jR_{Hj})}{\partial r} \Phi_{H_j} - \dfrac{i}{r} e^{-i\varphi} jR_{Hj} \dfrac{\partial\left(\Phi_{H_j}\right)}{\partial \varphi} - jk(R_H \Phi_H)i - \omega\left(jR_{Ej} \Phi_{E_j}\right) i = 0 \end{cases}$$

To simplify an exponential from the right we admit that it is (same hypotheses as above):

$$\begin{cases} \Phi_E = e^{in\varphi} \\ \Phi_H = e^{i(n+1)\varphi} \end{cases}$$

$$\begin{cases} \Phi_{E_j} = e^{i(n+1)\varphi} \\ \Phi_{H_j} = e^{in\varphi} \end{cases}$$

Substituting

$$\begin{cases} e^{i\varphi} \dfrac{\partial(jR_{Ej})}{\partial r} e^{i(n+1)\varphi} + \dfrac{i}{r} e^{i\varphi} jR_{Ej} \dfrac{\partial\left(e^{i(n+1)\varphi}\right)}{\partial \varphi} + jk(R_E e^{in\varphi})i - \omega(jR_{Hj} e^{in\varphi})i = 0 \\ e^{-i\varphi} \dfrac{\partial(jR_{Hj})}{\partial r} e^{in\varphi} - \dfrac{i}{r} e^{-i\varphi} jR_{Hj} \dfrac{\partial(e^{in\varphi})}{\partial \varphi} - jk\left(R_H e^{i(n+1)\varphi}\right)i - \omega\left(jR_{Ej} e^{i(n+1)\varphi}\right)i = 0 \end{cases}$$

and then executing the derivatives with respect to phi

$$\begin{cases} e^{i\varphi}\dfrac{\partial(jR_{Ej})}{\partial r}e^{i(n+1)\varphi}+\dfrac{i}{r}e^{i\varphi}jR_{Ej}i(n+1)\left(e^{i(n+1)\varphi}\right)+jk\left(R_E e^{in\varphi}\right)i-\omega\left(jR_{H_j}e^{in\varphi}\right)i=0 \\ e^{-i\varphi}\dfrac{\partial(jR_{Hj})}{\partial r}e^{in\varphi}-\dfrac{i}{r}e^{-i\varphi}jR_{Hj}in\left(e^{in\varphi}\right)-jk\left(R_H e^{i(n+1)\varphi}\right)i-\omega\left(jR_{Ej}e^{i(n+1)\varphi}\right)i=0 \end{cases}$$

you can simplify the exponential from the right and you get to this megasimplification

$$\begin{cases} \dfrac{\partial(jR_{Ej})}{\partial r}+\dfrac{i}{r}jR_{Ej}i(n+1)+jk(R_E)i-\omega\left(jR_{Hj}\right)i=0 \\ \dfrac{\partial(jR_{Hj})}{\partial r}-\dfrac{i}{r}jR_{Hj}in-jk(R_H)i-\omega\left(jR_{Ej}\right)i=0 \end{cases}$$

(2a)

$$\begin{cases} \dfrac{\partial(R_{Ej})}{\partial r}+\dfrac{1}{r}R_{Ej}(n+1)+k(R_E)i-\omega\left(R_{Hj}\right)i=0 \\ \dfrac{\partial(R_{Hj})}{\partial r}-\dfrac{1}{r}R_{Hj}n-k(R_H)i-\omega\left(R_{Ej}\right)i=0 \end{cases}$$

Be now

$$R_{H_j}e^{in\varphi}=\frac{k}{(\omega+\omega_0)}R_E e^{in\varphi}$$

and also

$$R_{E_j}e^{i(n+1)\varphi}=\frac{-k}{(\omega+\omega_0)}R_H e^{i(n+1)\varphi}$$

(which is also compatible with the previous positions on $e^{i(n+1)\varphi}$ and $e^{in\varphi}$).
Let be more in particular

$$R_{H_j}=\frac{k}{(\omega+\omega_0)}R_E$$

$$R_{E_j}=\frac{-k}{(\omega+\omega_0)}R_H$$

and I'm going to make these replacements.

I now do all the replacement steps in (2a) .
Step by step I get

$$R_{H_j} = \frac{k}{(\omega + \omega_0)} R_E$$

$$R_{E_j} = \frac{-k}{(\omega + \omega_0)} R_H$$

(2a)

$$\begin{cases} \frac{\partial (R_{Ej})}{\partial r} + \frac{1}{r} R_{Ej}(n+1) + k(R_E)i - \omega(R_{Hj})i = 0 \\ \frac{\partial (R_{Hj})}{\partial r} - \frac{1}{r} R_{Hj} n - k(R_H)i - \omega(R_{Ej})i = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \left(-\frac{k}{(\omega+\omega_0)} R_H\right)}{\partial r} + \frac{1}{r} - \frac{k}{(\omega+\omega_0)} R_H(n+1) + k(R_E)i - \omega\left(\frac{k}{(\omega+\omega_0)} R_E\right) i = 0 \\ \frac{\partial \left(\frac{k}{(\omega+\omega_0)} R_E\right)}{\partial r} - \frac{1}{r} \frac{k}{(\omega+\omega_0)} R_E n - k(R_H)i - \omega\left(-\frac{k}{(\omega+\omega_0)} R_H\right) i = 0 \end{cases}$$

$$\begin{cases} \frac{\partial (-kR_H)}{\partial r} - \frac{1}{r} kR_H(n+1) + k(\omega+\omega_0)(R_E)i - \omega(kR_E)i = 0 \\ \frac{\partial (kR_E)}{\partial r} - \frac{1}{r} kR_E n - k(\omega+\omega_0)(R_H)i + \omega(kR_H)i = 0 \end{cases}$$

$$\begin{cases} \frac{\partial (-R_H)}{\partial r} - \frac{1}{r} R_H(n+1) + (\omega+\omega_0)(R_E)i - \omega(R_E)i = 0 \\ \frac{\partial (R_E)}{\partial r} - \frac{1}{r} R_E n - (\omega+\omega_0)(R_H)i + \omega(R_H)i = 0 \end{cases}$$

and finally:
(2b)

$$\begin{cases} \frac{\partial (R_H)}{\partial r} + \frac{1}{r} R_H(n+1) - \omega_0(R_E)i = 0 \\ \frac{\partial (R_E)}{\partial r} - \frac{1}{r} R_E n - \omega_0(R_H)i = 0 \end{cases}$$

Instead, substituting step by step in (1a) we obtain

$$R_{H_j} = \frac{k}{(\omega + \omega_0)} R_E$$

$$R_{E_j} = \frac{-k}{(\omega + \omega_0)} R_H$$

(1a)

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r} R_E n - k(R_{Ej})i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r} R_H(n+1) + k(R_{Hj})i - \omega(R_E)i = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r} R_E n - k\left(\dfrac{-k}{(\omega + \omega_0)} R_H\right)i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r} R_H(n+1) + k\left(\dfrac{k}{(\omega + \omega_0)} R_E\right)i - \omega(R_E)i = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r} R_E n + \left(\dfrac{k^2}{(\omega + \omega_0)} R_H\right)i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r} R_H(n+1) + \left(\dfrac{k^2}{(\omega + \omega_0)} R_E\right)i - \omega(R_E)i = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r} R_E n + (\omega - \omega_0)(R_H)i - \omega(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r} R_H(n+1) + (\omega - \omega_0)(R_E)i - \omega(R_E)i = 0 \end{cases}$$

and finally:
(1b)

$$\begin{cases} \dfrac{\partial(R_E)}{\partial r} - \dfrac{1}{r} R_E n - \omega_0(R_H)i = 0 \\ \dfrac{\partial(R_H)}{\partial r} + \dfrac{1}{r} R_H(n+1) - \omega_0(R_E)i = 0 \end{cases}$$

Summarizing, to solve the equation

$$\partial^* F = 0$$

I place

$$F = F_1(x, y)e^{-i(\omega\tau - kz)}$$

with

$$F_1 = (E + jE_j) + T(H + jH_j)$$

and with

$$R_{H_j}e^{in\varphi} = \frac{k}{(\omega + \omega_0)} R_E e^{in\varphi}$$

and also

$$R_{E_j}e^{i(n+1)\varphi} = \frac{-k}{(\omega + \omega_0)} R_H e^{i(n+1)\varphi}$$

I get the following conditions (the 1b and 2b , coinciding with each other) :

$$\begin{cases} \dfrac{\partial R_H}{\partial r} + \dfrac{1}{r} R_H(n + 1) - \omega_0 R_E i = 0 \\ \dfrac{\partial R_E}{\partial r} - \dfrac{1}{r} n R_E - \omega_0 R_H i = 0 \end{cases}$$

Solution: Bessel functions.
Indeed, place

$$R_H = iJ_{n+1}(\omega_0 r)$$

$$R_E = J_n(\omega_0 r)$$

the equations become

$$\begin{cases} \dfrac{\partial J_{n+1}(\omega_0 r)}{\partial r} + \dfrac{(n + 1)}{r} J_{n+1}(\omega_0 r) - \omega_0 J_n(\omega_0 r) = 0 \\ \dfrac{\partial J_n(\omega_0 r)}{\partial r} - \dfrac{n}{r} J_n(\omega_0 r) + \omega_0 J_{n+1}(\omega_0 r) = 0 \end{cases}$$

wich, with an appropriate change of indices,

$$\begin{cases} \dfrac{\partial J_n(\omega_0 r)}{\partial r} + \dfrac{n}{r} J_n(\omega_0 r) - \omega_0 J_{n-1}(\omega_0 r) = 0 \\ \dfrac{\partial J_n(\omega_0 r)}{\partial r} - \dfrac{n}{r} J_n(\omega_0 r) + \omega_0 J_{n+1}(\omega_0 r) = 0 \end{cases}$$

coincides with recursive relationships
(D.35)

$$\frac{dZ_n(kx)}{dx} + \frac{n}{x}Z_n(kx) - kZ_{n-1}(kx) = 0$$

(D.36)

$$\frac{dZ_n(kx)}{dx} - \frac{n}{x}Z_n(kx) + kZ_{n+1}(kx) = 0$$

From [7] "Introductory Applications of Partial Differential Equations: with Emphasis on Wave Propagation and Diffusion"
by G. L. Lamb, Jr.

https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781118032831.app4

D.7 RECURRENCE RELATIONS

Bessel's functions of neighboring order are interrelated in ways that prove to be extremely useful. Some of these relations are listed below (Hildebrand, 1976, p. 149) where $Z_n(x)$ refers to one of the Bessel functions J, Y, $H^{(1)}$, $H^{(2)}$, I, or K:

$$\frac{d}{dx}[x^n Z_n(kx)] = \begin{cases} kx^n Z_{n-1}(kx), & Z = J, Y, I, H^{(1)}, H^{(2)} \\ -kx^n Z_{n-1}(kx), & Z = K \end{cases} \tag{D.34}$$

$$\frac{d}{dx}[x^{-n} Z_n(kx)] = \begin{cases} -kx^{-n} Z_{n+1}(kx), & Z = J, Y, K, H^{(1)}, H^{(2)} \\ kx^{-n} Z_{n+1}(kx), & Z = I \end{cases} \tag{D.35}$$

$$\frac{d}{dx}Z_n(kx) = \begin{cases} kZ_{n-1}(kx) - \frac{n}{x}Z_n(kx), & Z = J, Y, I, H^{(1)}, H^{(2)} \\ -kZ_{n-1}(kx) - \frac{n}{x}Z_n(kx), & Z = K \end{cases}$$

(D.36)

$$\frac{d}{dx}Z_n(kx) = \begin{cases} -kZ_{n+1}(kx) + \frac{n}{x}Z_n(kx), & Z = J, Y, K, H^{(1)}, H^{(2)} \\ kZ_{n+1}(kx) + \frac{n}{x}Z_n(kx), & Z = I \end{cases}$$

3 - SOLUTION

Summarizing things from the beginning, one has, in succession:

$$F = \left\{ \left[R_E \Phi_E + j R_{Ej} \Phi_{E_j} \right] + T \left[R_H \Phi_H + j R_{Hj} \Phi_{H_j} \right] \right\} e^{-i(\omega\tau - kz)}$$

$$R_E \Phi_E = J_n(\omega_0 r) e^{in\varphi}$$

$$j R_{Ej} \Phi_{E_j} = j \frac{-k}{(\omega + \omega_0)} i J_{n+1}(\omega_0 r) e^{i(n+1)\varphi}$$

$$R_H \Phi_H = i J_{n+1}(\omega_0 r) e^{i(n+1)\varphi}$$

$$j R_{Hj} \Phi_{H_j} = j \frac{k}{(\omega + \omega_0)} J_n(\omega_0 r) e^{in\varphi}$$

and replacing everything we get:

$$F = \left\{ \left[J_n(\omega_0 r) e^{in\varphi} - j \frac{k}{(\omega + \omega_0)} i J_{n+1}(\omega_0 r) e^{i(n+1)\varphi} \right] + T \left[i J_{n+1}(\omega_0 r) e^{i(n+1)\varphi} + j \frac{k}{(\omega + \omega_0)} J_n(\omega_0 r) e^{in\varphi} \right] \right\} e^{-i(\omega\tau - kz)}$$

which is reordered like this:

$$F = \left(J_n(\omega_0 r) e^{in\varphi} + T i J_{n+1}(\omega_0 r) e^{i(n+1)\varphi} \right) e^{-i(\omega\tau - kz)} + \left(i j J_{n+1}(\omega_0 r) e^{i(n+1)\varphi} + T j J_n(\omega_0 r) e^{in\varphi} \right) \frac{k}{(\omega + \omega_0)} e^{-i(\omega\tau - kz)}$$

4 - DISCUSSION

The 'at rest' solution already examined in the Manuscripts [1] reappears

$$F = \left(J_n(\omega_0 r)e^{in\varphi} + TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau - kz)}$$

To this is added the component due to motion

$$F = +\left(ijJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi} + TjJ_n(\omega_0 r)e^{in\varphi}\right)\frac{k}{(\omega + \omega_0)}e^{-i(\omega\tau - kz)}$$

In this part, the portion with index Tj simply represents the magnetic field components Hx and Hy

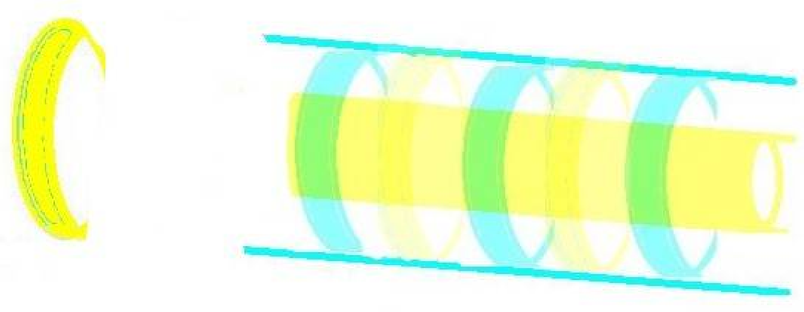
$$F = \left(TjJ_n(\omega_0 r)e^{in\varphi}\right)\frac{k}{(\omega + \omega_0)}e^{-i(\omega\tau - kz)}$$

The part with index ij shows the presence of a magnetic charge

$$F = \left(ijJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)\frac{k}{(\omega + \omega_0)}e^{-i(\omega\tau - kz)}$$

The T field component responsible for the electric charge is also a function of z when in motion, which involves electric current in the z direction

$$F = \left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau - kz)}$$



5 - GENERALIZATION

The solution found opens up a series of interesting possibilities, since each component of the solution found lends itself to be interpreted as (more precisely: it is) a harmonic potential.

$$F = \left(J_n(\omega_0 r)e^{in\varphi} + TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau - kz)} + \left(ijJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi} + TjJ_n(\omega_0 r)e^{in\varphi}\right)\frac{k}{(\omega + \omega_0)}e^{-i(\omega\tau - kz)}$$

Using cylindrical coordinates

$$\partial_{xy} = e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right)$$

$$\partial^*{}_{xy} = e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi}\right)$$

I can directly find a series of analytic functions .
For example, using the derivation operator ∂ this function is certainly analytic

$$F = \partial\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)e^{-i(\omega\tau - kz)}\right]$$

$$\partial = e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right) - j\frac{\partial}{\partial z} - T\frac{\partial}{\partial\tau}$$

indeed it is analytic (I presume) the function that arises from the harmonic potential

$$A = \left(J_n(\omega_0 r)e^{in\varphi}\right)\cos\left(\omega\tau - kz\right)$$

which generalizes exercise 5 of the Manuscripts [1].

Calculations.

$$F = \partial A$$

$$F = e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\varphi}\right)\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right] + \left(-j\frac{\partial}{\partial z} - T\frac{\partial}{\partial\tau}\right)\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right]$$

$$F = e^{-i\varphi}\frac{\partial}{\partial r}\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right] - e^{-i\varphi}\frac{i}{r}\frac{\partial}{\partial\varphi}\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right] - j\frac{\partial}{\partial z}\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right] - T\frac{\partial}{\partial\tau}\left[\left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau - kz)\right]$$

Indices 1, i, j, T.
Ex, Ey, Ez, Htau.

Then the function deriving from this harmonic potential is analytic (I presume)

$$A = \left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}$$

This generalizes exercise 1 of the Manuscripts [1].

Calculations.

$$F = \partial\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right]$$

$$\partial = e^{-i\varphi}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right)-j\frac{\partial}{\partial z}-T\frac{\partial}{\partial\tau}$$

$$F = e^{-i\varphi}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right] + \left(-j\frac{\partial}{\partial z}-T\frac{\partial}{\partial\tau}\right)\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right]$$

$$F = e^{-i\varphi}\frac{\partial}{\partial r}\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right] - e^{-i\varphi}\frac{i}{r}\frac{\partial}{\partial\varphi}\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right] - j\frac{\partial}{\partial z}\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right] - T\frac{\partial}{\partial\tau}\left[\left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}\right]$$

Indices T, Ti, Tji, Tj
Hx, Hy, Hz, Htau.

From the examination of these last two potentials it seems to be able to conclude that there is a potential to which correspond mass charge, spin, magnetic moment, and <u>without</u> magnetic charge. It is

$$A = \left(J_n(\omega_0 r)e^{in\varphi}\right)\cos(\omega\tau-kz) + \left(TiJ_{n+1}(\omega_0 r)e^{i(n+1)\varphi}\right)e^{-i(\omega\tau-kz)}$$

6 - EXAMPLE

One of the most interesting aspects is to verify if it is harmonic $[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]$

Let's calculate the Laplacian directly

$$\partial\partial^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}$$

$$\partial\partial^* f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial \tau^2}$$

$$\begin{aligned}\partial\partial^*[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]\\ &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial \tau^2}\right][(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]\\ &+ \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] + \frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]\\ &- \frac{\partial^2}{\partial \tau^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)]\end{aligned}$$

The individual parts are

(3)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] = e^{in\varphi}\cos(\omega\tau - kz)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_n(\omega_0 r)}{\partial r}\right)$$

(4)

$$\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] = J_n(\omega_0 r)\cos(\omega\tau - kz)\frac{1}{r^2}\frac{\partial^2 e^{in\varphi}}{\partial \varphi^2}$$

(5)

$$\frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] = (J_n(\omega_0 r)e^{in\varphi})\frac{\partial^2 \cos(\omega\tau - kz)}{\partial z^2}$$

(6)

$$\frac{\partial^2}{\partial \tau^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] = (J_n(\omega_0 r)e^{in\varphi})\frac{\partial^2 \cos(\omega\tau - kz)}{\partial z^2}$$

Development part (3)

(3)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] = e^{in\varphi}cos\,(\omega\tau - kz)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_n(\omega_0 r)}{\partial r}\right)$$

The second derivative is calculated in the Appendix and the result is

$$\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} = -\frac{1}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} + \left(\frac{n^2}{r^2} - {\omega_0}^2\right)J_n(\omega_0 r)$$

So replacing

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_n(\omega_0 r)}{\partial r}\right) = \frac{1}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} - \frac{1}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} + \left(\frac{n^2}{r^2} - {\omega_0}^2\right)J_n(\omega_0 r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_n(\omega_0 r)}{\partial r}\right) = \left(\frac{n^2}{r^2} - {\omega_0}^2\right)J_n(\omega_0 r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] = J_n(\omega_0 r)\left(\frac{n^2}{r^2} - {\omega_0}^2\right)e^{in\varphi}cos\,(\omega\tau - kz)$$

Therefore from all four we simplify $J_n(\omega_0 r)$

(3a)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] \propto \left(\frac{n^2}{r^2} - {\omega_0}^2\right)e^{in\varphi}cos\,(\omega\tau - kz)$$

(4a)

$$\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] \propto cos\,(\omega\tau - kz)\frac{1}{r^2}\frac{\partial^2 e^{in\varphi}}{\partial\varphi^2}$$

(5a)

$$\frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] \propto e^{in\varphi}\frac{\partial^2 cos\,(\omega\tau - kz)}{\partial z^2}$$

(6a)

$$\frac{\partial^2}{\partial\tau^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] \propto e^{in\varphi}\frac{\partial^2 cos\,(\omega\tau - kz)}{\partial z^2}$$

Continuing still I run the derivative with respect to phi.

$$\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto \cos(\omega\tau - kz)\frac{1}{r^2}\frac{\partial^2 e^{in\varphi}}{\partial\varphi^2}$$

$$\frac{\partial e^{in\varphi}}{\partial\varphi} = ine^{in\varphi}$$

$$\frac{1}{r^2}\frac{\partial^2 e^{in\varphi}}{\partial\varphi^2} = \frac{1}{r^2}\frac{\partial}{\partial\varphi}(ine^{in\varphi}) = -\frac{n^2}{r^2}e^{in\varphi}$$

So I replace in (4 a) and then I simplify $e^{in\varphi}$ everywhere

(3b)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto \left(\frac{n^2}{r^2} - {\omega_0}^2\right)\cos(\omega\tau - kz)$$

(4b)

$$\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto -\frac{n^2}{r^2}\cos(\omega\tau - kz)$$

(5b)

$$\frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto \frac{\partial^2\cos(\omega\tau - kz)}{\partial z^2}$$

(6b)

$$\frac{\partial^2}{\partial\tau^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto \frac{\partial^2\cos(\omega\tau - kz)}{\partial\tau^2}$$

I run the latest derivatives of $\cos(\omega\tau - kz)$

$$\frac{\partial}{\partial z}\cos(\omega\tau - kz) = k\sin(\omega\tau - kz)$$

$$\frac{\partial^2 \cos(\omega\tau - kz)}{\partial z^2} = -k^2\cos(\omega\tau - kz)$$

$$\frac{\partial}{\partial \tau}\cos(\omega\tau - kz) = -\omega\sin(\omega\tau - kz)$$

$$\frac{\partial^2 \cos(\omega\tau - kz)}{\partial \tau^2} = -\omega^2\cos(\omega\tau - kz)$$

So I simplify again $\cos(\omega\tau - kz)$

(3c)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto \left(\frac{n^2}{r^2} - {\omega_0}^2\right)$$

(4c)

$$\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto -\frac{n^2}{r^2}$$

(5c)

$$\frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto -k^2$$

(6c)

$$\frac{\partial^2}{\partial \tau^2}[(J_n(\omega_0 r)e^{in\varphi})\cos(\omega\tau - kz)] \propto -\omega^2$$

Finally in the Laplacian

$$\partial\partial^*[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)]$$
$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial\tau^2}\right][(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)]$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)]$$
$$+ \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] + \frac{\partial^2}{\partial z^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)]$$
$$- \frac{\partial^2}{\partial\tau^2}[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)]$$

I simplify everywhere by means of (3c), (4c), (5c), (6c).
Result:

$$\partial\partial^*[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] \propto \left(\frac{n^2}{r^2} - {\omega_0}^2\right) - \frac{n^2}{r^2} - k^2 + \omega^2 = -{\omega_0}^2 - k^2 + \omega^2 = 0$$

So ultimately

$$\partial\partial^*[(J_n(\omega_0 r)e^{in\varphi})cos\,(\omega\tau - kz)] = 0$$

The function $J_n(\omega_0 r)e^{in\varphi}cos\,(\omega\tau - kz)$ **is a harmonic function..**
As it was intended to prove.

7 - CONCLUSIONS

In conclusion, the existence of solutions of Cauchy Riemann's conditions in 4D occurs or, if you wish, of "waves", what I have called elsewhere "waves of charge" . These, whose physical existence remains to be demonstrated, are nothing more than a generalization of ordinary electromagnetic waves. They can therefore be "neutral", or equipped in various ways with mass, charge, spin. I have made some examples but much more could be studied.
In physics, someone said, what is not strictly prohibited happens.

8 - BIBLIOGRAPHY

[1] G. Bettini,
"Clifford Algebra, 3 and 4-Dimensional Analytic Functions with Applications. Manuscripts of the Last Century ", http://vixra.org/abs/1107.0060

[2] D. Hestenes "Space-Time Algebra", Gordon and Breach Science Publishers, 2nd edition (1966)

[3] David Hestenes. Quantum Mechanics from Self-interaction. Foundations of Physics, 15 (1): 63_87, 1985.

[4] G. Bettini, "Can Electromagnetic Scalar Waves be Radiated by a Metal Sphere?", http://vixra.org/abs/1109.0034

[5] Celani F Di Tommaso A and Vassallo G , 2017, " Maxwell's equations and Occam's razor", Condensed Matter Nucl Sci 25 1-29 https://www.researchgate.net/publication/320274378_Maxwell's_Equations_and_Occam's_Razor

[6] Francesco Celani, Antonino Oscar Di Tommaso, Giorgio Vassallo, "The_Electron_and_Occam's_Razor" https://www.researchgate.net/publication/320274514_The_Electron_and_Occam's_Razor

[7] G. L. Lamb, Jr., "Introductory Applications of Partial Differential Equations: with Emphasis on Wave Propagation and Diffusion", 1995 John Wiley & Sons, Inc. https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781118032831.app4

9 - APPENDIX

I use the formulas (see paragraph 2):

$$\begin{cases} \dfrac{\partial J_{n+1}(\omega_0 r)}{\partial r} + \dfrac{(n+1)}{r} J_{n+1}(\omega_0 r) - \omega_0 J_n(\omega_0 r) = 0 \\ \dfrac{\partial J_n(\omega_0 r)}{\partial r} - \dfrac{n}{r} J_n(\omega_0 r) + \omega_0 J_{n+1}(\omega_0 r) = 0 \end{cases}$$

Let's pick up $J_{n+1}(\omega_0 r)$ from the second and replace in the first

$$J_{n+1}(\omega_0 r) = -\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n}{\omega_0 r} J_n(\omega_0 r)$$

$$\frac{\partial}{\partial r}\left[-\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n}{\omega_0 r} J_n(\omega_0 r)\right] + \frac{(n+1)}{r}\left[-\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n}{\omega_0 r} J_n(\omega_0 r)\right] - \omega_0 J_n(\omega_0 r) = 0$$

$$\frac{\partial}{\partial r}\left[-\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n}{\omega_0 r} J_n(\omega_0 r)\right] = -\frac{1}{\omega_0}\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} - \frac{n}{\omega_0 r^2} J_n(\omega_0 r) + \frac{n}{\omega_0 r}\frac{\partial J_n(\omega_0 r)}{\partial r}$$

$$-\frac{1}{\omega_0}\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} - \frac{n}{\omega_0 r^2} J_n(\omega_0 r) + \frac{n}{\omega_0 r}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{(n+1)}{r}\left[-\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n}{\omega_0 r} J_n(\omega_0 r)\right] - \omega_0 J_n(\omega_0 r) = 0$$

$$-\frac{1}{\omega_0}\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} - \frac{n}{\omega_0 r^2} J_n(\omega_0 r) + \frac{n}{\omega_0 r}\frac{\partial J_n(\omega_0 r)}{\partial r} - \frac{(n+1)}{r}\frac{1}{\omega_0}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{(n+1)}{r}\frac{n}{\omega_0 r} J_n(\omega_0 r) - \omega_0 J_n(\omega_0 r) = 0$$

$$\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} = -\frac{n}{r^2} J_n(\omega_0 r) + \frac{n}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} - \frac{(n+1)}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{(n+1)}{r}\frac{n}{r} J_n(\omega_0 r) - {\omega_0}^2 J_n(\omega_0 r)$$

Conclusion:

$$\frac{\partial^2 J_n(\omega_0 r)}{\partial r^2} = -\frac{1}{r}\frac{\partial J_n(\omega_0 r)}{\partial r} + \frac{n^2}{r^2} J_n(\omega_0 r) - {\omega_0}^2 J_n(\omega_0 r)$$

Note: nothing but the Bessel second-order differential equation.