ON THE ERDŐS DISTANCE PROBLEM

T. AGAMA

In loving memory of Doctor Margaret Lesley McIntyre

Abstract. In this paper, using the method of compression, we recover the lower bound for the Erdős unit distance problem and provide an alternative proof to the distinct distance conjecture. In particular, we show that for sets of points $\mathbb{E} \subset \mathbb{R}^k$ concentrated around the origin with $\#\mathbb{E} \cap \mathbb{N}^k = \frac{n}{2}$, we have

$$\#\left\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^k,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n\right\} \gg_k \frac{\sqrt{k}}{2} n^{1+o(1)}.$$

We also show that

$$\#\left\{d_j : d_j = ||\vec{x_s} - \vec{y_t}||,\ d_j \neq d_i,\ 1 \leq s, t \leq n\right\} \gg_k \frac{\sqrt{k}}{2} n^{\frac{2}{k} - o(1)}.$$

1. **Introduction**

The Erdős distinct distance conjecture is the assertion that

Conjecture 1.1. The number of distinct distances that can be formed from n points in the plane should at least be $n^{1-o(1)}$.

Progress on this conjecture has developed overtime. Let us denote $g(n)$ the counting function for such construction. Then the first lower bound of the form

$$g(n) \gg n^{\frac{2}{3}}$$

was given in [3], which improves on an earlier version of Erdős. This was eventually improved to

$$g(n) \gg \frac{n^{\frac{4}{5}}}{\log n}$$

in [2] and

$$g(n) \gg n^{\frac{6}{7}}$$

in [4]. The best currently known lower bound can be found in [1], which essentially solves the problem. In this paper by using the method of compression and its accompanied estimates, we provide an alternative solution to the conjecture in the following result:

Theorem 1.1.

$$\#\left\{d_j : d_j = ||\vec{x_s} - \vec{y_t}||,\ d_j \neq d_i,\ 1 \leq s, t \leq n,\ \vec{x}, \vec{y} \in \mathbb{R}^k\right\} \gg_k \frac{\sqrt{k}}{2} n^{\frac{2}{k} - o(1)}.$$

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Using this method, we provide a lower bound for the Erdős unit distance problem, that takes into consideration the dimension of the space in which the points reside in the form:

Theorem 1.2. *Let $\mathbb{E} \subset \mathbb{R}^k$ be a set of points concentrated around the origin with $\#\mathbb{E} \cap \mathbb{N}^k = \frac{n}{2}$ and $\mathcal{I} = \left\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^k,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n\right\}$, then we have*

$$\#\mathcal{I} \gg_k \frac{\sqrt{k}}{2} n^{1+o(1)}.$$

2. **Compression**

In this section we launch the notion of compression of points in space. We study the mass of compression and its accompanied estimates. These estimates turn out to be useful for estimating the gap of compression, which we will launch in the sequel.

Definition 2.1. By the compression of scale $m \geq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $m \geq 1$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map. In particular the compression $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map of order 2.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. The latter claim follows by noting that $\mathbb{V}_m^2[\vec{x}] = \vec{x}$. □

2.1. **The mass of compression estimates.**

In this section we study the mass of a compression in a given scale. We use the upper and lower estimates of the mass of compression to establish corresponding estimates for the gap of compression. These estimates will form an essential tool for establishing the main result of this paper.

Definition 2.3. By the mass of a compression of scale $m \geq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Lemma 2.4. *The estimate remain valid*

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

Proposition 2.2 (The mass of compression estimates)**.** *Let* $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ *with* $x_i \neq x_j$ *for* $1 \leq i, j \leq n$ *with* $i \neq j$*, then the estimates holds*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \ll m\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum by appealing to Lemma 2.4. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

□

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

2.2. **Compression gap estimates.** In this section we recall the notion of the gap of compression and its various estimates. We prove upper and lower bounding the gap of a point under compression of any scale.

Definition 2.6. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale m $\mathbb{V}_m$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

Proposition 2.3. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, we have the estimate

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\Bigg(m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\Bigg).$$

Proposition 2.3 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

if and only if $||\vec{x}|| \leq ||\vec{y}||$ for $\vec{x}, \vec{y} \in \mathbb{N}^n$. This important transference principle will be mostly put to use in obtaining our results.

Lemma 2.7 (Compression gap estimates)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1} - 2mn.$$

Proof. The estimates follows by leveraging the estimates in Proposition 2.2 and noting that

$$n\mathrm{Inf}(x_j^2) \ll \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] \ll n\mathrm{sup}(x_j^2).$$

□

3. Application to the Erdős unit distance and distinct distance conjecture

In this section we leverage the estimate of the gap of compression to study the problem of determining the number of unit distances that can be formed from n points. We state our main theorem that takes into consideration the dimension of the space in which the points reside.

Theorem 3.1. *Let $\mathbb{E} \subset \mathbb{R}^k$ be a set of points concentrated around the origin with $\#\mathbb{E} \cap \mathbb{N}^k = \frac{n}{2}$ and $\mathcal{I} = \bigg\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^k,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n\bigg\}$, then we have*

$$\#\mathcal{I} \gg_k \frac{\sqrt{k}}{2} n^{1+o(1)}.$$

Proof. We notice that

$$\begin{aligned}\#\mathcal{I} &= \#\bigg\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^k,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n\bigg\}\\ &\geq \#\bigg\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^k,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n,\\ &\min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}} = n^{o(1)}\bigg\}\\ &\geq \#\bigg\{||\vec{x_j} - \vec{x_t}|| : \vec{x_j} \in \mathbb{N}^k \cap \mathbb{E},\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq j \leq \frac{n}{2},\ \mathbb{V}_1[\vec{x_j}] = \vec{x_t},\\ &\min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}} = n^{o(1)}\bigg\}.\end{aligned}$$

The right-hand side is basically the sum

$$\begin{aligned}\sum_{\substack{\mathcal{G} \circ \mathbb{V}_1[\vec{x_j}] = 1\\ 1 \leq j \leq \frac{n}{2}\\ \min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}} = n^{o(1)}}} 1 &= \sum_{\substack{1 \leq j \leq \frac{n}{2}\\ \min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}} = n^{o(1)}}} \mathcal{G} \circ \mathbb{V}_1[\vec{x_j}]\\ &\gg_k \sum_{\substack{1 \leq j \leq \frac{n}{2}\\ \min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}} = n^{o(1)}}} \mathrm{Inf}(x_{j_s})_{1 \leq s \leq k} \sqrt{k}\\ &\gg_k \frac{n\sqrt{k}}{2} \min\{\mathrm{Inf}(x_{j_s})\}_{\substack{1 \leq j \leq \frac{n}{2}\\ 1 \leq s \leq k}}\\ &= \frac{\sqrt{k}}{2} n^{1+o(1)}\end{aligned}$$

by an application of Lemma 2.7 and the proof of the theorem is complete. □

It is important to point out that the lower estimate provided in Theorem 3.1 is only valid for sets $\mathbb{E}$ of points that are concentrated around the origin and more crucially contains the required number of integer lattice points. This requirement is underscored in the assumption $\#\mathbb{E} \cap \mathbb{N}^k = \frac{n}{2}$. We state the second theorem as an application.

Theorem 3.2.

$$\#\{d_j : d_j = ||\vec{x_s} - \vec{y_t}||,\ d_j \neq d_i,\ 1 \leq s, t \leq n,\ \vec{x}, \vec{y} \in \mathbb{R}^k\} \gg_k \frac{\sqrt{k}}{2} n^{\frac{2}{k} - o(1)}.$$

Proof. First let $\{d_j : d_j = ||\vec{x_s} - \vec{y_t}||,\ d_j \neq d_i,\ 1 \leq s,t \leq n,\ \vec{x},\vec{y} \in \mathbb{R}^k\} = \mathcal{R}$, then we notice that

$$\begin{aligned}
\#\mathcal{R} &\geq \#\left\{d_j : d_j = ||\vec{x_s} - \vec{y_t}||,\ d_j \neq d_i, i \neq j,\ 1 \leq s,t \leq n,\ \vec{x},\vec{y} \in \mathbb{R}^k,\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)}\right\}\\
&\geq \#\left\{d_j : d_j = \mathcal{G} \circ \mathbb{V}_1[\vec{x_j}],\ d_j \neq d_i,\ 1 \leq j \leq \frac{n}{2},\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)},\ \vec{x_j} \in \mathbb{N}^k,\ \mathbb{V}[\vec{x_j}] = \vec{x_t}\right\}\\
&= \sum_{\substack{d_j = \mathcal{G}\circ\mathbb{V}_1[\vec{x_j}]\\ 1 \leq j \leq \frac{n}{2}\\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)}\\ \vec{x_j} \in \mathbb{N}^k\\ d_i \neq d_j\\ i \neq j}} 1\\
&= \sum_{\substack{1 \leq j \leq \frac{n}{2}\\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)}\\ \vec{x_j} \in \mathbb{N}^k\\ d_i \neq d_j\\ i \neq j}} \frac{\mathcal{G} \circ \mathbb{V}_1[\vec{x_j}]}{d_j}\\
&\gg_k \sqrt{k} \sum_{\substack{1 \leq j \leq \frac{n}{2}\\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)}\\ \vec{x_j} \in \mathbb{N}^k\\ d_i \neq d_j\\ i \neq j}} \frac{\mathrm{Inf}(x_{j_s})_{1 \leq s \leq k}}{d_j}\\
&\geq \sqrt{k} \sum_{\substack{1 \leq j \leq \frac{n}{2}\\ \sup(d_j) = n^{1-\frac{2}{k}+o(1)}\\ d_i \neq d_j\\ i \neq j}} \frac{1}{d_j}\\
&\gg_k \sum_{\sup(d_j) = n^{1-\frac{2}{k}+o(1)}} \sqrt{k}\frac{\frac{n}{2}}{\sup(d_j)}_{1 \leq j \leq \frac{n}{2}}\\
&\gg_k \frac{\sqrt{k}}{2} n^{\frac{2}{k}-o(1)}
\end{aligned}$$

and the claimed lower bound follows by Lemma 2.7. □

It needs to be said that the result in Theorem 3.2 can be viewed as providing an alternate solution to the Erdős distinct distance problem, that takes into consideration the dimension of the space in which the points reside. The lower bound of this type, It has to be said, exists in the literature (See [1]). But the method employed is completely different from the one we have used here. Theorem 3.1 and Theorem 3.2 can be considered as a generalization of the solution to both versions of the Erdős distance problem to any euclidean space of dimension $k \geq 2$. In particular we have the following theorems as consequences of the main results of this paper.

Theorem 3.3. *The number of distinct distances that can be formed from n points in a euclidean space $\mathbb{R}^n$ for $n \geq 2$ is at least*

$$\gg \frac{n^{\frac{2}{n}+\frac{1}{2}-o(1)}}{2}.$$

Theorem 3.4. *The number of distinct distances that can be formed from n points in any euclidean space $\mathbb{R}^{2n}$ for $n \geq 2$ is at least*

$$\gg \frac{\sqrt{2}}{2} n^{\frac{1}{n}+\frac{1}{2}-o(1)}.$$

Theorem 3.5. *The number of distinct distances that can be formed from n points in a euclidean space of dimension n^2 for $n \geq 2$ is at least*

$$\gg \frac{n^{\frac{2}{n^2}+1-o(1)}}{2}.$$

[1].

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Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: theophilus@aims.edu.gh/emperordagama@yahoo.com

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