A thesis about Newtonian mechanics rotations and about differential operators:

Louiz Akram
The Higher Institute for Maritime Studies, Morocco – Naval engineering. Casablanca
louiz.akram@gmail.com

Abstract:
This work dates from January 2016 as a result of my remarks related to physics made during my high studies.
I try in this work to explain the cause behind the inability of Newtonian mechanics to describe correctly many phenomena where the studied object rotates at a very high linear speed.
I proved that, in this case, the velocity field is not equiprojective and that the famous formula for changing the reference frame is not correct.
I made an application to the case of the GPS system satellites, then I presented a new method for studying a rotating system velocity without needing the conventional steps of changing reference frames.
I finished my work by demonstrating the formulas of the main differential operators and I presented them with all the related steps and calculations by using the elementary surfaces.
I am eager to discuss the results of this work further with physics and mathematics specialists, and I hope that my formulas will help to simplify the study of many difficult physics phenomena.
Keywords: Newtonian mechanics, Rotations, derivative, perpendicularity, Nabla, Gradient, Curl, Divergence, differential operators, GPS, reference frame.

Introduction:

Even nowadays, there are many electromagnetic phenomena where we will always make mistakes by using the normal mathematical tools in their study. They are in general all the phenomena where the studied object rotates at a high linear speed, especially when its speed exceeds three quarters of the light speed. Because in this case the study mistake becomes very coarse and impossible to neglect.
Furthermore, the scientists are obliged to use the relativity of time as a difficult explanation or statistical physics as a solution to the contradictions found by the conventional change of reference frames. This problematic is described as a historical crisis of Newtonial classical mechanics. [1]
I proved first in the part A of my thesis that in this case the velocity field is not equiprojective and that the famous conventional fsteps for changing the reference frame is not correct.
I presented also a method for studying rotating systems velocity without needing the normal change of reference frames that requires the perpendicularity of a rotating vector and its derivative. [2]
I presented also an application to the case of the GPS system satellites, then I finished my work by demonstrating geometrically, in the part B, the formulas of the three important differential operators: Gradient (nabla), Divergence and Curl.
I presented those differential operators with all the related steps and calculations by using the elementary surfaces. However, the results found will make the convinced readers change their vector and matrix calculations especially with the famous navier-stokes equations. [3]
I remind the readers that this work is a revision of personal previous work made since January 2016 entitled “mémoire en physiques”. [4]

Part A: Rotations of Newtonian mechanics:

1. Introduction:

We know that: $f'(x)=\lim_{dx\to 0}\frac{df}{dx}$. However, in physics, the infinitesimal variation dx is never null when we study the movement in a physics phenomenon except at rest. Consequently, we shouldn’t use f’ the derivative of the function f. We should use $\frac{df}{dx}$ instead which is the differential divided by the infinitesimal variation.

2. Remarks:

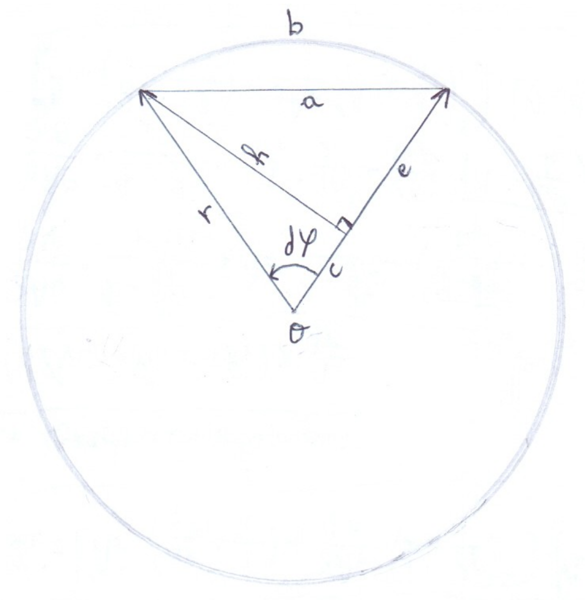

Fig.1: *The trigonometric circle of radius r=1.*

Let's consider the circle of radius r=1 in the figure 1, where b is the length of the circular arc and a is the circular arc chord and dφ is the circular arc infinitesimal angle.
Consequently b= dφ and S is the area of the circular segment that is the area of the surface between a and b, where $S=\frac{r^2}{2}\times(d\varphi-\sin(d\varphi))$.
And we consider that h=sin(dφ) and c=cos(dφ) and c+e=1.

Remark 1:

If dφ is small enough, then $\frac{\sin(d\varphi)}{d\varphi}\approx 1 \Leftrightarrow \sin(d\varphi)\approx d\varphi$ (1)

and in this case S≈0 therefore a and b become combined which means a≈b. Also, in this case, h≈ dφ therefore by using Pythagoras' theorem we conclude that: $e^2+h^2=a^2 \Leftrightarrow d\varphi^2+e^2\approx d\varphi^2$
consequently: $e^2\approx 0 \Leftrightarrow e\approx 0$ and thus e disappears and cos(dφ)=c≈1.
We conclude that: $\sin(d\varphi)\approx d\varphi \Rightarrow \cos(d\varphi)\approx 1$ (2)
and we remark that in this case $\cos(d\varphi)^2+\sin(d\varphi)^2\neq 1$ consequently, trigonometric identities are unusable when sin(dφ)≈ dφ.
By Al-Kashi's theorem we prove also that a≈0 which implies that dφ≈0. And this is remarked also with: $\cos(d\varphi)\approx 1 \Leftrightarrow d\varphi\approx 0$ and this is absurd since dφ exists during the study of the rotation in a physics phenomenon where we should have normally cos(dφ) < 1.
We conclude that the approximation with sin(dφ)≈dφ is unusable.
If we use sin(dφ)≈ dφ and $\cos(d\varphi)\approx\sqrt{1-d\varphi^2}$ (3)
by using Pythagoras' theorem in the trigonometric circle, then in this case:
$e\approx 1-\sqrt{1-d\varphi^2} \Rightarrow e^2\approx 2-d\varphi^2-2\sqrt{1-d\varphi^2}$
Hence: $e^2+h^2\approx d\varphi^2\approx 2-2\sqrt{1-d\varphi^2}$
Consequently: $2\approx d\varphi^2+2\sqrt{1-d\varphi^2} \Rightarrow 4\approx d\varphi^4+4-4d\varphi^2+4d\varphi^2\sqrt{1-d\varphi^2}$
We conclude that: $1\approx\frac{d\varphi^2}{4}+\sqrt{1-d\varphi^2} \Rightarrow \left(1-\frac{d\varphi^2}{4}\right)^2\approx 1-d\varphi^2$

And thus: $1+\frac{d\varphi^4}{16}-\frac{d\varphi^2}{2}\approx 1-d\varphi^2 \Rightarrow d\varphi^2\approx -8$

We proved that if we use sin(dφ)≈ dφ and $\cos(d\varphi)\approx\sqrt{1-d\varphi^2}$, then this implies that $d\varphi^2\approx -8$ which is a contradiction. Hence, the use of this approximation is false and can cause errors in the study of the rotation in a phenomenon.

<u>Remark 2:</u>

Now let's use the limit of $\frac{1-\cos(d\varphi)}{d\varphi^2}$ when dφ tends to zero.

We have consequently when dφ is small enough: $\cos(d\varphi)\approx 1-\frac{d\varphi^2}{2}$. (4)

Consequently, by using Pythagoras' theorem in the trigonometric circle:

$$\sin(d\varphi)\approx d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}} \quad . \tag{5}$$

In this case S≠0 even if we notice that a tends to dφ (a≈ dφ) when we use Al-Kashi's theorem in the triangle of the figure 1. S≠0 because a equals approximately dφ but b equals exactly dφ and the formula $S=\frac{r^2}{2}\times(d\varphi-\sin(d\varphi))$ prevents us to consider that a=b.

This approximation is correct and causes no contradictions. Consequently, we can use it without risking any errors.

All the trigonometric identities are usable in this case since $\cos(d\varphi)^2+\sin(d\varphi)^2=1$ and since the triangles of the trigonometric circle in figure 1 stay all valid.

<u>Remark 3:</u>

$$\cos(d\varphi)\approx 1-\frac{d\varphi^2}{2}\Leftrightarrow d\varphi^2\approx 2(1-\cos(d\varphi))\approx 4\sin\left(\frac{d\varphi}{2}\right)^2\Leftrightarrow \sin\left(\frac{d\varphi}{2}\right)\approx\frac{d\varphi}{2} \tag{6}$$

consequently trigonometric identities are not usable for $\frac{d\varphi}{2}$. We conclude that we should fix dφ as the smallest detectable variation in the phenomenon studied rotation which depends on the detection technology in order to avoid mathematical contradictions.

The fixed dφ will respect the formulas: $\cos(d\varphi)\approx 1-\frac{d\varphi^2}{2}$ and $\sin(d\varphi)\approx d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}$.

However, in this case we should use dt as the variable.

During the rotation in Newtonian mechanics, we have a fixed infinitesimal variation which defines the fixed change dφ that happens in different changing durations Δt. For example, in a polar coordinate system (ρ,φ,z=0), we will have:

$$\frac{d\vec{e}_\rho(\varphi)}{dt}=\frac{d\vec{e}_\rho(\varphi)}{\Delta t}=\frac{d\vec{e}_\rho(\varphi)}{\Delta t\times\dot{\varphi}}\times\dot{\varphi}=\frac{d\vec{e}_\rho(\varphi)}{d\varphi}\times\dot{\varphi} \tag{7}$$

where: $\dot{\varphi}=\frac{d\varphi}{\Delta t}$ and dt=Δt is the variable duration of each dφ. You will find in this document an example of this method application that deals with the case of GPS systems.

<u>Important:</u>

These approximations imply that: $\frac{de^{i\varphi}}{d\varphi}\neq i(\cos(\varphi)+i\sin(\varphi))$

consequently: $e^{i\varphi}\neq\cos(\varphi)+i\sin(\varphi)$

And thus: $\cos(\varphi) \neq \frac{e^{i\varphi}}{2} + \frac{e^{-i\varphi}}{2}$ and: $\sin(\varphi) \neq \frac{e^{i\varphi}}{2i} - \frac{e^{-i\varphi}}{2i}$

Remark 4:

In order to simplify the results, we will consider that $\cos(d\varphi) = 1 - \frac{d\varphi^2}{2}$ and $\sin(d\varphi) = d\varphi \times \sqrt{1 - \frac{d\varphi^2}{4}}$ (not only approximations).

We deduce that:

$$\cos(\varphi + d\varphi) = \cos(\varphi) \times \cos(d\varphi) - \sin(\varphi) \times \sin(d\varphi) = (1 - \frac{d\varphi^2}{2}) \times \cos(\varphi) - d\varphi \times \sqrt{1 - \frac{d\varphi^2}{4}} \times \sin(\varphi) \quad (8)$$

and:

$$\sin(\varphi + d\varphi) = \sin(\varphi) \times \cos(d\varphi) + \cos(\varphi) \times \sin(d\varphi) = (1 - \frac{d\varphi^2}{2}) \times \sin(\varphi) + d\varphi \times \sqrt{1 - \frac{d\varphi^2}{4}} \times \cos(\varphi) \quad (9)$$

Consequently:

$$\frac{d\cos(\varphi)}{d\varphi} = \frac{\cos(\varphi + d\varphi) - \cos(\varphi)}{d\varphi} = \frac{-d\varphi}{2} \times \cos(\varphi) - \sqrt{1 - \frac{d\varphi^2}{4}} \times \sin(\varphi) \quad (10)$$

and: $$\frac{d\sin(\varphi)}{d\varphi} = \frac{\sin(\varphi + d\varphi) - \sin(\varphi)}{d\varphi} = \frac{-d\varphi}{2} \times \sin(\varphi) + \sqrt{1 - \frac{d\varphi^2}{4}} \times \cos(\varphi) \quad (11)$$

and: $$\frac{d\tan(\varphi)}{d\varphi} = \frac{\sqrt{1 - \frac{d\varphi^2}{4}}}{(1 - \frac{d\varphi^2}{2}) \times \cos(\varphi)^2 - \frac{d\varphi}{2} \times \sqrt{1 - \frac{d\varphi^2}{4}} \times \sin(2\varphi)} \quad (12)$$

and: $$\frac{d\cot(\varphi)}{d\varphi} = \frac{-\sqrt{1 - \frac{d\varphi^2}{4}}}{(1 - \frac{d\varphi^2}{2}) \times \sin(\varphi)^2 + \frac{d\varphi}{2} \times \sqrt{1 - \frac{d\varphi^2}{4}} \times \sin(2\varphi)} \quad (13)$$

where we can check that:

$\sin'(\varphi) = \lim_{d\varphi \to 0} \frac{d\sin(\varphi)}{d\varphi} = \cos(\varphi)$ and: $\cos'(\varphi) = \lim_{d\varphi \to 0} \frac{d\cos(\varphi)}{d\varphi} = -\sin(\varphi)$

and: $\tan'(\varphi) = \lim_{d\varphi \to 0} \frac{d\tan(\varphi)}{d\varphi} = \frac{1}{\cos(\varphi)^2}$ and: $\cot'(\varphi) = \lim_{d\varphi \to 0} \frac{d\cot(\varphi)}{d\varphi} = \frac{-1}{\sin(\varphi)^2}$

We can also deduce by using these approximations that:

$$\int_A^B \sin(\varphi) d\varphi = -\sqrt{1 - \frac{d\varphi^2}{4}} \times (\cos(B) - \cos(A)) - \frac{d\varphi}{2} \times (\sin(B) - \sin(A)) \quad (14)$$

and: $$\int_A^B \cos(\varphi) d\varphi = \frac{-d\varphi}{2} \times (\cos(B) - \cos(A)) + \sqrt{1 - \frac{d\varphi^2}{4}} \times (\sin(B) - \sin(A)) \quad (15)$$

by integrating $\frac{d\cos(\varphi)}{d\varphi}$ and $\frac{d\sin(\varphi)}{d\varphi}$ calculated above.

These two integrals are only approximate sums since dφ is fixed as $d\varphi > 0$. However Riemann’s definition of integrals requires that dφ tends exactly to zero.

However, we can also check from above that:

$$\lim_{d\varphi\to 0}\int_A^B \sin(\varphi)d\varphi=\cos(A)-\cos(B) \quad \text{and:} \quad \lim_{d\varphi\to 0}\int_A^B \cos(\varphi)d\varphi=\sin(B)-\sin(A)$$

$\int_A^B \tan(\varphi)d\varphi$ and $\int_A^B \cot(\varphi)d\varphi$ **are left as a challenge to the readers!!!**

<u>Remark 5:</u>

We can prove easily that:

$$\cos(\varphi)=-d\cos(\varphi)-\frac{d^2\cos(\varphi)}{d\varphi^2} \tag{16}$$

and: $$\sin(\varphi)=-d\sin(\varphi)-\frac{d^2\sin(\varphi)}{d\varphi^2} \tag{17}$$

Consequently: $$\tan(\varphi)=\frac{d^2\sin(\varphi)+d\varphi^2\times d\sin(\varphi)}{d^2\cos(\varphi)+d\varphi^2\times d\cos(\varphi)} \tag{18}$$

and: $$\cot(\varphi)=\frac{d^2\cos(\varphi)+d\varphi^2\times d\cos(\varphi)}{d^2\sin(\varphi)+d\varphi^2\times d\sin(\varphi)} \tag{19}$$

And also: $$\left(\frac{d\sin(\varphi)}{d\varphi}\right)^2+\left(\frac{d\cos(\varphi)}{d\varphi}\right)^2=1 \tag{20}$$

and: $$\left(\frac{d^2\sin(\varphi)}{d\varphi^2}\right)^2+\left(\frac{d^2\cos(\varphi)}{d\varphi^2}\right)^2=1 \quad . \tag{21}$$

<u>3. Conclusion 1:</u>

Let's consider that $\vec{u}(\varphi)$ is a rotating vector that belongs to the plane of its rotation φ.

Hence:

$$\frac{d\vec{u}(\varphi)}{d\varphi}=\frac{\vec{u}(\varphi+d\varphi)-\vec{u}(\varphi)}{d\varphi}=-\|\vec{u}(\varphi)\|\times\left(\frac{d\varphi}{2}\times(\cos(\varphi)\vec{i}+\sin(\varphi)\vec{j})+\sqrt{1-\frac{d\varphi^2}{4}}\times(\sin(\varphi)\vec{i}-\cos(\varphi)\vec{j})\right)=\frac{-d\varphi}{2}\times\vec{u}(\varphi)+\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{u}\left(\varphi+\frac{\pi}{2}\right) \tag{22}$$

Consequently: $$\frac{d\vec{u}(\varphi)}{d\varphi}\times\vec{u}(\varphi)=\frac{-d\varphi}{2}\times\|\vec{u}(\varphi)\|^2\neq 0 \tag{23}$$

because dφ exists during the study.

Where: $$\left\|\frac{d\vec{u}(\varphi)}{d\varphi}\right\|=\|\vec{u}(\varphi)\| \quad \text{and:} \quad \widehat{\left(\frac{d\vec{u}(\varphi)}{d\varphi},\vec{u}(\varphi)\right)}=\arccos\left(\frac{-d\varphi}{2}\right) \tag{24}$$

<u>4. Results and applications:</u>

<u>Result 1:</u>

When using the approximations: $\cos(d\varphi)=1-\frac{d\varphi^2}{2}$ and $\sin(d\varphi)=d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}$, if $\vec{u}(\varphi)$ is a rotating vector that belongs to the plane of its rotation φ, then $\frac{d\vec{u}(\varphi)}{d\varphi}$ is not orthogonal to $\vec{u}(\varphi)$. Consequently, the famous method of reference frames change becomes false and unusable in Newtonian mechanics. Furthermore, the velocity field becomes not equiprojective for solid mechanics.

Result 2:

In cylindrical coordinates (ρ,φ,z):

$$\frac{d\vec{e}_\rho(\varphi)}{d\varphi}=\frac{-d\varphi}{2}\times\vec{e}_\rho(\varphi)+\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\varphi(\varphi) \quad (25)$$

and: $$\frac{d\vec{e}_\varphi(\varphi)}{d\varphi}=\frac{-d\varphi}{2}\times\vec{e}_\varphi(\varphi)-\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\rho(\varphi) \quad (26)$$

Important example:

We will study the case of The GPS system satellites by proving the real time at a given satellite. We will need no time dilation in this proof.

Let's consider that the speed V of the satellite is constant $V=k_1$,

By using the correct approximations above:

$$\frac{d\overrightarrow{OM}}{dt}=\frac{d\rho(t)}{dt}\times\overrightarrow{e_\rho(t)}-\frac{d\varphi}{2}\times\dot{\varphi}\times\rho(t)\times\overrightarrow{e_\rho(t)}+\rho(t)\times\dot{\varphi}\times\sqrt{1-\frac{d\varphi^2}{4}}\times\overrightarrow{e_\varphi(t)} \quad (27)$$

Where M stands for the position of the satellite that has a circular orbit of angle φ.

Since the speed vector given to the satellite is tangent to the circle of the wanted trajectory, we should have: $$\frac{d\rho(t)}{dt}\times\overrightarrow{e_\rho(t)}-\frac{d\varphi}{2}\times\dot{\varphi}\times\rho(t)\times\overrightarrow{e_\rho(t)}=\vec{0} \quad (28)$$

Hence: $$\frac{d\rho(t)}{dt}=\frac{d\varphi}{2}\times\dot{\varphi}\times\rho(t) \quad (29)$$

Consequently: $$\rho(t)=\exp(\frac{d\varphi}{2}\times\varphi+\ln(h))=h\times\exp(\frac{d\varphi}{2}\times\varphi) \quad (30)$$

Where: h is the initial altitude of the satellite.

The linear speed V of the GPS satellite is the constant speed that equals the initial speed given to the satellite in order to start orbiting.

Consequently: $$\vec{V}=\rho(t)\times\dot{\varphi}\times\sqrt{1-\frac{d\varphi^2}{4}}\times\overrightarrow{e_\varphi(t)} \quad (31)$$

And thus: $$\frac{d\varphi}{dt}=\frac{V}{h\times\sqrt{1-\frac{d\varphi^2}{4}}}\times\exp(\frac{-d\varphi}{2}\times\varphi)=\exp(\frac{-d\varphi}{2}\times\varphi+\ln(\frac{V}{h\times\sqrt{1-\frac{d\varphi^2}{4}}})) \quad (32)$$

We should remark that:

- ρ(t) increases in the beginning of the GPS satellite lifetime with a very slight change thanks to the initial launching power from earth.
- φ=0 at the point where the satellite started orbiting after being launched.
- φ is always increasing and exceeds 2π, consequently φ is always positive and $\frac{d\varphi}{dt}\geqslant 0$.
- dφ is the constant smallest variation of the satellite angle that we can detect, and dt is the time needed for that variation.

Since we consider in our study that dφ is constant and dt is the variable, we conclude that:

$$dt=\frac{h}{V}\times\sqrt{1-\frac{d\varphi^2}{4}}\times d\varphi\times\exp(\frac{d\varphi}{2}\times\varphi) \quad (33)$$

and thus dt depends on the satellite rotation angle φ.

application:

The real time ΔT needed by the satellite to make the first lap around the earth after it is launched is:

$$\Delta T=\sum_{i=1}^{K}(\frac{h}{V}\times\sqrt{1-\frac{d\varphi^2}{4}}\times d\varphi\times\exp(i\frac{d\varphi}{2}\times d\varphi))+\Delta t=\frac{h}{V}\times\sqrt{1-\frac{d\varphi^2}{4}}\times d\varphi\times\sum_{i=1}^{K}(\exp(i\frac{d\varphi^2}{2}))+\Delta t \quad (34)$$

Where: $K=\lfloor \frac{2\pi}{d\varphi} \rfloor$ (the floor of $\frac{2\pi}{d\varphi}$) (35)

and $\Delta t=\frac{h}{V}\times\sqrt{1-\frac{d\varphi^2}{4}}\times d\varphi_1\times\exp(\frac{d\varphi_1}{2}\times 2\pi)$ (36)

and $d\varphi_1=2\pi-K\times d\varphi$. (37)

<u>Result 3:</u>

In spherical coordinates(r,θ,φ):

$d\vec{e}_r=\frac{\partial\vec{e}_r}{\partial\varphi}d\varphi+\frac{\partial\vec{e}_r}{\partial\theta}\theta$ and: $d\vec{e}_\theta=\frac{\partial\vec{e}_\theta}{\partial\varphi}d\varphi+\frac{\partial\vec{e}_\theta}{\partial\theta}\theta$ and: $d\vec{e}_\varphi=\frac{\partial\vec{e}_\varphi}{\partial\varphi}d\varphi+\frac{\partial\vec{e}_\varphi}{\partial\theta}\theta$

since $(\vec{e}_r,\vec{e}_\theta,\vec{e}_\varphi)$ don't change by the variation of r but only θ and φ.

Also: $\frac{\partial\vec{e}_r}{\partial\theta}=\frac{-d\theta}{2}\times\vec{e}_r+\sqrt{1-\frac{d\theta^2}{4}}\times\vec{e}_\theta$ (38)

and: $\frac{\partial\vec{e}_\theta}{\partial\theta}=\frac{-d\theta}{2}\times\vec{e}_\theta-\sqrt{1-\frac{d\theta^2}{4}}\times\vec{e}_r$ (39)

since: $\vec{e}_r$ and $\vec{e}_\theta$ are always in the rotation plane θ.

And also: $\frac{\partial\vec{e}_\varphi}{\partial\theta}=\vec{0}$ since $\vec{e}_\varphi$ is always perpendicular to the plane of the rotation θ.

Consequently, $\vec{e}_\varphi$ doesn't change by the variation of θ.

In order to find: $\frac{\partial\vec{e}_r}{\partial\varphi}$, $\frac{\partial\vec{e}_\theta}{\partial\varphi}$ and $\frac{\partial\vec{e}_\varphi}{\partial\varphi}$, we should make the projections of $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_\varphi$ in the suitable cylindrical coordinates (ρ,φ,z) with the same plane of rotation φ that contains always $\vec{e}_\rho$ and the same vector $\vec{e}_\varphi$ of the spherical coordinates.

Hence: $\vec{e}_r=\sin(\theta)\times\vec{e}_\rho+\cos(\theta)\times\vec{k}$ and: $\vec{e}_\theta=\cos(\theta)\times\vec{e}_\rho-\sin(\theta)\times\vec{k}$

and: $\vec{e}_\varphi=\vec{e}_r\wedge\vec{e}_\theta$ where: $\vec{e}_\varphi$ is the same in the two coordinates systems.

Also: $\frac{\partial\vec{k}}{\partial\varphi}=\vec{0}$ consequently: $\frac{\partial\vec{e}_r}{\partial\varphi}=\sin(\theta)\times\frac{\partial\vec{e}_\rho}{\partial\varphi}=\sin(\theta)\times(\frac{-d\varphi}{2}\times\vec{e}_\rho+\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\varphi)$ (40)

And: $\frac{\partial\vec{e}_\theta}{\partial\varphi}=\cos(\theta)\times\frac{\partial\vec{e}_\rho}{\partial\varphi}=\cos(\theta)\times(\frac{-d\varphi}{2}\times\vec{e}_\rho+\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\varphi)$ (41)

with: $\vec{e}_\rho=\sin(\theta)\times\vec{e}_r+\cos(\theta)\times\vec{e}_\theta$

and thus:

$\frac{\partial\vec{e}_r}{\partial\varphi}=\frac{-d\varphi}{2}\times\sin(\theta)^2\times\vec{e}_r-\frac{d\varphi}{4}\times\sin(2\theta)\times\vec{e}_\theta+\sin(\theta)\times\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\varphi$ (42)

and: $\frac{\partial\vec{e}_\theta}{\partial\varphi}=\frac{-d\varphi}{4}\times\sin(2\theta)\times\vec{e}_r-\frac{d\varphi}{2}\times\cos(\theta)^2 e_\theta+\cos(\theta)\times\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\varphi$ (43)

and: $\frac{\partial\vec{e}_\varphi}{\partial\varphi}=-\sin(\theta)\times\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_r-\cos(\theta)\times\sqrt{1-\frac{d\varphi^2}{4}}\times\vec{e}_\theta-\frac{d\varphi}{2}\times\vec{e}_\varphi$ (44)

<u>5. Conclusion 2:</u>

$$d\vec{e}_r=-(\frac{d\theta^2}{2}+\frac{d\varphi^2}{2}\times\sin(\theta)^2)\times\vec{e}_r+(d\theta\times\sqrt{1-\frac{d\theta^2}{4}}-\frac{d\varphi^2}{4}\times\sin(2\theta))\times\vec{e}_\theta+d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}\times\sin(\theta)\times\vec{e}_\varphi$$

(45)

and:

$$d\vec{e}_\theta=-(d\theta\times\sqrt{1-\frac{d\theta^2}{4}}-\frac{d\varphi^2}{4}\times\sin(2\theta))\times\vec{e}_r-(\frac{d\theta^2}{2}+\frac{d\varphi^2}{2}\times\cos(\theta)^2)\times\vec{e}_\theta+d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}\times\cos(\theta)\times\vec{e}_\varphi$$

(46)

and also: $d\vec{e}_{\varphi}=-d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}\times\sin(\theta)\times\vec{e}_r-d\varphi\times\sqrt{1-\frac{d\varphi^2}{4}}\times\cos(\theta)\times\vec{e}_{\theta}-\frac{d\varphi^2}{2}\times\vec{e}_{\varphi}$ (47)

6. A new method to avoid the conventional method of changing reference frames:

We will present a new method to avoid the famous method of changing reference frames that needs the perpendicularity between a rotating vector and its derivative.
When a given reference frame with orthonormal direct axes is making any revolution, this revolution can be decomposed to three simple revolutions, and each one of these simple revolutions is around one of the three axes. However each axis changes only by two simple revolutions that are the simple revolutions around the two other axes and not by the revolution around itself.
Consequently, let's consider that α is the angle of the simple revolution around the axis $\vec{i}$, β is the angle of the simple revolution around the axis $\vec{j}$, and ɣ is the angle of the simple revolution around the axis $\vec{k}$ where $(\vec{i},\vec{j},\vec{k})$ are the three orthonormal direct axes of the reference frame that makes any given revolution. This revolution is composed of the three simple revolutions of angles: α, β and ɣ. Let's study each one of the three axes independently from each other. The simple revolution doesn't influence its axis but the two others. Hence each axis is influenced by two simple revolutions and thus we can use a spherical coordinates system to study each axis.
We will calculate $\frac{d\vec{i}}{dt}$, $\frac{d\vec{j}}{dt}$ and $\frac{d\vec{k}}{dt}$ which are the derivatives of $\vec{i}$, $\vec{j}$ and $\vec{k}$ in an absolute fixed reference frame $(\vec{Ox},\vec{Oy},\vec{Oz})$.
This method is useful in order to avoid the difficult projections of $\vec{i}$, $\vec{j}$ and $\vec{k}$ in the absolute fixed reference frame. Furthermore, the three derivatives will be calculated without making any change of reference frames. However, the functions α(t), β(t) and ɣ(t) of the three simple revolutions must be well known before making this study.

Step 1: The study of the vector $\vec{i}$:

Let's consider that $\vec{i}=\vec{e}_r$, $\vec{j}=\vec{e}_{\theta}$ and $\vec{k}=\vec{e}_{\varphi}$ where: $\vec{e}_r$, $\vec{e}_{\theta}$ and $\vec{e}_{\varphi}$ are the vectors of a spherical coordinates system.
We conclude that: ɣ=θ and α has no effects on $\vec{i}$. However, we should find the relation between φ and both β and ɣ.

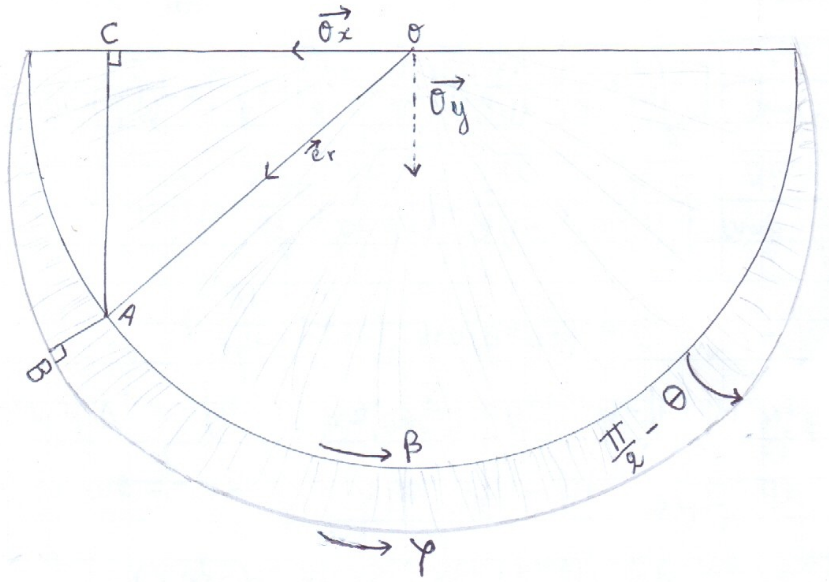

Fig.2 : *The circle of rotation β and the ellipse of rotation φ.*

The figure 2 shows that the intersection of the rotation β plane and the rotation φ plane is the axis $\overrightarrow{Ox}$ of a Cartesian coordinates system. And the orthogonal projection of the circle made by the rotation β inside the plane of $\vec{e_\varphi}$ is an ellipse made by the rotation φ. We consider that O is the center of the ellipse whereas φ is the angle between $\vec{e_\varphi}$ and $\overrightarrow{Ox}$.

$\overrightarrow{AB}$ is perpendicular to the plane of the rotation φ which is the plane of $\vec{e_\varphi}$ that is the plane of the two Cartesian vectors $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$.

$\overrightarrow{AC}$ is in the plane of the rotation β which is the plane of $\vec{i}$ and $\overrightarrow{AC}$ is perpendicular to the axis $\overrightarrow{Ox}$.

We can prove that: $AC=\sin(\beta)$ and $AB=\sin(\beta)\times\cos(\theta)$ (48)

$\overrightarrow{OB}$ is in the plane of $\vec{e_\varphi}$ with: $OB=\sqrt{1-AB^2}=\sqrt{1-\sin(\beta)^2\times\cos(\theta)^2}$ (49)

And the equation of the ellipse is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+1$

where a= (the half of the ellipse major axis), and b= (the half of the ellipse minor axis).

Consequently: a=1 and b= sin(θ)

And also: $x=OB\times\cos(\varphi)$ and: $y=OB\times\sin(\varphi)$

Therefore, the ellipse equation becomes:

$$\cos(\varphi)^2\times(1-\cos(\theta)^2\times\sin(\beta)^2)+\frac{\sin(\varphi)^2}{\cos(\theta)^2}\times(1-\cos(\theta)^2\times\sin(\beta)^2)=1 \quad (50)$$

Consequently: $$\cos(\varphi)^2=\frac{\tan(\theta)^2}{(\cos(\theta)^2\times\sin(\beta)^2)-1}+\frac{1}{\cos(\theta)^2}=\frac{\tan(\gamma)^2}{(\cos(\gamma)^2\times\sin(\beta)^2)-1}+\frac{1}{\cos(\gamma)^2} \quad (51)$$

and this equation doesn't change even if: $\varphi>\frac{\pi}{2}$ and: $\beta>\frac{\pi}{2}$.

Since we know exactly the functions β(t) and ɣ(t), we can deduce easily the function φ(t) in order to be able to calculate dφ(t), especially that: $0\leqslant\beta\leqslant\frac{\pi}{2}\Rightarrow 0\leqslant\varphi\leqslant\frac{\pi}{2}$ and: $\frac{\pi}{2}\leqslant\beta\leqslant\pi\Rightarrow\frac{\pi}{2}\leqslant\varphi\leqslant\pi$ and also: $\pi\leqslant\beta\leqslant\frac{3\pi}{2}\Rightarrow\pi\leqslant\varphi\leqslant\frac{3\pi}{2}$ and also: $\frac{3\pi}{2}\leqslant\beta\leqslant 2\pi\Rightarrow\frac{3\pi}{2}\leqslant\varphi\leqslant 2\pi$.

However we can only use this formula when $\gamma\neq 0$ and $\gamma\neq\frac{\pi}{2}$.

But we notice that if $\gamma=\frac{\pi}{2}$ and ɣ stays fixed, then always φ=β and thus we can study $\vec{i}$ easily by using cylindrical coordinates system.

Finally, by using the formula of $d\vec{e_r}$, we conclude that:

$$\frac{d\vec{i}}{dt}=-(\frac{d\theta}{2}\times\frac{d\theta}{dt}+\frac{d\varphi}{2}\times\frac{d\varphi}{dt}\times\sin(\theta)^2)\times\vec{i}+(\frac{d\theta}{dt}\times\sqrt{1-\frac{d\theta^2}{4}}-\frac{d\varphi}{4}\times\frac{d\varphi}{dt}\times\sin(2\theta))\times\vec{j}+\frac{d\varphi}{dt}\times\sqrt{1-\frac{d\varphi^2}{4}}\times\sin(\theta)\times\vec{k} \quad (52)$$

Hence:

$$\frac{d\vec{i}}{dt}=-(\frac{d\gamma}{2}\times\frac{d\gamma}{dt}+\frac{d\varphi_1}{2}\times\frac{d\varphi_1}{dt}\times\sin(\gamma)^2)\times\vec{i}+(\frac{d\gamma}{dt}\times\sqrt{1-\frac{d\gamma^2}{4}}-\frac{d\varphi_1}{4}\times\frac{d\varphi_1}{dt}\times\sin(2\gamma))\times\vec{j}+\frac{d\varphi_1}{dt}\times\sqrt{1-\frac{d\varphi_1^2}{4}}\times\sin(\gamma)\times\vec{k} \quad (53)$$

where: $\varphi_1(t)=\varphi(t)$ in step 1.

<u>Step 2: The study of the vector $\vec{j}$:</u>

By following the same method, let's consider that $\vec{i}=\vec{e_\varphi}$, $\vec{j}=\vec{e_r}$ and $\vec{k}=\vec{e_\theta}$ where: $\vec{e_r}$, $\vec{e_\theta}$ and $\vec{e_\varphi}$ are the vectors of an other spherical coordinates system.

We conclude that: α=θ and β has no effects on $\vec{j}$. However, we should find the relation between φ and both α and ɣ. And Let's consider that: $\varphi_2(t)=\varphi(t)$ in step 2.

The ellipse equation gives: $$\cos(\varphi_2)^2=\frac{\tan(\alpha)^2}{(\cos(\alpha)^2\times\sin(\gamma)^2)-1}+\frac{1}{\cos(\alpha)^2} \quad (54)$$

Consequently, we can deduce easily the function $\varphi_2(t)$.
And thus by using the formula of $d\vec{e}_r$ again, we conclude that:

$$\frac{d\vec{j}}{dt}=\frac{d\varphi_2}{dt}\times\sqrt{1-\frac{d\varphi_2^2}{4}}\times\sin(\alpha)\vec{i}-(\frac{d\alpha}{2}\times\frac{d\alpha}{dt}+\frac{d\varphi_2}{2}\times\frac{d\varphi_2}{dt}\times\sin(\alpha)^2)\times\vec{j}+(\frac{d\alpha}{dt}\times\sqrt{1-\frac{d\alpha^2}{4}}-\frac{d\varphi_2}{4}\times\frac{d\varphi_2}{dt}\times\sin(2\alpha))\times\vec{k} \quad (55)$$

<u>Step 3: The study of the vector $\vec{k}$:</u>

By following the same method, let's consider that $\vec{i}=\vec{e}_\theta$, $\vec{j}=\vec{e}_\varphi$ and $\vec{k}=\vec{e}_r$ where: $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_\varphi$ are the vectors of an other spherical coordinates system.
We conclude that: β=θ and ɣ has no effects on $\vec{k}$. However, we should find the relation between φ and both α and β. And Let's consider that: $\varphi_3(t)=\varphi(t)$ in step 3.

The ellipse equation gives: $\cos(\varphi_3)^2=\frac{\tan(\beta)^2}{(\cos(\beta)^2\times\sin(\alpha)^2)-1}+\frac{1}{\cos(\beta)^2}$ (56)

Consequently, we can deduce easily the function $\varphi_3(t)$.
And thus by using the formula of $d\vec{e}_r$ again, we conclude that:

$$\frac{d\vec{k}}{dt}=(\frac{d\beta}{dt}\times\sqrt{1-\frac{d\beta^2}{4}}-\frac{d\varphi_3}{4}\times\frac{d\varphi_3}{dt}\times\sin(2\beta))\vec{i}+\frac{d\varphi_3}{dt}\times\sqrt{1-\frac{d\varphi_3^2}{4}}\times\sin(\beta)\times\vec{j}-(\frac{d\beta}{2}\times\frac{d\beta}{dt}+\frac{d\varphi_3}{2}\times\frac{d\varphi_3}{dt}\times\sin(\beta)^2)\times\vec{k} \quad (57)$$

<u>7. An important advice:</u>
The result of this vector study should preferably be used with the formula: $\Delta E_c=\int\vec{v}\times\overrightarrow{dp}$ during the energetic study of a system, then simplifications can be made in order to find the correct expression of the kinetic energy variation ΔE_c .

<u>Part B: Differential operators:</u>
The differential operators gradient (nabla), divergence, curl and Laplacian provide information about about a field of scalars or vectors situated immediately in front of the studied point M according to the orientations of the axes of the used coordinates system.
Let's prove the formulas of each differential operator geometrically by using the three coordinates systems in an Euclidean space where the field lines are considered continuous vector functions.

<u>1. The operator gradient (nabla):</u>
Let's consider a function of locations points: $f:E\rightarrow\mathbb{R}$. Where E is the Euclidean space and the function f is differentiable and thus continuous.

Consequently: $f:\begin{matrix}E\rightarrow & \mathbb{R}\\ M\rightarrow & f(M)\end{matrix}$ makes a scalar field.

In each of the three coordinates system, the operator gradient has this form: $df=\overrightarrow{grad}f\times\overrightarrow{dM}$.

<u>The Cartesian coordinates System (x,y,z):</u>

$$df=\frac{\partial f}{\partial x}.dx+\frac{\partial f}{\partial y}.dy+\frac{\partial f}{\partial z}.dz=(\frac{\partial f}{\partial x}.\vec{e}_x+\frac{\partial f}{\partial y}.\vec{e}_y+\frac{\partial f}{\partial z}.\vec{e}_z)\times(dx.\vec{e}_x+dy.\vec{e}_y+dz.\vec{e}_z)=(\frac{\partial f}{\partial x}.\vec{e}_x+\frac{\partial f}{\partial y}.\vec{e}_y+\frac{\partial f}{\partial z}.\vec{e}_z)\times\overrightarrow{dM} \quad (58)$$

Consequently: $\vec{\nabla}f=\overrightarrow{grad}f=\frac{\partial f}{\partial x}.\vec{e}_x+\frac{\partial f}{\partial y}.\vec{e}_y+\frac{\partial f}{\partial z}.\vec{e}_z$ (59)

<u>The cylindrical coordinates system (ρ,φ,z):</u>

$$df=\frac{\partial f}{\partial\rho}.d\rho+\frac{\partial f}{\partial\varphi}.d\varphi+\frac{\partial f}{\partial z}.dz \quad (60)$$

and $\overrightarrow{dM}=d\rho.\vec{e}_\rho+\rho.d\varphi.\vec{e}_\varphi+dz.\vec{e}_z$ (61)

and: $\vec{\nabla}f=\overrightarrow{grad}f=A.\vec{e}_\rho+B.\vec{e}_\varphi+C.\vec{e}_z$ where A,B and C are the coordinates of $\vec{\nabla}f$.

Consequently: $df=\overrightarrow{grad}\,f\times\overrightarrow{dM}=A.d\rho+B.\rho.d\varphi+C.dz$ (62)

We conclude this identification: $A=\frac{\partial f}{\partial\rho}$ (63)

and: $B=\frac{1}{\rho}\times\frac{\partial f}{\partial\varphi}$ (64)

and $C=\frac{\partial f}{\partial z}$. (65)

And thus: $\vec{\nabla}f=\overrightarrow{grad}\,f=\frac{\partial f}{\partial\rho}.\vec{e}_\rho+\frac{\partial f}{\partial\varphi}\times\frac{1}{\rho}.\vec{e}_\varphi+\frac{\partial f}{\partial z}.\vec{e}_z$ (66)

<u>The spherical coordinates(r,θ,φ):</u>

$df=\frac{\partial f}{\partial r}.dr+\frac{\partial f}{\partial\theta}.d\theta+\frac{\partial f}{\partial\varphi}.d\varphi$ (67)

and $\overrightarrow{dM}=dr.\vec{e}_r+r.d\theta.\vec{e}_\theta+\sin(\theta).r.d\varphi.\vec{e}_\varphi$ (68)

and: $\vec{\nabla}f=\overrightarrow{grad}\,f=A.\vec{e}_r+B.\vec{e}_\theta+C.\vec{e}_\varphi$ where A,B and C are the coordinates of $\vec{\nabla}f$.

Consequently: $df=\overrightarrow{grad}\,f\times\overrightarrow{dM}=A.dr+B.r.d\theta+C.\sin(\theta).r.d\varphi$ (69)

We conclude this identification: $A=\frac{\partial f}{\partial r}$ (70)

and: $B=\frac{1}{r}\times\frac{\partial f}{\partial\theta}$ (71)

and $C=\frac{1}{r.\sin(\theta)}\times\frac{\partial f}{\partial\varphi}$. (72)

And thus: $\vec{\nabla}f=\overrightarrow{grad}\,f=\frac{\partial f}{\partial r}.\vec{e}_r+\frac{1}{r}\times\frac{\partial f}{\partial\theta}.\vec{e}_\theta+\frac{1}{r.\sin(\theta)}\times\frac{\partial f}{\partial\varphi}.\vec{e}_\varphi$ (73)

<u>Clarifications:</u>

- $\overrightarrow{dM}$ is an infinitesimal displacement that depends on the used coordinates system.
- The studied function f must be expressed according to the coordinates system of the used reference frame, then we use the coordinates of M in the final expression.
- $\overrightarrow{grad}\,f$ characterizes the variation of f in the space for a given displacement $\overrightarrow{dM}$.
- df changes depending on M the studied point of the space, because to each point M corresponds a value f(M), and also, $\overrightarrow{dM}$ is immediately in front of the studied point M according to the orientations of the axes of the used coordinates system . Consequently, $\overrightarrow{grad}\,f$ is a vector field that depends on the value f(M) at the location of the point M and also on the coordinates system being used.
- $df=\overrightarrow{grad}\,f\times\overrightarrow{dM}=\|\overrightarrow{grad}\,f\|\times\|\overrightarrow{dM}\|\times\cos(\Omega)$ consequently $\overrightarrow{grad}\,f$ is located at a rotation angle τ from $\overrightarrow{dM}$ in the anticlockwise orientation: $\cos(\Omega)=(\widehat{\overrightarrow{grad}\,f,\overrightarrow{dM}})$.
- The level surfaces are the space surfaces where f stays constant. And $\overrightarrow{grad}\,f$ is perpendicular to its level surfaces.

<u>2.The flow and the operator divergence:</u>

The elementary flow is: $d\phi=\vec{A}\times dS\times\vec{n}=div\,\vec{A}\times d\tau$

$div\,\vec{A}$ is a scalar field where:

- $\vec{A}$ is a vector field
- dS and dτ are consecutively the elementary surface and volume of the used coordinates system.
- $\vec{n}$ is the unit normal vector to dS.

For a closed surface we orientate $\vec{n}$ towards outside the surface. Also, dS and dτ change from a coordinates system to an other. Consequently, $div\,\vec{A}$ depends on the coordinates system being used.

The Cartesian coordinates System (x,y,z):

We remark that: dτ=dx.dy.dz and $dS_1=dS_2=dx.dy$ and $dS_3=dS_4=dy.dz$ and $dS_5=dS_6=dx.dz$.

And also: $d\phi_1=\vec{A}\times dS_1\times\vec{n}_1=-A_z\times dx\times dy$ (74)

and: $d\phi_2=\vec{A}^+\times dS_2\times\vec{n}_2=A^+{}_z\times dx\times dy$ (75)

where: $A^+{}_z=A_z+\frac{\partial A_z}{\partial z}.dz$ (76)

Consequently: $d\phi_z=d\phi_1+d\phi_2=\frac{\partial A_z}{\partial z}.dx.dy.dz=\frac{\partial A_z}{\partial z}.d\tau$ (77)

because: $\vec{n}_2=-\vec{n}_1=\vec{k}$.

And also: $d\phi_3=\vec{A}\times dS_3\times\vec{n}_3=-A_x\times dy\times dz$ (78)

and: $d\phi_4=\vec{A}^+\times dS_4\times\vec{n}_4=A^+{}_x\times dy\times dz$ (79)

where: $A^+{}_x=A_x+\frac{\partial A_x}{\partial x}.dx$ (80)

Consequently: $d\phi_x=d\phi_3+d\phi_4=\frac{\partial A_x}{\partial x}.dx.dy.dz=\frac{\partial A_x}{\partial x}.d\tau$ (81)

because: $\vec{n}_4=-\vec{n}_3=\vec{i}$.

And also: $d\phi_5=\vec{A}\times dS_5\times\vec{n}_5=-A_y\times dx\times dz$ (82)

and: $d\phi_6=\vec{A}^+\times dS_6\times\vec{n}_6=A^+{}_y\times dx\times dz$ (83)

where: $A^+{}_y=A_y+\frac{\partial A_y}{\partial y}.dy$ (84)

Consequently: $d\phi_y=d\phi_5+d\phi_6=\frac{\partial A_y}{\partial y}.dx.dy.dz=\frac{\partial A_y}{\partial y}.d\tau$ (85)

because: $\vec{n}_6=-\vec{n}_5=\vec{j}$.

And thus: $d\phi=d\phi_x+d\phi_y+d\phi_z=(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z})\times d\tau$ (86)

We finally conclude that: $div\vec{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ (87)

The cylindrical coordinates system (ρ,φ,z):

In the cylindrical coordinates system: $\vec{OM}=\rho\times\vec{e}_\rho+z\times\vec{k}$ where M is a point of the space.

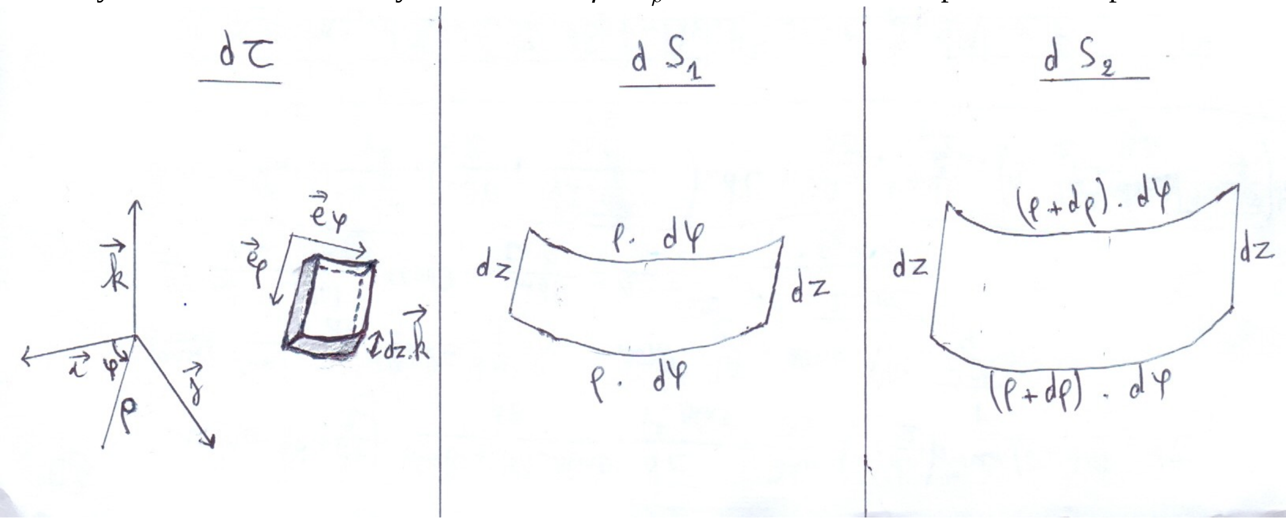

Fig.3 : *The elementary volume dτ and the elementary surfaces dS_1 and dS_2.*

In the figure 3, the elementary volume is: $d\tau=\rho\times d\varphi\times d\rho\times dz$. (88)

Also: $dS_1=\rho\times d\varphi\times dz$ (89)

and $dS_2=(\rho+d\rho)\times d\varphi\times dz$ (90)

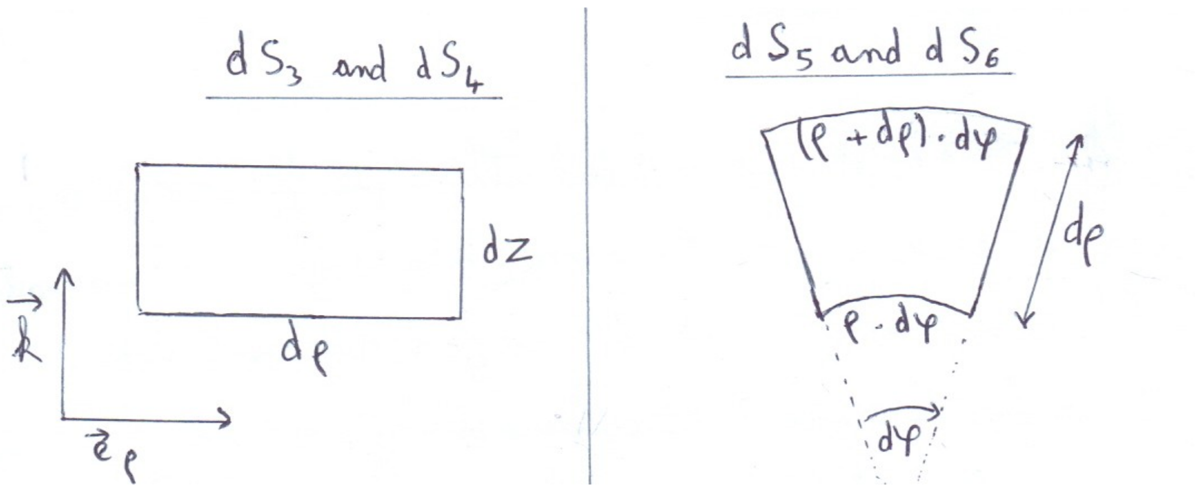

Fig.4 : *The elementary similar surfaces dS_3, dS_4, and the elementary similar surfaces dS_5 and dS_6.*

In the figure 3, we prove by using circular sectors that: $dS_3 = dS_4 = d\rho \times dz$ (91)

And: $dS_5 = dS_6 = d\rho \times d\varphi \times (\rho + \frac{d\rho}{2})$ (92)

Hence: $d\phi_1 = \vec{A} \times dS_1 \times \vec{n}_1 = -A_\rho \times \rho \times d\varphi \times dz$ (93)

and: $d\phi_2 = \overrightarrow{A^+} \times dS_2 \times \vec{n}_2 = A^+{}_\rho \times (\rho + d\rho) \times d\varphi \times dz$ (94)

where: $A^+{}_\rho = A_\rho + \frac{\partial A_\rho}{\partial \rho} . d\rho$ (95)

Consequently: $d\phi_\rho = d\phi_1 + d\phi_2 = (\frac{A_\rho}{\rho} + \frac{\partial A_\rho}{\partial \rho} \times (1 + \frac{d\rho}{\rho})) . d\tau$ (96)

because: $\vec{n}_2 = -\vec{n}_1 = \vec{e}_\rho$.

And also: $d\phi_3 = \vec{A} \times dS_3 \times \vec{n}_3 = -A_\varphi \times d\rho \times dz$ (97)

and: $d\phi_4 = \overrightarrow{A^+} \times dS_4 \times \vec{n}_4 = A^+{}_\varphi \times d\rho \times dz$ (98)

where: $A^+{}_\varphi = A_\varphi + \frac{\partial A_\varphi}{\partial \varphi} . d\varphi$ (99)

Consequently: $d\phi_\varphi = d\phi_3 + d\phi_4 = \frac{\partial A_\varphi}{\partial \varphi} \times d\varphi \times d\rho \times dz = \frac{1}{\rho} \times \frac{\partial A_\varphi}{\partial \varphi} \times d\tau$ (100)

because: $\vec{n}_4 = -\vec{n}_3 = \vec{e}_\varphi$.

And also: $d\phi_5 = \vec{A} \times dS_5 \times \vec{n}_5 = -A_z \times d\rho \times d\varphi \times (\rho + \frac{d\rho}{2})$ (101)

and: $d\phi_6 = \overrightarrow{A^+} \times dS_6 \times \vec{n}_6 = A^+{}_z \times d\rho \times d\varphi \times (\rho + \frac{d\rho}{2})$ (102)

where: $A^+{}_z = A_z + \frac{\partial A_z}{\partial z} . dz$ (103)

Consequently: $d\phi_z = d\phi_5 + d\phi_6 = \frac{\partial A_z}{\partial z} \times d\varphi \times d\rho \times dz \times (\rho + \frac{d\rho}{2}) = (1 + \frac{d\rho}{2\rho}) \times \frac{\partial A_z}{\partial z} \times d\tau$ (104)

because: $\vec{n}_6 = -\vec{n}_5 = \vec{k}$.

Hence: $d\phi = d\phi_\rho + d\phi_\varphi + d\phi_z = (\frac{A_\rho}{\rho} + (1 + \frac{d\rho}{\rho}) \times \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \times \frac{\partial A_\varphi}{\partial \varphi} + (1 + \frac{d\rho}{2\rho}) \times \frac{\partial A_z}{\partial z}) \times d\tau$ (105)

We finally conclude that: $div\,\vec{A} = \frac{A_\rho}{\rho} + (1 + \frac{d\rho}{\rho}) \times \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \times \frac{\partial A_\varphi}{\partial \varphi} + (1 + \frac{d\rho}{2\rho}) \times \frac{\partial A_z}{\partial z}$ (106)

<u>The spherical coordinates(r,θ,φ):</u>

In the spherical coordinates: $\overrightarrow{OM} = r \times \vec{e}_r$ and $\vec{e}_r$ is always in the plane of the rotation θ. And the elementary volume is: $d\tau = r^2 \times \sin(\theta) \times d\theta \times d\varphi \times dr$. (107)

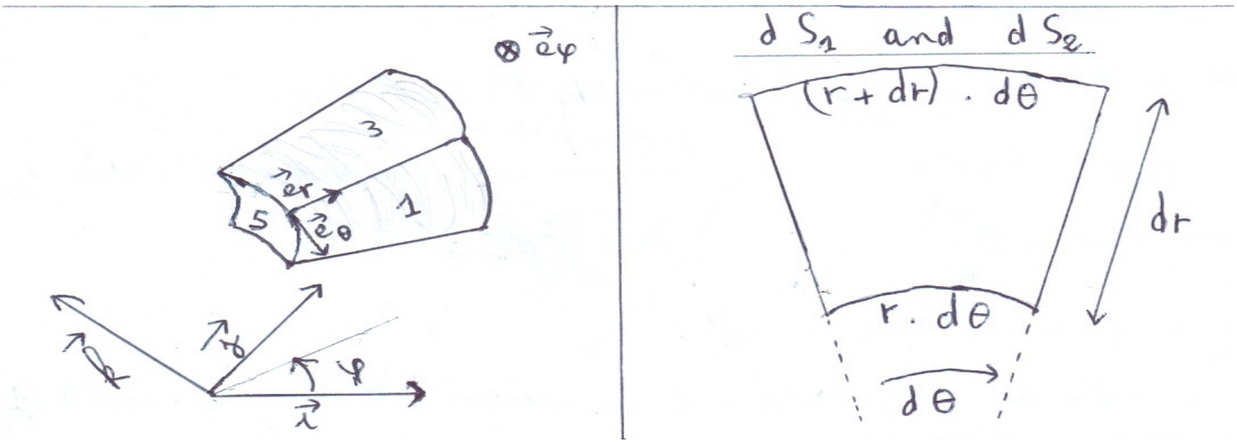

Fig.5 : *The six elementary surfaces forming an infinitesimal volume. And the elementary similar surfaces dS_1 and dS_2.*

In the figure 5, the vector that is the radius of the circular arc made by dφ is always in the plane of $\vec{e}_\varphi$. And by using circular sectors: $dS_1 = dS_2 = (r+\frac{dr}{2}) \times d\,\theta \times dr$. (108)

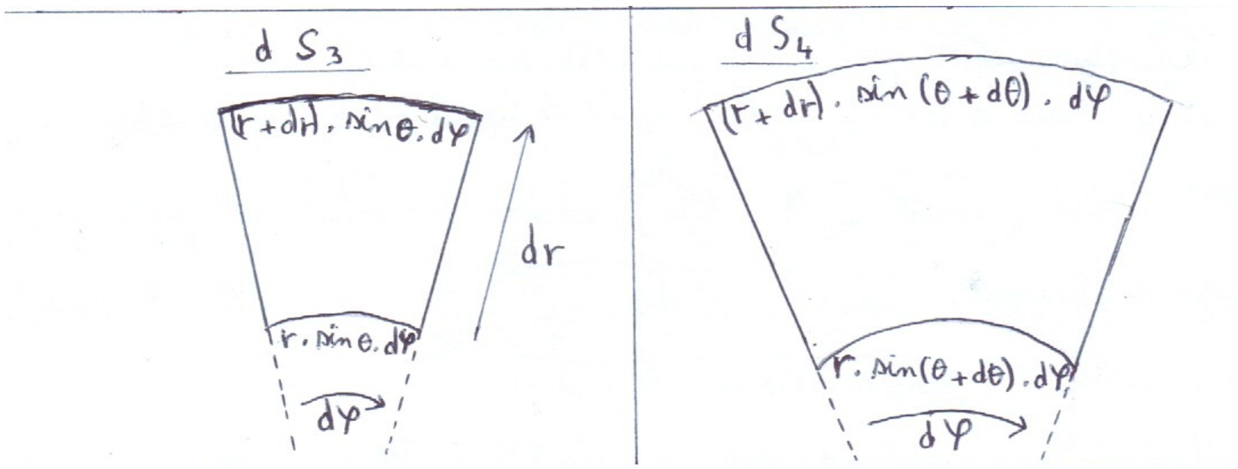

Fig.6 : *The elementary surfaces dS_3 and dS_4.*

In the figure 6: $dS_3 = (r+\frac{dr}{2}) \times \sin(\theta)^2 \times d\,\varphi \times dr$ (109)

and: $dS_4 = (r+\frac{dr}{2}) \times \sin(\theta + d\,\theta)^2 \times d\,\varphi \times dr$ (110)

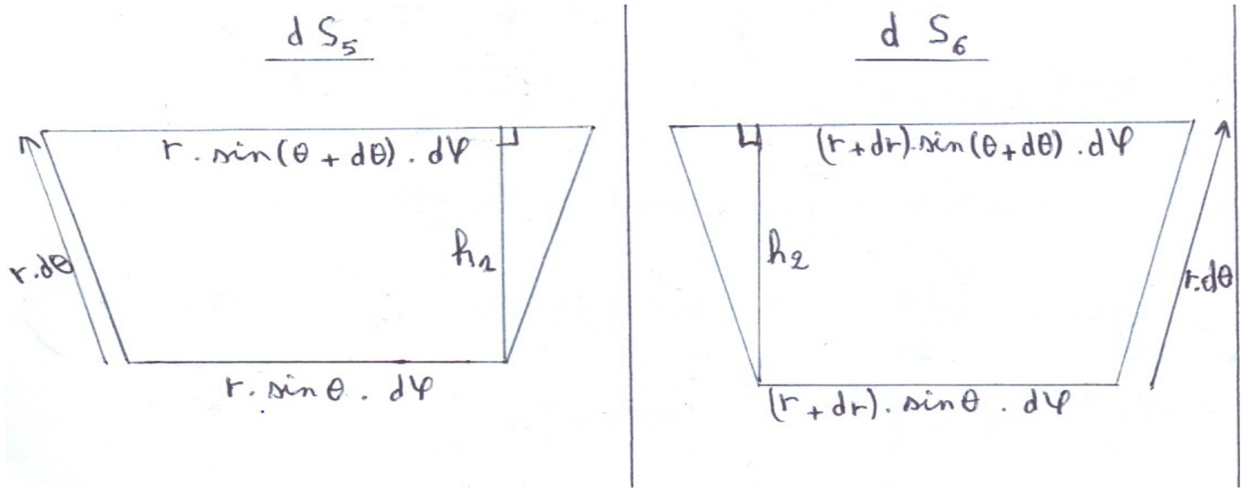

Fig.7 : *The elementary surfaces dS_5 and dS_6.*

In the figure 7, we flatten the surface dS_5 consequently we get a trapezoid shape that has the following height h_1:

$$h_1=(r^2\times d\theta^2+r^2\times d\varphi^2\times(\sin(\theta+d\theta)-\sin(\theta))^2)^{\frac{1}{2}} \quad (111)$$

Hence:

$$dS_5=\frac{h_1\times(r\times d\varphi\times\sin(\theta+d\theta)+r\times d\varphi\times\sin(\theta))}{2}=\frac{r^2}{4}\times d\varphi\times d\theta\times(4(\sin(\theta+d\theta)+\sin(\theta))^2-d\varphi^2\times(\frac{\sin(\theta+d\theta)^2-\sin(\theta)^2}{d\theta})^2)^{\frac{1}{2}}$$

(112)

We make the same with the surface dS_6 and we get a trapezoid shape that has the following height h_2: $h_2=((r+dr)^2\times d\theta^2+(r+dr)^2\times d\varphi^2\times(\sin(\theta+d\theta)-\sin(\theta))^2)^{\frac{1}{2}}$ (113)

And:

$$dS_6=\frac{h_2\times((r+dr)\times d\varphi\times\sin(\theta+d\theta)+(r+dr)\times d\varphi\times\sin(\theta))}{2}=\frac{(r+dr)^2}{4}\times d\varphi\times d\theta\times(4(\sin(\theta+d\theta)+\sin(\theta))^2-d\varphi^2\times(\frac{\sin(\theta+d\theta)^2-\sin(\theta)^2}{d\theta})^2)^{\frac{1}{2}}$$

(114)

We have: $d\phi_1=\vec{A}\times dS_1\times\vec{n}_1=-A_\varphi\times d\theta\times dr\times(r+\frac{dr}{2})$ (115)

and: $d\phi_2=\overrightarrow{A^+}\times dS_2\times\vec{n}_2=A^+{}_\varphi\times d\theta\times dr\times(r+\frac{dr}{2})$ (116)

where: $A^+{}_\varphi=A_\varphi+\frac{\partial A_\varphi}{\partial\varphi}.d\varphi$ (117)

Consequently: $d\phi_\varphi=d\phi_1+d\phi_2=\frac{\partial A_\varphi}{\partial\varphi}\times d\theta\times d\varphi\times dr\times(r+\frac{dr}{2})=(\frac{1+\frac{dr}{2r}}{r\times\sin(\theta)})\times\frac{\partial A_\varphi}{\partial\varphi}\times d\tau$ (118)

because: $\vec{n}_2=-\vec{n}_1=\vec{e}_\varphi$.

And also: $d\phi_3=\vec{A}\times dS_3\times\vec{n}_3=-A_\theta\times\sin(\theta)^2\times d\varphi\times dr\times(r+\frac{dr}{2})$ (119)

and: $d\phi_4=\overrightarrow{A^+}\times dS_4\times\vec{n}_4=A^+{}_\theta\times\sin(\theta+d\theta)^2\times d\varphi\times dr\times(r+\frac{dr}{2})$ (120)

because: $\vec{n}_4=-\vec{n}_3=\vec{e}_\theta$ where: $A^+{}_\theta=A_\theta+\frac{\partial A_\theta}{\partial\theta}.d\theta$ (121)

Consequently:

$$d\phi_\theta=d\phi_3+d\phi_4=-A_\theta\times\sin(\theta)^2\times d\varphi\times dr\times(r+\frac{dr}{2})+A_\theta\times\sin(\theta+d\theta)^2\times d\varphi\times dr\times(r+\frac{dr}{2})+\frac{\partial A_\theta}{\partial\theta}\times\sin(\theta+d\theta)^2\times d\theta\times d\varphi\times dr\times(r+\frac{dr}{2})$$

(122)

Hence:

$$d\phi_\theta=A_\theta\times d\varphi\times dr\times(r+\frac{dr}{2})\times(\sin(\theta+d\theta)^2-\sin(\theta)^2)+\frac{\partial A_\theta}{\partial\theta}\times\frac{\sin(\theta+d\theta)^2}{\sin(\theta)\times r^2}\times(r+\frac{dr}{2})\times d\tau=A_\theta\times(r+\frac{dr}{2})\times\frac{(\sin(\theta+d\theta)^2-\sin(\theta)^2)}{d\theta}\times\frac{d\tau}{\sin(\theta)\times r^2}+\frac{\partial A_\theta}{\partial\theta}\times\frac{\sin(\theta+d\theta)^2}{\sin(\theta)\times r^2}\times(r+\frac{dr}{2})\times d\tau$$

(123)

Let's consider that:

$$B(\theta)=\frac{\sin(\theta+d\theta)^2-\sin(\theta)^2}{d\theta}=\frac{d\sin(\theta)^2}{d\theta}=2\times\sin(\theta)\times\frac{d\sin(\theta)}{d\theta} \quad (124)$$

And also: $D(\theta)=\frac{\sin(\theta+d\theta)^2}{\sin(\theta)}$ (125)

We conclude that:

$$d\phi_\theta=A_\theta\times(\frac{1}{r}+\frac{dr}{2r^2})\times B(\theta)\times\frac{d\tau}{\sin(\theta)}+\frac{\partial A_\theta}{\partial\theta}\times D(\theta)\times(\frac{1}{r}+\frac{dr}{2r^2})\times d\tau \quad (126)$$

For: dS_5 and dS_6 :

Let's consider that: $F(\theta)=(\sin(\theta+d\theta)+\sin(\theta))^2$ (127)

and: $G(\theta)=(\frac{\sin(\theta+d\theta)^2-\sin(\theta)^2}{d\theta})^2=(\frac{d(\sin(\theta)^2)}{d\theta})^2=(2\times\sin(\theta)\times\frac{dsin(\theta)}{d\theta})^2$ (128)

Consequently: $d\phi_5=\vec{A}\times dS_5\times\vec{n_5}=-A_r\times\frac{r^2}{4}\times d\varphi\times d\theta\times(4F(\theta)-d\varphi^2\times G(\theta))^{\frac{1}{2}}$ (129)

and: $d\phi_6=\vec{A^+}\times dS_6\times\vec{n_6}=A^+{}_r\times\frac{(r+dr)^2}{4}\times d\varphi\times d\theta\times(4F(\theta)-d\varphi^2\times G(\theta))^{\frac{1}{2}}$ (130)

because: $\vec{n_6}=-\vec{n_5}=\vec{e_r}$ and: $A^+{}_r=A_r+\frac{\partial A_r}{\partial r}.dr$ (131)

We conclude that: $d\phi_r=(A_r\times(\frac{dr}{4r^2}+\frac{1}{2r})+\frac{\partial A_r}{\partial r}\times(\frac{1}{2}+\frac{dr}{2r})^2)\times\frac{(4.F(\theta)-d\varphi^2\times G(\theta))^{\frac{1}{2}}}{\sin(\theta)}\times d\tau$ (132)

And thus:

$$d\phi=d\phi_r+d\phi_\theta+d\phi_\varphi=(\frac{1+\frac{dr}{2r}}{r\times\sin(\theta)})\times\frac{\partial A_\varphi}{\partial\varphi}\times d\tau+A_\theta\times(\frac{1}{r}+\frac{dr}{2r^2})\times B(\theta)\times\frac{d\tau}{\sin(\theta)}+\frac{\partial A_\theta}{\partial\theta}\times D(\theta)\times(\frac{1}{r}+\frac{dr}{2r^2})\times d\tau+(A_r\times(\frac{dr}{4r^2}+\frac{1}{2r})+\frac{\partial A_r}{\partial r}\times(\frac{1}{2}+\frac{dr}{2r})^2)\times\frac{(4.F(\theta)-d\varphi^2\times G(\theta))^{\frac{1}{2}}}{\sin(\theta)}\times d\tau$$

(133)

We finally conclude that:

$$div\vec{A}=(\frac{1+\frac{dr}{2r}}{r\times\sin(\theta)})\times\frac{\partial A_\varphi}{\partial\varphi}+A_\theta\times(\frac{1}{r}+\frac{dr}{2r^2})\times\frac{B(\theta)}{\sin(\theta)}+\frac{\partial A_\theta}{\partial\theta}\times D(\theta)\times(\frac{1}{r}+\frac{dr}{2r^2})+(A_r\times(\frac{dr}{4r^2}+\frac{1}{2r})+\frac{\partial A_r}{\partial r}\times(\frac{1}{2}+\frac{dr}{2r})^2)\times\frac{(4.F(\theta)-d\varphi^2\times G(\theta))^{\frac{1}{2}}}{\sin(\theta)}$$

(134)

<u>By using the demonstrated approximations:</u>

$$B(\theta)=-d\theta\times\sin(\theta)^2+\sqrt{1-\frac{d\theta^2}{4}}\times 2\times\sin(\theta)\times\cos(\theta)=\frac{-d\theta}{2}\times(1-\cos(2\theta))+\sqrt{1-\frac{d\theta^2}{4}}\times\sin(2\theta)$$

(135)

And:

$$D(\theta)=(1-\frac{d\theta^2}{2})^2\times\sin(\theta)+2d\theta\times(1-\frac{d\theta^2}{2})\times\sqrt{1-\frac{d\theta^2}{4}}\times\cos(\theta)+d\theta^2\times(1-\frac{d\theta^2}{4})\times\frac{\cos(\theta)}{\tan(\theta)}$$ (136)

And Also:

$$F(\theta)=(\sqrt{2}-\frac{d\theta^2}{2\sqrt{2}})^2\times(1-\cos(2\theta))+\frac{d\theta^2}{2}\times(1-\frac{d\theta^2}{4})\times(1+\cos(2\theta))+(4-3d\theta^2+\frac{3}{4}\times d\theta^4-\frac{d\theta^6}{16})^{\frac{1}{2}}\times d\theta\times\sin(2\theta)$$

(137)

And:

$$G(\theta)=B(\theta)^2=(\frac{d\theta^2}{4}+\frac{1}{2})+\cos(4\theta)\times(\frac{d\theta^2}{4}-\frac{1}{2})-\frac{d\theta^2}{2}\times\cos(2\theta)+\frac{d\theta}{2}\times\sqrt{1-\frac{d\theta^2}{4}}\times\sin(4\theta)-d\theta\times\sqrt{1-\frac{d\theta^2}{4}}\times\sin(2\theta)$$

(138)

<u>Clarifications:</u>

- In order to calculate the divergence of $\vec{A}$ in a point of the space M(x,y,z), we should replace by the data of $\vec{A}$ in the expression of the chosen reference frame coordinates system. Then, we integrate the final expression by using the coordinates of the studied part of the space if the integration is possible.
- The six vectors $\vec{n_i}$ that are normal to the elementary surfaces are oriented towards outside. Consequently, for a uniform field $\vec{A}$, when $d\phi>0$, the field vectors that are in the studied part of the space, in the orientation of the used reference frame axes, have the same

orientations of the vectors $\vec{n}_i$. And thus the field $\vec{A}$ is divergent in the studied part of the space and $\frac{d\phi}{d\tau}=div\vec{A}>0$.

- In the case when $d\phi<0$, the field vectors in the studied part of the space have orientations opposite to the vectors $\vec{n}_i$ orientations. Consequently the field $\vec{A}$ is convergent in the studied part of the space and $\frac{d\phi}{d\tau}=div\vec{A}<0$.
- In the case when $d\phi=0$, and the field $\vec{A}$ exists in the studied part of the space, this means that every field vector has its opposite across each elementary surface. In this case $\frac{d\phi}{d\tau}=div\vec{A}=0$, otherwise the field is tangent to all the elementary surfaces and thus it is a rotational field.

3. The operator curl:

In an orthonormed direct reference frame of axes $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$, we can define the operator curl as:

$$\vec{rot}\,\vec{A}=\begin{matrix}\vec{rot}\,\vec{A}.\vec{e}_1\\ \vec{rot}\,\vec{A}.\vec{e}_2\\ \vec{rot}\,\vec{A}.\vec{e}_3\end{matrix}$$

And it is a vector field that respects the following Stokes' theorem:
$$\begin{matrix}\vec{rot}\,\vec{A}.\vec{e}_1.dS_{23}=dC_1\\ \vec{rot}\,\vec{A}.\vec{e}_2.dS_{13}=dC_2\\ \vec{rot}\,\vec{A}.\vec{e}_3.dS_{12}=dC_3\end{matrix}$$

Where: dS_{12}, is the elementary surface in the plane $(\vec{e}_1,\vec{e}_2)$,dS_{13} is the elementary surface in the plane $(\vec{e}_1,\vec{e}_3)$ and dS_{23} is the elementary surface in the plane $(\vec{e}_2,\vec{e}_3)$, and Also: dC_1,dC_2 and dC_3 are respectively their closed boundaries of the corresponding elementary surfaces in the reference frame planes. These boundaries are outlines oriented anticlockwise when we are observing the surfaces from the inside (center) of the reference frame.

We prove that:
$$\begin{matrix}dC_1=\overrightarrow{A_{11}}\times d_{11}\times\vec{e}_2+\overrightarrow{A_{12}}\times d_{12}\times\vec{e}_3-\overrightarrow{A_{13}}\times d_{13}\times\vec{e}_2-\overrightarrow{A_{14}}\times d_{14}\times\vec{e}_3\\ dC_2=\overrightarrow{A_{21}}\times d_{21}\times\vec{e}_3+\overrightarrow{A_{22}}\times d_{22}\times\vec{e}_1-\overrightarrow{A_{23}}\times d_{23}\times\vec{e}_3-\overrightarrow{A_{24}}\times d_{24}\times\vec{e}_1\\ dC_3=\overrightarrow{A_{31}}\times d_{31}\times\vec{e}_2+\overrightarrow{A_{32}}\times d_{32}\times\vec{e}_1-\overrightarrow{A_{33}}\times d_{33}\times\vec{e}_2-\overrightarrow{A_{34}}\times d_{34}\times\vec{e}_1\end{matrix} \quad (139)$$

Where: d_{ij} is the elementary length of the elementary surface sides. And A_{ij} is the field vector of $\vec{A}$ that coincides with d_{ij}.

The boundaries are drawed by starting from the studied point M(x,y,z). Consequently, the sides of the elementary surfaces have elementary sides d_{ij} in common.

The Cartesian coordinates System (x,y,z):

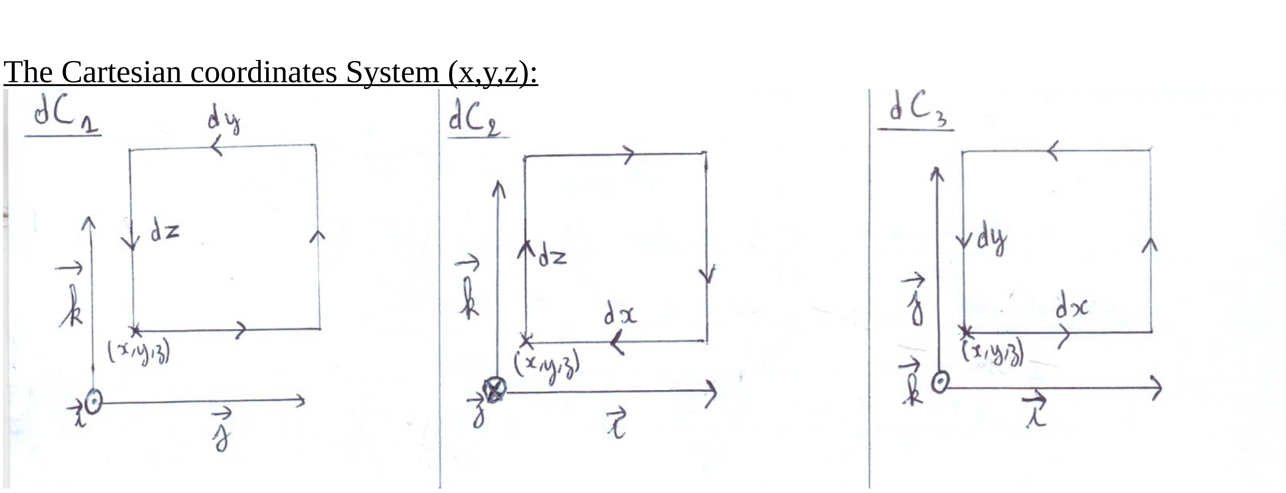

Fig.8 : *The boundaries dC_1, dC_2 and dC_3 of the cartesian coordinates.*

In the figure 8: dS_{23}=dy.dz (140)
and: dS_{13}=dx.dz (141)
and dS_{12}=dx.dy (142)

And: $dC_1 = \overrightarrow{A_{11}} \times d_{11} \times \vec{e_2} + \overrightarrow{A_{12}} \times d_{12} \times \vec{e_3} - \overrightarrow{A_{13}} \times d_{13} \times \vec{e_2} - \overrightarrow{A_{14}} \times d_{14} \times \vec{e_3}$ (143)

Consequently: $dC_1 = -A_{z_1} \times dz - A^+_{y_1} \times dy + A^+_{z_1} \times dz + A_{y_1} \times dy$ (144)

By following the same method we prove that:

$dC_2 = -A_{x_2} \times dx - A^+_{z_2} \times dz + A^+_{x_2} \times dx + A_{z_2} \times dz$ (145)

And: $dC_3 = A_{x_3} \times dx + A^+_{y_3} \times dy - A^+_{x_3} \times dx - A_{z_3} \times dz$ (146)

Where: $A^+_{y_1} = A_{y_1} + \frac{\partial A_{y_1}}{\partial z} . dz$ (147)

and: $A^+_{z_1} = A_{z_1} + \frac{\partial A_{z_1}}{\partial y} . dy$ (148)

and also: $A^+_{x_2} = A_{x_2} + \frac{\partial A_{x_2}}{\partial z} . dz$ (150)

and: $A^+_{z_2} = A_{z_2} + \frac{\partial A_{z_2}}{\partial x} . dx$ (152)

and also: $A^+_{x_3} = A_{x_3} + \frac{\partial A_{x_3}}{\partial y} . dy$ (153)

and: $A^+_{y_3} = A_{y_3} + \frac{\partial A_{y_3}}{\partial x} . dx$ (154)

with: $A_x = A_{x_2} = A_{x_3}$ and: $A_y = A_{y_1} = A_{y_2}$ and: $A_z = A_{z_1} = A_{z_3}$ (155)

because the sides of the elementary surfaces have elementary sides d_{ij} in common.

And thus:

$$dC_1 = -A_z \times dz - (A_y + \frac{\partial A_y}{\partial z} . dz) \times dy + (A_z + \frac{\partial A_z}{\partial y} . dy) \times dz + A_y \times dy$$

$$dC_2 = -A_x \times dx - (A_z + \frac{\partial A_z}{\partial x} . dx) \times dz + (A_x + \frac{\partial A_x}{\partial z} . dz) \times dx + A_z \times dz \quad (156)$$

$$dC_3 = A_x \times dx + (A_y + \frac{\partial A_y}{\partial x} . dx) \times dy - (A_x + \frac{\partial A_x}{\partial y} . dy) \times dx - A_z \times dz$$

Consequently: $dC_1 = \frac{-\partial A_y}{\partial z} . dz \times dy + \frac{\partial A_z}{\partial y} . dy \times dz = \overrightarrow{rot}\, \vec{A} \times \vec{i} \times dS_{23}$ (157)

where: dS_{23}=dy.dz (158)

And: $dC_2 = \frac{-\partial A_z}{\partial x} . dz \times dx + \frac{\partial A_x}{\partial z} . dx \times dz = \overrightarrow{rot}\, \vec{A} \times \vec{j} \times dS_{13}$ (159)

where: dS_{13}=dx.dz (160)

And also: $dC_3 = \frac{-\partial A_x}{\partial y} . dy \times dx + \frac{\partial A_y}{\partial x} . dx \times dy = \overrightarrow{rot}\, \vec{A} \times \vec{k} \times dS_{12}$ (161)

where: dS_{12}=dx.dy (162)

We conclude finally that:

$$\overrightarrow{rot}\, \vec{A} = \begin{matrix} \frac{-\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} \\ \frac{-\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} \\ \frac{-\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} \end{matrix} \quad (163)$$

The cylindrical coordinates system (ρ,φ,z):

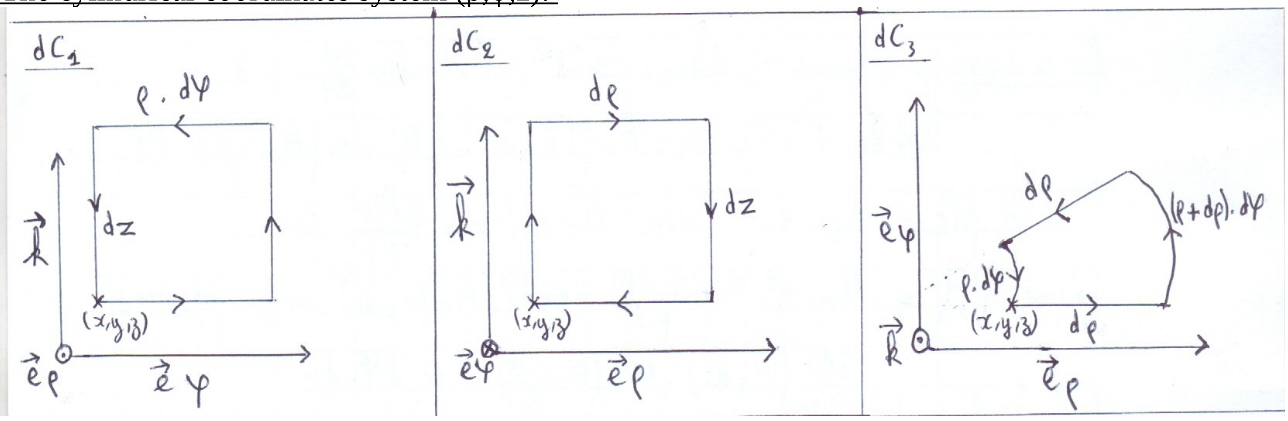

Fig.9 : *The boundaries dC_1, dC_2 and dC_3 of the cylindrical coordinates.*

In the figure 9: $dS_{23}=\rho\times d\varphi\times dz$ (164)

and: $dS_{13}=d\rho\times dz$ (165)

and: $dS_{12}=(\rho+\frac{d\rho}{2})\times d\varphi\times d\rho$ (166)

And: $dC_1=A_{\varphi_1}\times\rho\times d\varphi+A^+_{z_1}\times dz-A^+_{\varphi_1}\times\rho\times d\varphi-A_{z_1}\times dz$ (167)

And also: $dC_2=A_{z_2}\times dz+A^+_{\rho_2}\times d\rho-A_{\rho_2}\times d\rho-A^+_{z_2}\times dz$ (168)

And also: $dC_3=A_{\rho_3}\times d\rho+A^+_{\varphi_3}\times(\rho+d\rho)\times d\varphi-A^+_{\rho_3}\times d\rho-A_{\varphi_3}\times\rho\times d\varphi$ (169)

Where: $A^+_{z_1}=A_{z_1}+\frac{\partial A_{z_1}}{\partial\varphi}.d\varphi$ (170)

and: $A^+_{\varphi_1}=A_{\varphi_1}+\frac{\partial A_{\varphi_1}}{\partial z}.dz$ (171)

and also: $A^+_{\rho_2}=A_{\rho_2}+\frac{\partial A_{\rho_2}}{\partial z}.dz$ (172)

and: $A^+_{z_2}=A_{z_2}+\frac{\partial A_{z_2}}{\partial\rho}.d\rho$ (173)

and also: $A^+_{\varphi_3}=A_{\varphi_3}+\frac{\partial A_{\varphi_3}}{\partial\rho}.d\rho$ (174)

and: $A^+_{\rho_3}=A_{\rho_3}+\frac{\partial A_{\rho_3}}{\partial\varphi}.d\varphi$ (175)

with: $A_\rho=A_{\rho_2}=A_{\rho_3}$ and: $A_\varphi=A_{\varphi_1}=A_{\varphi_3}$ and: $A_z=A_{z_1}=A_{z_2}$ (176)

because the sides of the elementary surfaces have elementary sides d_{ij} in common.

And thus:

$$dC_1=A_\varphi\times\rho\times d\varphi+(A_z+\frac{\partial A_z}{\partial\varphi}.d\varphi)\times dz-(A_\varphi+\frac{\partial A_\varphi}{\partial z}.dz)\times\rho\times d\varphi-A_z\times dz$$

$$dC_2=A_z\times dz+(A_\rho+\frac{\partial A_\rho}{\partial z}.dz)\times d\rho-(A_z+\frac{\partial A_z}{\partial\rho}.d\rho)\times dz-A_\rho\times d\rho$$

$$dC_3=A_\rho\times d\rho+(A_\varphi+\frac{\partial A_\varphi}{\partial\rho}.d\rho)\times(\rho+d\rho)\times d\varphi-(A_\rho+\frac{\partial A_\rho}{\partial\varphi}.d\varphi)\times d\rho-A_\varphi\times\rho\times d\varphi$$

(177)

Consequently: $dC_1=\frac{-\partial A_\varphi}{\partial z}.dz\times\rho\times d\varphi+\frac{\partial A_z}{\partial\varphi}.d\varphi\times dz=\vec{rot}\,\vec{A}\times\vec{e_\rho}\times dS_{23}$ (178)

Where: $dS_{23}=\rho\times d\varphi\times dz$ (179)

And: $dC_2=\frac{-\partial A_z}{\partial\rho}.dz\times d\rho+\frac{\partial A_\rho}{\partial z}.d\rho\times dz=\vec{rot}\,\vec{A}\times\vec{e_\varphi}\times dS_{13}$ (180)

Where: $dS_{13}=d\rho\times dz$ (181)

And also:

$$dC_3 = \frac{-\partial A_\rho}{\partial \varphi}.d\varphi \times d\rho + \frac{\partial A_\varphi}{\partial \rho}.d\rho \times \rho \times d\varphi + (A_\varphi + \frac{\partial A_\varphi}{\partial \rho} \times d\rho) \times d\rho \times d\varphi = \vec{rot}\,\vec{A} \times \vec{k} \times dS_{12} \quad (182)$$

Where: $dS_{12} = (\rho + \frac{d\rho}{2}) \times d\varphi \times d\rho$ (183)

Hence: $\frac{-\partial A_\varphi}{\partial z} + \frac{1}{\rho}.\frac{\partial A_z}{\partial \varphi} = \vec{rot}\,\vec{A} \times \vec{e_\rho}$ (184)

And: $\frac{-\partial A_z}{\partial \rho} + \frac{\partial A_\rho}{\partial z} = \vec{rot}\,\vec{A} \times \vec{e_\varphi}$ (185)

And also: $\frac{-\partial A_\rho}{\partial \varphi}.\frac{2}{2\rho + d\rho} + \frac{\partial A_\varphi}{\partial \rho}.\frac{2\rho}{2\rho + d\rho} + \frac{\partial A_\varphi}{\partial \rho}.\frac{2d\rho}{2\rho + d\rho} + A_\varphi.\frac{2}{2\rho + d\rho} = \vec{rot}\,\vec{A} \times \vec{k}$ (186)

We conclude finally that: $\vec{rot}\,\vec{A} = \begin{matrix} \frac{-\partial A_\varphi}{\partial z} + \frac{1}{\rho}.\frac{\partial A_z}{\partial \varphi} \\ \frac{-\partial A_z}{\partial \rho} + \frac{\partial A_\rho}{\partial z} \\ \frac{-2}{2\rho + d\rho}.\frac{\partial A_\rho}{\partial \varphi} + \frac{2(\rho + d\rho)}{2\rho + d\rho}.\frac{\partial A_\varphi}{\partial \rho} + \frac{2}{2\rho + d\rho}.A_\varphi \end{matrix}$ (187)

¿

<u>The spherical coordinates(r,θ,φ):</u>

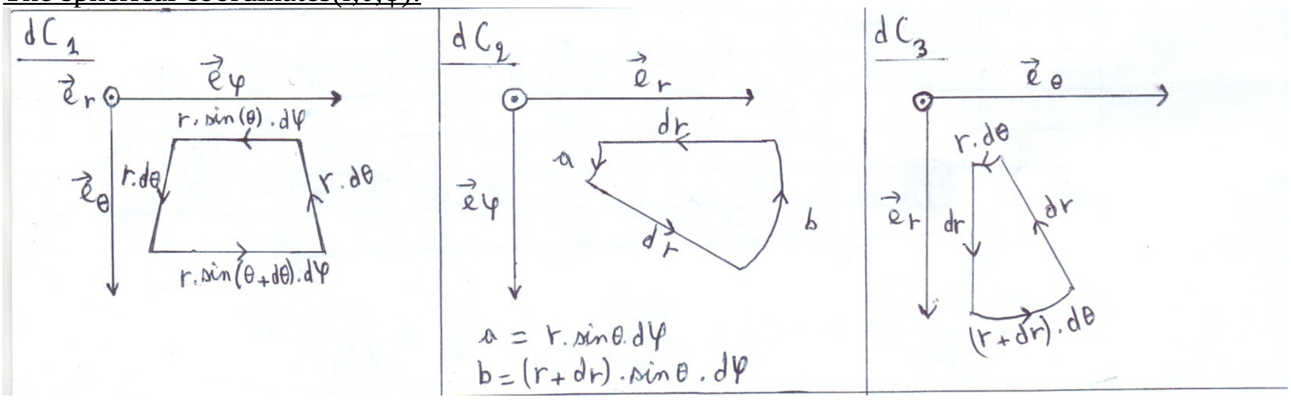

Fig.10 : *The boundaries dC_1, dC_2 and dC_3 of the spherical coordinates.*

In the figure 10:

$$dS_{23} = \frac{h_1 \times (r \times d\varphi \times \sin(\theta + d\theta) + r \times d\varphi \times \sin(\theta))}{2} = \frac{r^2}{4} \times d\varphi \times d\theta \times (4(\sin(\theta + d\theta) + \sin(\theta))^2 - d\varphi^2 \times (\frac{\sin(\theta + d\theta)^2 - \sin(\theta)^2}{d\theta})^2)^{\frac{1}{2}}$$

(188)

Consequently, by considering that:

$F(\theta) = (\sin(\theta + d\theta) + \sin(\theta))^2$ (189)

and: $G(\theta) = (\frac{\sin(\theta + d\theta)^2 - \sin(\theta)^2}{d\theta})^2$ (190)

We find that: $dS_{23} = \frac{r^2}{4} \times d\varphi \times d\theta \times (4F(\theta) - d\varphi^2 \times G(\theta))^{\frac{1}{2}}$ (191)

We remind that the calculations from before gave:

$$F(\theta) = (\sqrt{2} - \frac{d\theta^2}{2\sqrt{2}})^2 \times (1 - \cos(2\theta)) + \frac{d\theta^2}{2} \times (1 - \frac{d\theta^2}{4}) \times (1 + \cos(2\theta)) + (4 - 3d\theta^2 + \frac{3}{4} \times d\theta^4 - \frac{d\theta^6}{16})^{\frac{1}{2}} \times d\theta \times \sin(2\theta)$$

(192)

And:

$$G(\theta)=\left(\frac{d\theta^2}{4}+\frac{1}{2}\right)+\cos(4\theta)\times\left(\frac{d\theta^2}{4}-\frac{1}{2}\right)-\frac{d\theta^2}{2}\times\cos(2\theta)+\frac{d\theta}{2}\times\sqrt{1-\frac{d\theta^2}{4}}\times\sin(4\theta)-d\theta\times\sqrt{1-\frac{d\theta^2}{4}}\times\sin(2\theta)$$ (193)

Also: $dS_{13}=\left(r+\frac{dr}{2}\right)\times\sin(\theta)^2\times d\varphi\times dr$ (194)

And also: $dS_{12}=\left(r+\frac{dr}{2}\right)\times d\theta\times dr$ (195)

We have: $dC_1=A_{\theta_1}\times r\times d\theta+A^+_{\varphi_1}\times r\times\sin(\theta+d\theta)\times d\varphi-A^+_{\theta_1}\times r\times d\theta-A_{\varphi_1}\times r\times\sin(\theta)\times d\varphi$ (196)

And: $dC_2=A_{\varphi_2}\times r\times\sin(\theta)\times d\varphi+A^+_{r_2}\times dr-A^+_{\varphi_2}\times(r+dr)\times\sin(\theta)\times d\varphi-A_{r_2}\times dr$ (197)

And also: $dC_3=A_{r_3}\times dr+A^+_{\theta_3}\times(r+dr)\times d\theta-A^+_{r_3}\times dr-A_{\theta_3}\times r\times d\theta$ (198)

Where: $A^+_{\varphi_1}=A_{\varphi_1}+\frac{\partial A_{\varphi_1}}{\partial\theta}.d\theta$ (199)

and: $A^+_{\theta_1}=A_{\theta_1}+\frac{\partial A_{\theta_1}}{\partial\varphi}.d\varphi$ (200)

and also: $A^+_{r_2}=A_{r_2}+\frac{\partial A_{r_2}}{\partial\varphi}.d\varphi$ (201)

and: $A^+_{\varphi_2}=A_{\varphi_2}+\frac{\partial A_{\varphi_2}}{\partial r}.dr$ (202)

and also: $A^+_{\theta_3}=A_{\theta_3}+\frac{\partial A_{\theta_3}}{\partial r}.dr$ (203)

and: $A^+_{r_3}=A_{r_3}+\frac{\partial A_{r_3}}{\partial\theta}.d\theta$ (204)

with: $A_r=A_{r_2}=A_{r_3}$ and: $A_\theta=A_{\theta_1}=A_{\theta_3}$ and: $A_\varphi=A_{\varphi_1}=A_{\varphi_2}$ (205)

because the sides of the elementary surfaces have elementary sides d_{ij} in common.

And thus:

$$dC_1=A_\theta\times r\times d\theta+\left(A_\varphi+\frac{\partial A_\varphi}{\partial\theta}.d\theta\right)\times r\times\sin(\theta+d\theta)\times d\varphi-\left(A_\theta+\frac{\partial A_\theta}{\partial\varphi}.d\varphi\right)\times r\times d\theta-A_\varphi\times r\times\sin(\theta)\times d\varphi$$

$$dC_2=A_\varphi\times r\times\sin(\theta)\times d\varphi+\left(A_r+\frac{\partial A_r}{\partial\varphi}.d\varphi\right)\times dr-\left(A_\varphi+\frac{\partial A_\varphi}{\partial r}.dr\right)\times d\varphi\times\sin(\theta)\times(r+dr)-A_r\times dr$$

$$dC_3=A_r\times dr+\left(A_\theta+\frac{\partial A_\theta}{\partial r}.dr\right)\times(r+dr)\times d\theta-\left(A_r+\frac{\partial A_r}{\partial\theta}.d\theta\right)\times dr-A_\theta\times r\times d\theta$$

(206)

Consequently:

$$dC_1=\left(A_\varphi+\frac{\partial A_\varphi}{\partial\theta}.d\theta\right)\times r\times\sin(\theta+d\theta)\times d\varphi-A_\varphi\times r\times\sin(\theta)\times d\varphi-\frac{\partial A_\theta}{\partial\varphi}\times r\times d\varphi\times d\theta=\frac{-\partial A_\theta}{\partial\varphi}\times r\times d\varphi\times d\theta+A_\varphi\times r\times(\sin(\theta+d\theta)-\sin(\theta))\times d\varphi+\frac{\partial A_\varphi}{\partial\theta}.d\theta\times r\times\sin(\theta+d\theta)\times d\varphi=\overrightarrow{rot}\,\vec{A}\times\vec{e}_r\times dS_{23}$$

(207)

Where: $dS_{23}=\frac{r^2}{4}\times d\varphi\times d\theta\times\left(4F(\theta)-d\varphi^2\times G(\theta)\right)^{\frac{1}{2}}$ (208)

And:

$$dC_2=\frac{\partial A_r}{\partial\varphi}.d\varphi\times dr-A_\varphi\times\sin(\theta)\times d\varphi\times dr-\frac{\partial A_\varphi}{\partial r}.(r+dr)\times\sin(\theta)\times d\varphi\times dr=\overrightarrow{rot}\,\vec{A}\times\vec{e}_\theta\times dS_{13}$$

(209)

Where: $dS_{13}=\left(r+\frac{dr}{2}\right)\times\sin(\theta)^2\times d\varphi\times dr$ (210)

And also: $dC_3=\frac{-\partial A_r}{\partial\theta}.d\theta\times dr+\frac{\partial A_\theta}{\partial r}.dr\times(r+dr)\times d\theta+A_\theta\times d\theta\times dr=\overrightarrow{rot}\,\vec{A}\times\vec{e}_\varphi\times dS_{12}$ (211)

Where: $dS_{12}=(r+\frac{dr}{2})\times d\,\theta\times dr$ (212)

We conclude that:

$$\vec{rot}\,\vec{A}\times\vec{e_r}=\frac{-\partial A_\theta}{\partial\varphi}\times\frac{1}{\frac{r}{4}\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}}+A_\varphi\times\frac{\sin(\theta+d\,\theta)-\sin(\theta)}{\frac{r}{4}\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}\times d\,\theta}+\frac{\partial A_\varphi}{\partial\theta}.\frac{\sin(\theta+d\,\theta)}{\frac{r}{4}\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}}$$

(213)

Consequently:

$$\vec{rot}\,\vec{A}\times\vec{e_r}=\frac{4}{r\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}}\times(-\frac{\partial A_\theta}{\partial\varphi}+\frac{\sin(\theta+d\,\theta)-\sin(\theta)}{d\,\theta}\times A_\varphi+\sin(\theta+d\,\theta)\times\frac{\partial A_\varphi}{\partial\theta})=\frac{4}{r\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}}\times(A_\varphi\times(\frac{-d\,\theta}{2}\times\sin(\theta)+\sqrt{1-\frac{d\,\theta^2}{4}}\times\cos(\theta))+\frac{\partial A_\varphi}{\partial\theta}\times((1-\frac{d\,\theta^2}{2})\times\sin(\theta)+\sqrt{1-\frac{d\,\theta^2}{4}}\times d\,\theta\times\cos(\theta))-\frac{\partial A_\theta}{\partial\varphi})$$

(214)

And: $$\vec{rot}\,\vec{A}\times\vec{e_\theta}=\frac{\frac{\partial A_r}{\partial\varphi}}{\sin(\theta)^2\times(r+\frac{dr}{2})}-\frac{A_\varphi}{(r+\frac{dr}{2})\times\sin(\theta)}-\frac{\partial A_\varphi}{\partial r}\times\frac{r+dr}{(r+\frac{dr}{2})\times\sin(\theta)}$$ (215)

And also: $$\vec{rot}\,\vec{A}\times\vec{e_\varphi}=\frac{-\frac{\partial A_r}{\partial\theta}}{r+\frac{dr}{2}}+\frac{\frac{\partial A_\theta}{\partial r}\times(r+dr)}{r+\frac{dr}{2}}+\frac{A_\theta}{r+\frac{dr}{2}}$$ (216)

We conclude finally that:

$$\vec{rot}\,\vec{A}=\begin{pmatrix}\frac{4}{r\times(4F(\theta)-d\,\varphi^2\times G(\theta))^{\frac{1}{2}}}\times(A_\varphi\times(\frac{-d\,\theta}{2}\times\sin(\theta)+\sqrt{1-\frac{d\,\theta^2}{4}}\times\cos(\theta))+\frac{\partial A_\varphi}{\partial\theta}\times((1-\frac{d\,\theta^2}{2})\times\sin(\theta)+\sqrt{1-\frac{d\,\theta^2}{4}}\times d\,\theta\times\cos(\theta))-\frac{\partial A_\theta}{\partial\varphi}) \\ \frac{\frac{\partial A_r}{\partial\varphi}}{\sin(\theta)^2\times(r+\frac{dr}{2})}-\frac{A_\varphi}{(r+\frac{dr}{2})\times\sin(\theta)}-\frac{\partial A_\varphi}{\partial r}\times\frac{r+dr}{(r+\frac{dr}{2})\times\sin(\theta)} \\ \frac{-\frac{\partial A_r}{\partial\theta}}{r+\frac{dr}{2}}+\frac{\frac{\partial A_\theta}{\partial r}\times(r+dr)}{r+\frac{dr}{2}}+\frac{A_\theta}{r+\frac{dr}{2}}\end{pmatrix}$$

(217)

<u>Clarifications:</u>

- During the study, we should replace by the field coordinates in the curl formulas demonstrated above. Then, we integrate according to the studied part of the space when the integration is possible.
- The operator curl informs about a part of the vicinity of a studied point. This part is the part of the space in front of the studied point in the orientation of the coordinates system being used.
- If a coordinate of the curl is positive, then the vector field $\vec{A}$ located at the studied point vicinity that is perpendicular to that coordinate axis is a vortex field. The vortex is anticlockwise around the positive coordinate axis.
- If that coordinate is negative then the vortex will be oriented clockwise.
- If that coordinate is null whereas the field exists, then the field will be uniform in the part of the studied point vicinity concerned by the null coordinate.

4. The operator Laplacian:

Let’s consider a function of locations points: $f : E \rightarrow \mathbb{R}$. Where E is the Euclidean space and the function f is differentiable and thus continuous.

The Laplacian of f is: $\Delta f = div(\overrightarrow{grad} f)$.

The Cartesian coordinates System (x,y,z):

$$\vec{\nabla} f = \overrightarrow{grad} f = \frac{\partial f}{\partial x}.\vec{e_x} + \frac{\partial f}{\partial y}.\vec{e_y} + \frac{\partial f}{\partial z}.\vec{e_z} \quad \text{and:} \quad div \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Where: $\vec{A}$ is a vector field.

Consequently: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (218)

The cylindrical coordinates system (ρ,φ,z):

$$\vec{\nabla} f = \overrightarrow{grad} f = \frac{\partial f}{\partial \rho}.\vec{e_\rho} + \frac{\partial f}{\partial \varphi} \times \frac{1}{\rho}.\vec{e_\varphi} + \frac{\partial f}{\partial z}.\vec{e_z}$$

and: $div \vec{A} = \frac{A_\rho}{\rho} + (1 + \frac{d\rho}{\rho}) \times \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho} \times \frac{\partial A_\varphi}{\partial \varphi} + (1 + \frac{d\rho}{2\rho}) \times \frac{\partial A_z}{\partial z}$

Where: $\vec{A}$ is a vector field.

Consequently: $\Delta f = \frac{1}{\rho} \times \frac{\partial f}{\partial \rho} + (1 + \frac{d\rho}{\rho}) \times \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2} \times \frac{\partial^2 f}{\partial \varphi^2} + (1 + \frac{d\rho}{2\rho}) \times \frac{\partial^2 f}{\partial z^2}$ (219)

The spherical coordinates(r,θ,φ):

$$\vec{\nabla} f = \overrightarrow{grad} f = \frac{\partial f}{\partial r}.\vec{e_r} + \frac{1}{r} \times \frac{\partial f}{\partial \theta}.\vec{e_\theta} + \frac{1}{r.\sin(\theta)} \times \frac{\partial f}{\partial \varphi}.\vec{e_\varphi}$$

and:

$$div \vec{A} = \left(\frac{1 + \frac{dr}{2r}}{r \times \sin(\theta)}\right) \times \frac{\partial A_\varphi}{\partial \varphi} + A_\theta \times \left(\frac{1}{r} + \frac{dr}{2r^2}\right) \times \frac{B(\theta)}{\sin(\theta)} + \frac{\partial A_\theta}{\partial \theta} \times D(\theta) \times \left(\frac{1}{r} + \frac{dr}{2r^2}\right) + \left(A_r \times \left(\frac{dr}{4r^2} + \frac{1}{2r}\right) + \frac{\partial A_r}{\partial r} \times \left(\frac{1}{2} + \frac{dr}{2r}\right)^2\right) \times \frac{(4.F(\theta) - d\varphi^2 \times G(\theta))^{\frac{1}{2}}}{\sin(\theta)}$$

Where: $\vec{A}$ is a vector field.

Consequently:

$$\Delta f = \left(\frac{1 + \frac{dr}{2r}}{r^2 \times \sin(\theta)^2}\right) \times \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial f}{\partial \theta} \times \frac{1 + \frac{dr}{2r}}{r^2} \times \frac{B(\theta)}{\sin(\theta)} + \frac{\partial^2 f}{\partial \theta^2} \times \frac{1 + \frac{dr}{2r}}{r^2} \times D(\theta) + \left(\frac{\partial f}{\partial r} \times \left(\frac{dr}{4r^2} + \frac{1}{2r}\right) + \frac{\partial^2 f}{\partial r^2} \times \left(\frac{1}{2} + \frac{dr}{2r}\right)^2\right) \times \frac{(4.F(\theta) - d\varphi^2 \times G(\theta))^{\frac{1}{2}}}{\sin(\theta)}$$

(220)

We remind that:

$$B(\theta) = -d\theta \times \sin(\theta)^2 + \sqrt{1 - \frac{d\theta^2}{4}} \times 2 \times \sin(\theta) \times \cos(\theta) = \frac{-d\theta}{2} \times (1 - \cos(2\theta)) + \sqrt{1 - \frac{d\theta^2}{4}} \times \sin(2\theta)$$

And:

$$D(\theta) = \left(1 - \frac{d\theta^2}{2}\right)^2 \times \sin(\theta) + 2d\theta \times \left(1 - \frac{d\theta^2}{2}\right) \times \sqrt{1 - \frac{d\theta^2}{4}} \times \cos(\theta) + d\theta^2 \times \left(1 - \frac{d\theta^2}{4}\right) \times \frac{\cos(\theta)}{\tan(\theta)}$$

And Also:

$$F(\theta) = \left(\sqrt{2} - \frac{d\theta^2}{2\sqrt{2}}\right)^2 \times (1 - \cos(2\theta)) + \frac{d\theta^2}{2} \times \left(1 - \frac{d\theta^2}{4}\right) \times (1 + \cos(2\theta)) + \left(4 - 3d\theta^2 + \frac{3}{4} \times d\theta^4 - \frac{d\theta^6}{16}\right)^{\frac{1}{2}} \times d\theta \times \sin(2\theta)$$

And:

$$G(\theta) = B(\theta)^2 = \left(\frac{d\theta^2}{4} + \frac{1}{2}\right) + \cos(4\theta) \times \left(\frac{d\theta^2}{4} - \frac{1}{2}\right) - \frac{d\theta^2}{2} \times \cos(2\theta) + \frac{d\theta}{2} \times \sqrt{1 - \frac{d\theta^2}{4}} \times \sin(4\theta) - d\theta \times \sqrt{1 - \frac{d\theta^2}{4}} \times \sin(2\theta)$$

Clarifications:

- During the study, we should replace by the studied function f in the Laplacian formulas demonstrated above. Then, we integrate according to the studied part of the space when the integration is possible.

- If $\Delta f=0$ then f behaves uniformly in the location that is immediately in front of the studied point M.
- If $\Delta f>0$ then a local minimum of f exists in the location that is immediately in front of the studied point M.
- If $\Delta f<0$ then a local maximum of f exists in the location that is immediately in front of the studied point M.
- When we are studying the Laplacian of a function f, the gradient vectors cross diagonally the volume made by the elementary surfaces of the divergence.

Remark:
The results found should make the convinced readers change their vector and matrix calculations especially with the famous navier-stokes equations.

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