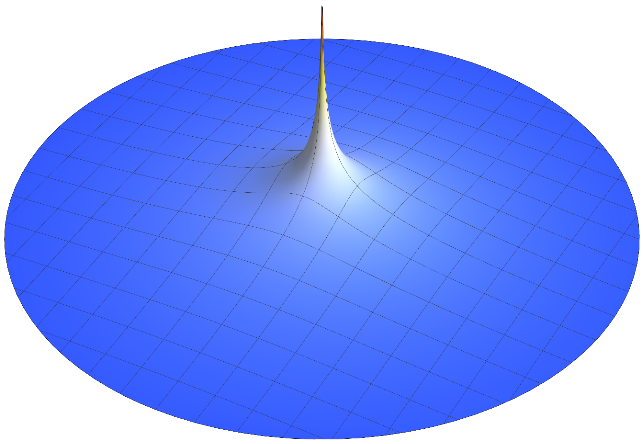
STATISTICAL PRINCIPLES OF NATURAL PHILOSOPHY

Guo Tao



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Title:

Statistical Principles of Natural Philosophy

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Abstract

Currently, natural philosophy (Physics) lacks the most fundamental model and a complete set of self-consistent explanations. This article attempts to discuss several issues related to this lack. Starting from the most basic philosophical paradoxes, I deduce a physical model (the natural philosophical outlook) to describe the laws governing the operation of the universe. Based on this model, a mathematical model ($\boldsymbol{i}\frac{\partial \mathcal{M}}{\partial t}=-\frac{\hbar e^{-\mathcal{M}}}{2m}\left[\Delta\mathcal{M}-T^2(\mathcal{M})\right]$) is established to describe the generalized diffusion behavior of a moving particle swarm, and its simple verification is carried out. In this article, the gravitational force and relativistic effects are interpreted for the first time as a statistical effect of randomly moving particles. Thus, the gravitational force and special relativistic effects are integrated into a single equation (achieved by selecting an initial wave function with a specific norm when solving it), and the cause of stable particle formation is also revealed. The derived equation and the method of acquiring the initial wave function are fully self-consistent with the hypotheses stated in the physical model, thereby also proving the reliability of the physical model to some extent. Some of these ideas may have potential value as a basis for understanding the essence of quantum mechanics, relativity and superstring theory, as well as for gaining a further understanding of nature and the manufacture of quantum computers.

1. Introduction

"Birds flock and sing when the wind is warm, Flower-shadows climb when the sun is high"[1]; the Earth, our home, is overflowing with vigor! However, light years away, dead silence seems to prevail; from the human perspective, the Earth appears vast, but at the scale of the Solar System, it is merely a "little blue dot". By what forces are these mysterious phenomena, which are as far apart as Heaven and Earth in our eyes, arranged? How enormous is the universe? Why is it like this? Through what mechanism does it operate? Is there a beginning or an end? Where does the vast amount of energy in our universe come from? Will it ever run out? How do the concepts of time, space and speed come into being? Will the total entropy in the universe continue to increase? Etc. Throughout the history of human existence, these have been difficult questions to answer. "Know the enemy and know yourself, and you can fight a hundred battles with no danger of defeat"[2]; exporing the origin of the universe is the only way for human beings to conquer nature.

Since ancient times, human beings have gradually deepened their understanding of the laws of nature and the universe through a continuous process of development that can be roughly divided into the following three stages:

In the initial period of Aristotle, Ptolemy, Copernicus, Kepler and others, people's explorations of nature were restricted not only by the level of technological development at that time but also by various political conditions[3]. The explorations of nature and the universe were slow, and the levels of understanding gained were also relatively shallow. By the time of Galileo and Newton, technology had greatly improved, and a framework of relatively strict logic and scientific thinking methods had also been developed. Under the guidance of Newtonian mechanics and calculus, the understanding of nature greatly improved. However, Newtonian mechanics held that gravitation was generated directly by mass and was not affected by motion or energy. The laws of gravitation, inertia and acceleration were all developed based on simple rules of experience from the perspective of philosophy (i.e., axioms; although the definition of universal gravitation was formulated by Newton, Galileo had already established empirical rules in accordance with observation, and the essential nature of inertia or acceleration was not clear), and the universe as a whole was considered to be relatively static.

In modern times, Einstein's theory of general relativity emerged, and humans' ability to understand natural laws and predict natural phenomena improved tremendously. According to general relativity, a gravitation or space-time field is affected by matter, energy and motion, which leads to apparent "magical" changes in motion. On this basis, the existence of black holes and other celestial bodies was predicted[3,4]. With the subsequent rapid development of quantum mechanics, the human understanding of the universe at the microscale greatly improved, resulting in a new era of philosophy (the Copenhagen interpretation of quantum mechanics) as well as a large number of modern technological advances[3].

However, what are the physical principles behind quantum mechanics? How should quantum entanglement[5] and Wheeler's delayed-choice experiment[6] be perceived, and is the Dirac equation with special relativistic effects essentially correct or not? What is the more fundamental reason behind the curved nature space-time and the principles of special relativity? Furthermore, how can dark matter, dark energy and inexplicable repulsion, which often arise in discussions of modern cosmology, be explained? Etc.

In all this time, humans have made no effort to explore the answers to these substantive questions (by establishing a more fundamental physical model) but rather have remained at the superficial surface of quantum physics. Based on classical physics (such as Newtonian mechanics), formulas have been deduced from a mathematical point of view, and the conclusions of special relativity and the constraints of Lorentz covariance have been added to various equations, yielding results that seem to be very fragmented (such as the Dirac equation and quantum field theory). All these practices have led to the emergence of various theories but have not fundamentally solved the problem[3,7]. The whole edifice of physics seems to have improved by virtue of various explanations, such as the so-called Standard Model of Particle Physics and superstring theory, but none of them is completely satisfactory. The Standard Model and various models of a Grand Unified Theory that have been developed to date merely integrate the previous models from the perspectives of mathematics and the surface nature of physical phenomena; as a result, they cannot perfectly explain gravitational effects (irreducible normalization after the introduction of gravitation). Superstring theory seems to encompass all known successful theories because it includes additional degrees of freedom (higher dimensions). However, the invocation of higher dimensions is not meaningful for solving more practical problems. Instead, because many additional false possibilities arise that make the equations extremely difficult to solve, the requirements in terms of the mathematical skills needed to pursue such theory have reached an amazing level. Moreover, a "string" is not and should not be considered the most basic physical morphology. In addition, the theory of loop quantum gravity is not perfect, and it seems to raise more difficulties than can be solved. In view of the above problems, it is necessary to further understand the essential nature of physical phenomena or physical constraints and to establish a more fundamental physical model.

Starting from the most basic philosophical paradoxes, this article probes into a series of even deeper and more essential problems in physics and attempts to establish a most fundamental physical model to describe the laws governing the operation of the universe. Based on this, a self-consistent mathematical equation is established in a concise form. The basic structure of the whole article is as follows: first, make sure that human beings can understand nature; second, extract philosophical contradictions from natural phenomena (axioms) acquired by human senses (the phenomena with contradictory constraints are reasonable); finally, based on philosophical contradictions, physical models are established (thus to determine time, space, and the coordinate system used to establish mathematical models, etc.) and based on the defined physical model, the mathematical model is deduced step by step. Before the derivation, two checks have been made: First, it is confirmed that the physical model contains the special relativistic effects; second, the Schrödinger equation can be derived from the physical model under certain conditions. The process of the two checks also makes it clearer how to derive the mathematical model, that is, the generalized diffusion equation. The process of deriving the generalized diffusion equation includes: (**i**) vector decomposition. The decomposition of nonmoving particles in space is extended to the decomposition of a 2-dimensional vector representing the sum of the 3-dimensional vector of moving particles at a certain point in space, which is the core of the whole derivation. (**ii**) The classic diffusion coefficient is reinterpreted and the essential key information is obtained. (**iii**) On the basis of (**i**) and (**ii**), the equations are assembled according to the classical diffusion principle to obtain the generalized diffusion equation. In addition, some important parts related to the equation are discussed and verified. For example, how to assign initial values to partial differential equations and some other key physical meanings represented by the equations. The following is a detailed description.

2. Methods

In this article, the physical model with a clear definition is derived employing the constraint of philosophical paradox, and then the mathematical model is obtained by logical derivation based on the physical model. Finally, it is also logically proven that all the typical features of the physical model can be extracted from the mathematical model. Mathematica 12.1.1.0 for Mac (*Wolfram Research Inc.*) was used for all of the mathematical calculations, and the operating system was macOS High Sierra 10.13.6. The solutions to each specific problem can be found in the Supplementary Information. If no specific parameters are specified, the default values in the software system were used. The effective number of significant figures in the numerical methods was no less than 6.

3. Results and Discussions

3.1 Can the World be Understood?

The innate knowledge possessed by human beings is perceptual knowledge that corresponds to external stimuli and is established through long-term interaction and internalization between an organism and its natural environment constrained by the elimination mechanism of nature[8,9]. Therefore, such innate knowledge shows excellent reliability. The acquired knowledge or experience accumulated by human beings through a model of innate cognition (even if such a cognitive model includes more or less subjective factors) should still be reliable and applicable in practice if such practice is based on the same cognitive model. Moreover, in view of the relative stability and repeatability of certain external conditions (i.e., the translation invariance of time and space), the innate knowledge and acquired experience possessed by human beings should also be reliable throughout the whole range of human practice.

Therefore, the theories established by human beings, even if they are cognitions only from the perspective of human beings on Earth, who in some sense are equivalent to cosmic dust, and even if they contain many limitations or mistakes (such "mistakes" are relative; they are related to the fact that the appearances and forms of things as reflected in the human consciousness are not, in fact, the original appearances and forms of those things), as long as they can effectively explain and predict the phenomena we observe, are successful theories, even though we cannot confirm whether they represent completely correct truth[8].

3.2 The World from the Perspective of Philosophical Paradoxes

The reason why the world has infinite energy and runs endlessly must be that there exists a series of philosophical paradoxes restricting each other[10,11]. Only under such contradictory constraints can the world become balanced and logical (self-consistent). Under the guidance of this perspective, this article summarizes three axioms, as follows:

AXIO 1: Substances exist in the world.

Whether substances exist in the world is an ancient topic of philosophical discussion. However, this debate serves as the original basis for all rational inference and logical extrapolation in this article. There are only two possible situations: either some substance exists in the world or there is no substance at all. The fact is obvious: there are some substances that exist in this world. On average, however, these substances are so sparse that they are almost nonexistent[12]. As a result, the world (or at least within the range of human observation) is as sparse as though it is without substance.

AXIO 2: These substances are inhomogeneous.

If the world is full of substance, then there are only two possibilities for its distribution: it is either homogeneous or inhomogeneous. Obviously, the distribution of substances in this world is inhomogeneous within the range of our observations. However, there is no reason that any one of these inhomogeneous substances should be favored more than another, that is, substances should have no greater opportunity to be distributed in one place than another. Therefore, it should be considered that the probability of the distribution of substances in every location (not limited to only 3 dimensions) is equal, or homogeneous, from a large-scale perspective[13,14]. To satisfy both of these properties of inequality of distribution and equality of probability, the substances in this world must exist in quantum form. This fact does not require discussion because it has been verified by various physical experiments. In addition, there is no reason for the world to "favor one substance more than another", and it should be probabilistically identical between different original "quantum dots" (called infinitesimal particles in the following). The fact that the above two properties of "inequality of distribution" and "equality of probability" are both satisfied also necessitates that the world is a paradoxical body with uniform probability but inhomogeneous characteristics at the microscale (or in several dimensions).

AXIO 3: These substances are moving.

This seems to be another topic of philosophical discussion, but I give it new connotation here. The substances observed in the world are moving, or from the perspective of human understanding, the substances that exist in the world are moving. In any case, the world can be interpreted as dynamic rather than static. Then, what is the most reasonable movement pattern?

The current understanding is as follows: photons, which have no stationary mass, are the fastest substance in the universe. It is impossible to accelerate species with stationary masses (such as electrons) to the speed of light. If they were to reach this speed, their masses would become infinite, and their energy consumption would also become infinite (according to the conclusions of relativity). Therefore, there is no species that can move faster than the speed of light, even if there is, it cannot transmit information. However, from this point of view, the abovementioned essential nature of quantum entanglement cannot be understood, the phenomenon of Wheeler's delayed-choice experiment is astonishing, and the mechanism by which the influence of gravitation can reach out beyond a black hole is not easy to explain.

Therefore, this point of view is abandoned here, and it is instead considered that photons are the fastest species that can transmit information that has been found or perceived by human beings at present, that photons have a light mass (or infinitesimal mass. This is the observation from another perspective, as discussed in Section **3.3.5.4**), and that motion cannot substantially change the physical mass of an object. Particles (particle swarms) at a smaller mass level than photons, even if they can transmit information, cannot be perceived (or consciously perceived) by human beings at present, and their limiting speed is faster. Therefore, once the speed of a particle (particle swarm) is sufficiently fast, it must "split" into particles (particle swarms) of lower mass levels, until the speed reaches infinity and the mass becomes infinitesimal (in the framework based on the assertion that "the substances in this world must exist in quantum form", Section **3.3.5.2** will confirm that it is possible that particles (particle swarms) of lower mass levels can form a particle (particle swarm) of a higher mass level and that the opposite process can also occur).

From this point of view, the whole universe will exhibit motion-related phenomena as follows: For a particle with infinitesimal mass, its speed can reach infinity. Therefore, no matter how large the space in which it exists, such an infinitesimal particle can instantaneously (no time consuming) exist at any position. Therefore, it can be everywhere at once, relative to it, any arbitrarily large space is also an infinitesimal space in which the concepts of time and distance do not apply, and such a particle is infinitely large relative to any such space, meaning that no motion in space can be perceived for space at all (particle scatters all over the space). Since there is no concept of space or time in the case of infinitesimal particles, there is also no concept of energy. If the universe is composed of infinitely many such moving particles (because they are infinitesimal particles, there are no collisions between them), then it will not consume any so-called energy and can continue to exist and run forever. However, once a particle of a larger mass level (a particle swarm of infinitesimal particles) is observed, its speed will decrease (in 3-dimensional space, the relationship between the mass of the particle swarm, which can be equivalent to the dominance of directional aggregation, and the average speed of the constituent particles obeys the Maxwell distribution; see Part 1 of the Supplementary Information for details). Simultaneously, the concepts of time, space, speed, mass and energy will arise. Therefore, there is no inherent concept of time, space or speed and no inherent concept of mass or energy in the universe; all of these concepts instead arise from the representations of the universe that are observed from various perspectives. Although a particle is infinitesimal, it is infinite relative to the universe; although the universe is infinitely great, it is infinitesimal relative to the infinitesimal particles. As the velocity of a particle approaches infinity, the very concept of motion will be lost. The universe is both large and small; substances both move and do not move within it; and the concepts of time, space, speed, mass and energy are both extant and absent. Thus, the nature of the universe is described by several pairs of mutually constraining paradoxes.

The validity of the above three axioms is obvious, and their existence depends on the constraints imposed by their counterparts in the mutually constraining paradoxes introduced above. Here, only the meaningful sides of these paradoxes are selected for further investigation. In addition, under the constraints of logic, the concepts derived from the 3 axioms are also contradictory constraints. Only logic that is constrained by paradoxes is complete and self-consistent. On the basis of the above three axioms, this article makes reasonable inferences and extracts the following 3 hypotheses:

HYPO 1: The universe is composed of infinitely many uniform particles with infinite speed and infinitesimal mass.

The concepts of "infinity" and "infinitesimal" are equivalent to those in mathematical analysis. The statement that the masses of the particles are uniform refers to these masses relative to their standard deviation, and the concept of "uniform particles" discussed below also has the same meaning.

HYPO 2: The speeds of these infinitesimal particles in space are equal, and the directions of their motion are random.

As mentioned in **AXIO 2**, these infinitesimal particles are formed in accordance with the same law; therefore, their masses and speeds (or norms of momenta) should be either strictly equal or equal relative to their standard deviations. The concepts of equal masses and speeds (or norms of momenta) discussed below also have the same meaning. The probabilities of the possible directions in each dimension are also equal because there is no reason for them to be uneven.

HYPO 3: There is no interaction between infinitesimal particles.

In the world we observe, interaction forces exist everywhere. However, this is not necessarily true for infinitesimal particles. For infinitesimal particles, it is assumed that there is no traditional interaction between them (such as gravitation) and that any observed macroscopic force (or interaction) is caused by a statistical effect of these infinitesimal moving particles. This assumption does not conflict with the classical force concept but will be helpful for establishing a general equation and expanding the self-consistent range of theory.

These are the 3 basic characteristics (hypotheses) extracted from the 3 basic axioms regarding the nature of the world. Next, a model will be built on the basis of these 3 hypotheses.

3.3 Model Building Based on Philosophical Paradoxes

On the basis of the above 3 axioms and 3 hypotheses (although the abovementioned can be examined in accordance with, but not limited to, the 3 + 1 dimension), this article infers that there are only four possibilities regarding the scale (large or small) of space (in any dimension) and the number (many or few) of particles in any local domain and that these four possibilities are independent in different local domains. This is because, in any dimension, the world is dynamic and inhomogeneous, and motion and inhomogeneity are two independent properties. Because of the movement of particles, when a certain number of particles are observed without any spatial differences, the concept of velocity will be generated in the world. In any dimension, the concept of velocity will be characterized in terms of the concepts of time and distance. Due to the inhomogeneity of the distribution of particles, when a certain number of particles are observed with spatial differences, the concept of density will be generated in the world. In any dimension, the concept of density will be characterized in terms of the concepts of scale (another single degree of freedom different from distance) and the number of particles with spatial differences (disguised distance in another degree of freedom). If the latter two degrees of freedom are fixed (that is, the two degrees of freedom or the entities they represent are used as references to determine the object under inspection), then they will be characterized in terms of the degrees of freedom of distance in the other two dimensions. Therefore, there are four independent dimensions in this world, with three dimensions characterized in terms of the concept of space in our consciousness and one dimension characterized in terms of the concept of time in our consciousness. In principle, these 3-dimensional space and 1-dimensional time coordinates can describe all natural phenomena. Even methods operating in the so-called multidimensional space of string theory have the ultimate purpose of solving problems in 4-dimensional space-time.

In view of the above discussion, the essence of time can be expounded here. If the world is regarded as a whole, there is no concept of time and space. If the substances in this world are observed separately from their environment, concepts such as time will be produced. Therefore, time is one of the independent degrees of freedom produced when the world is thought by "one divides into two". It is the concept of time when this independent degree of freedom is reflected in people's consciousness. This article will not discuss it in depth, but treats it as a classical concept of time.

To understand the world more easily and intuitively, humans tend to project various abstract results and conclusions into the world we are familiar with. In principle, if a 4-dimensional curvilinear coordinate system is adopted through coordinate transformation, the necessary mathematical operations may be simple, but this conceptualization will lead to difficulties in understanding the problem. Einstein's general relativity uses a 4-dimensional curvilinear coordinate system (space-time), which is an "immersive perspective" with a sense of participation. Although individual immersive physical events (such as the constant speed of light) are consistent with physical observations, difficulties will eventually arise in understanding the essence of physical problems. In absolute space-time, the coordinate system consisting of 3 spatial dimensions and 1 time dimension is the "God perspective", which is helpful in allowing people to look at and understand problems from a macroscopic perspective. Of course, no matter which perspective is adopted, it does not affect the descriptions of physical phenomena in 4-dimensional space-time. Finally, the evolution of various phenomena should ultimately be measured and understood in the flat coordinate system that we are familiar with at present.

It should be emphasized that the "God perspective" (or "absolute space-time") mentioned here also has relativity. When absolute space-time is used as the reference system, the particles in it should satisfy the conditions given by 3-dimensional **HYPO 1**–**3** (this system can also be regarded as the classical "inertial reference system" here). This means that if the entire swarm of existing particles moves as a whole, then absolute space-time will also move with it; it would be meaningless if absolute space-time (or the corresponding absolute coordinate system) did not follow the overall movement of the particle swarm.

Since our goal is to understand and grasp the world, it is unnecessary to use a relatively variable view of space-time. Sometimes, the concept of absolute space-time is more advantageous for building models and understanding laws. In view of the above analysis, the physical and mathematical models presented in this paper will be established in the 4-dimensional (3 dimensions of space plus 1 dimension of time) absolute coordinate system.

3.3.1 Physical Model

Here, the abovementioned conclusions are combined to form the physical model considered in this article: The universe is composed of infinitely many uniform particles with infinite speed and infinitesimal mass. The speeds of these infinitesimal particles in 3-dimensional space are equal, and the directions of their motion are random. There is no interaction between infinitesimal particles. No additional rules are needed.

3.3.2 Special Relativistic Effects on Infinitesimal Particles

It will be proven that (special) relativistic effects exist in the abovementioned physical model (Section **3.3.1**). Once again, it is emphasized that the speeds of these particles (throughout this article, the "infinitesimal particles" described in the above physical model are called "particles", "1st-order particles" or "tiny particles", while larger finite-mass-level particles composed of *k* particles are called "*k*th-order particles") are exactly the same (or $\sigma_1 \ll c$, where *c* is the mean value of the particle speeds and σ_1 is their standard deviation), and the directions of their motions in 3-dimensional space are random. Therefore, these particles can be represented by random vectors with equal norms in Euclidean space. When the position of particles in a certain background (environment) domain is well distributed and particles in a certain subdomain of the background region have the phenomenon of velocity direction aggregation, we call it the situation that the velocity direction aggregation is dominant in the subdomain. When the velocity direction of particles in a certain background (environment) domain is well distributed and the particles in a certain subdomain of the background domain have the phenomenon of position aggregation, we say that the position aggregation is dominant in the subdomain. They are the statistical expression of a large number of particles, and diffusion has no discrimination to these two kinds of aggregation or their influence on diffusion is equivalent, so they can be equivalent to each other. The case of velocity direction aggregation being dominant of particles is studied here for the time being. In this article, statistical methods will be used to prove the existence of special relativistic effects in the vector swarm (with velocity direction aggregation being dominant) composed of such a group of vectors. When a group of particles in the same 3-

dimensional space is moving in one direction on average (i.e., their centroid is moving in one direction), they will lose some probability of movement in other directions due to statistical effects, i.e., the movement trends in other directions will decrease, giving rise to a special relativistic effect. This phenomenon will be quantitatively explained in detail below.

Note that the velocity of a kth-order particle is the velocity of the overall center of mass of the k particles, which is the average of the velocity vectors of all these particles. Moreover, the projection of the velocity vector of a kth-order particle onto one of the three equivalent coordinate axes of the 3-dimensional Cartesian coordinate system is the mean value of the projection (onto the same axis) of the velocity vectors of the 1st-order particles forming the kth-order particle, which follow the same distribution; therefore, when k is a large value, it approximately follows a normal distribution (central limit theorem). There are three equivalent (approximate) normal distributions, one on each of the three axes, which are not completely independent. However, James Clerk Maxwell[15] and Ludwig Boltzmann[16] proved that these distributions can, in fact, be equivalently treated as completely independent. This is because randomly selecting a vector is equivalent to randomly determining a three-axis coordinate (each 3-dimensional random vector consists of three random dimensions); moreover, the problem of the momentum transfer of gas molecules participating in random collisions is also equivalent to the problem discussed in this article. Accordingly, the speeds of kth-order particles follow the Maxwell distribution. Suppose that the standard deviation of the projection (treated as a random variable; the same is done below) of the velocity of any one of the k equivalent particles forming a kth-order particle onto each equivalent coordinate axis is σ. Then, the standard deviation of the projection of the velocity of a kth-order particle onto each equivalent coordinate axis is $\frac{\sigma}{\sqrt{k}}$, namely, the projection onto each coordinate axis follows a normal distribution with a standard deviation of $\frac{\sigma}{\sqrt{k}}$. As a result, the speed of kth-order particles follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$ (see Part 1 of the Supplementary Information for details).

As already mentioned, it is assumed that the speed of all particles is c ($c > 0$) and that the directions of their movement are evenly distributed in 3-dimensional space.

Among the possible particle swarms composed of randomly moving particles, the particle swarm with an average velocity of 0 (i.e., the "absolute space-time" mentioned earlier) is called the stationary reference system (denoted by $\mathcal{R}_0$), and a 3-dimensional Cartesian (rectangular) coordinate system *Oxyz* is established for it. A particle swarm formed by a subset of particles in a certain period of time and moving at an average velocity *u* is called a moving reference system (denoted by $\mathcal{R}_u$). Let the direction of the velocity of $\mathcal{R}_u$ be parallel to the *z*-axis in the direction of increasing *z*. Then, the mean value of the velocity component of the particles in $\mathcal{R}_u$ along the *z*-axis must be *u*. Under the assumptions that all particles in $\mathcal{R}_u$ are represented by vectors with their starting points at the origin of the coordinate system and that the point (0, 0, *u*) is taken as the dividing point of the *z*-axis, the vectors in $\mathcal{R}_u$ can be separated into two groups: the components of the vectors above this dividing point and the components of the vectors below it. These vectors randomly enter $\mathcal{R}_u$ from $\mathcal{R}_0$ with equal probability. Therefore, the distribution of the vectors in $\mathcal{R}_u$ can be thought of as a mixed distribution of the vector distribution of the components above the dividing point and the vector distribution of the components below the dividing point. When the mean value of the components on the *z*-axis of this mixed distribution is *u*, the mixture weights *w* can be determined. With this value (*w*) as the reference, the component distribution of the vectors that form the mixed distribution on the *x*-axis (or *y*-axis) can be determined; thus, the standard deviation σ_u of these components can also be obtained. When the standard deviation of the components on the *z*-axis of this mixed distribution is also σ_u (this will be proved below), then the speed of *k*th-order particles (of mass μk, where μ is the mass of a single particle; the same is true below) in $\mathcal{R}_u$ follows the Maxwell distribution with scale parameter $\sigma_{u,k}$, where

$$\sigma_{u,k} = \frac{\sigma_u}{\sqrt{k}}. \tag{1}$$

Therefore, $\sigma_{u,k}$ is directly proportional to the average speed $\bar{v}_{u,k}$ of the *k*th-order particle (μk), namely,

$$\bar{v}_{u,k} = 2\sqrt{\frac{2}{\pi}}\sigma_{u,k}. \tag{2}$$

By substituting Eq. 1 into Eq. 2, we obtain

$$\bar{v}_{u,k} = 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma_u}{\sqrt{k}}. \tag{3}$$

The distribution of components of vectors forming $\mathcal{R}_0$ on each axis is relatively simple (all of them are approximately normal distributions). Suppose that the standard deviation of their components on the *x*-axis (or *y*- or *z*-axis) is σ_0; similarly, the average velocity of the *k*th-order particles (μk) that is formed by them is

$$\overline{v}_{0,k} = 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma_0}{\sqrt{k}}. \tag{4}$$

When (*k*th-order) particles (μk) of the same mass level are formed in both $\mathcal{R}_u$ and $\mathcal{R}_0$, the ratio (Eq. 3 to Eq. 4) between their average speeds (relative to the coordinate system in $\mathcal{R}_0$) is

$$\frac{\overline{v}_{u,k}}{\overline{v}_{0,k}} = \frac{\sigma_u}{\sigma_0}. \tag{5}$$

Therefore, the ratio of σ_u to σ_0 is the ratio between the average speeds (relative to the coordinate system in $\mathcal{R}_0$) of (*k*th-order) particles of higher mass levels in $\mathcal{R}_u$ and $\mathcal{R}_0$. A more detailed introduction will be presented in the following.

As mentioned above, in the 3-dimensional Cartesian coordinate system constructed in the stationary reference system $\mathcal{R}_0$, if the moving reference system $\mathcal{R}_u$ moves along the *z*-axis at velocity *u*, then the *x*- and *y*-coordinates are equivalent; hence, only the *x*-coordinate is considered in the following. In view of the nature of probability theory, in $\mathcal{R}_0$, if the components of these vectors (with norms being *c*) along the *z*-axis are uniformly distributed in the interval [–*c*, *c*], then the probability density of the components on the *x*-axis is

$$\mathcal{D}(\theta,\eta) = c \cdot \cos\theta \cdot \sin\cos^{-1}\eta, \tag{6}$$

where the random variables are $\Theta \sim U(-\pi, \pi)$ and $H \sim U(-1, 1)$. Note that in this article, random variables (vectors) are expressed in capital Greek letters, and the values of random variables (vectors) are expressed in the corresponding lower-case letters. The component distribution of the vectors whose components are above (0, 0, *u*) on the *x*-axis is denoted by $\mathcal{D}_1$, and its probability density is written as

$$\mathcal{D}_1(\theta,\eta) = c \cdot \cos\theta \cdot \sin\cos^{-1}\eta, \tag{7}$$

where the random variables are $\Theta \sim U(-\pi, \pi)$ and $H \sim U(\frac{u}{c}, 1)$. Correspondingly, the component distribution of these vectors on the *z*-axis is denoted by $\mathcal{D}_3$, namely, $\mathcal{D}_3 \sim U(u, c)$. The component distribution of the vectors whose components are below (0, 0, *u*) on the *x*-axis is denoted by $\mathcal{D}_2$, and its probability density is written as

$$\mathcal{D}_2(\theta,\eta) = c \cdot \cos\theta \cdot \sin\cos^{-1}\eta, \tag{8}$$

where the random variables are $\Theta \sim U(-\pi, \pi)$ and $H \sim U(-1, \frac{u}{c})$. Correspondingly, the component distribution of these vectors on the z-axis is denoted by $\mathcal{D}_4$, namely, $\mathcal{D}_4 \sim U(-c, u)$. When the mean value of the components of the mixed distribution consisting of $\mathcal{D}_3$ and $\mathcal{D}_4$ on the z-axis is u, the corresponding mixture weights are $\frac{c+u}{2c}$ and $\frac{c-u}{2c}$, respectively. Note that $\mathcal{D}_1$ and $\mathcal{D}_2$ are randomly selected from the random vector swarms with the same characteristics as $\mathcal{D}_3$ and $\mathcal{D}_4$, respectively. When $\mathcal{D}_1$ or $\mathcal{D}_2$ is taken from all the samples from its population, $\mathcal{D}_3$ or $\mathcal{D}_4$ is also taken from all the samples from its population. The determination of each random 3-dimensional vector is the random determination of three coordinates. Therefore, the mixed distribution of the components of these vectors on the x-axis is only affected by the mixing weights of the two vectors. Thus, the mixed distribution consisting of $\mathcal{D}_1$ and $\mathcal{D}_2$ can be calculated in accordance with the abovementioned two weights (the analytical form of this mixed distribution cannot be given in this article at present); then, it can be found that the standard deviation of the velocity components on the x-axis of the particles in $\mathcal{R}_u$ is

$$\sigma_u = \frac{\sqrt{c^2 - u^2}}{\sqrt{3}}. \tag{9}$$

By evaluating the ratio between Eq. 9 and the standard deviation of the velocity components on the x-axis of the particles in $\mathcal{R}_0$, we can obtain the corresponding scale factor, namely,

$$\frac{\sqrt{c^2 - u^2}}{c}. \tag{10}$$

This equals the additive inverse of the Lorentz factor when c represents the speed of light. Obviously, the ratio of the standard deviations of the velocity components on the y-axis is also this scale factor, as shown in Eq. 10. This same factor can also be obtained by evaluating the ratio of the standard deviation of the velocity components (of particles forming $\mathcal{R}_u$) on the z-axis of the mixed distribution in $\mathcal{R}_u$ to the standard deviation of the velocity components (of particles forming $\mathcal{R}_0$) on the z-axis in $\mathcal{R}_0$. The detailed Mathematica code for the above calculation can be found in Part 2 of the Supplementary Information. This result implies that when a subset of the particles in the reference system $\mathcal{R}_0$ composed of particles moving at the same speed (such as c) and in (spatial) random directions forms a reference system $\mathcal{R}_u$ moving at speed u, the speed of the particles or their forming kth-order ($k \in \mathbb{N}$ and $k \neq 1$) particles in $\mathcal{R}_u$ will be relatively decreased, with a degree of deceleration corresponding to the value determined by the scale factor given by Eq. 10.

The abovementioned results prove that vectors with equal norms in Euclidean space exhibit special relativistic effects. In a stationary (inertial) reference system, if particles of different mass levels are moving in accordance with the relationship determined by Eq. 40 below, they will be considered to have different average velocities based on the corresponding Maxwell distributions. When the average velocity of a larger-mass-level particle composed of $\mathcal{K}$th-order particles is measured in a moving reference system $\mathcal{R}_u$ with velocity u, the corresponding degree of deceleration is determined by the average speed $c_{\mathcal{K}}$ of the $\mathcal{K}$th-order particles in accordance with the scale factor $\frac{\sqrt{{c_{\mathcal{K}}}^2 - u^2}}{c_{\mathcal{K}}}$, and when the average velocity of a larger-mass-level particle composed of $\mathcal{L}$th-order particles is measured similarly, the corresponding degree of deceleration is determined by the average speed $c_{\mathcal{L}}$ of the $\mathcal{L}$th-order particles in accordance with the scale factor $\frac{\sqrt{{c_{\mathcal{L}}}^2 - u^2}}{c_{\mathcal{L}}}$. If a moving species in a moving reference system $\mathcal{R}_u$ consists entirely of photons (an energy group of photons), then the degree of reduction in their average velocity is calculated using the Lorentz factor given in Eq. 10 (or determined by special relativity). At present, human beings can detect only photons and photon-level formations (such as electromagnetic waves and atomic clocks); from this point of view, the quantitative relationship given by special relativity is extremely accurate!

It is also noted that in $\mathcal{R}_u$, the slowdown on all three axes is the same. This means that there is no difference in physical laws that can be perceived between $\mathcal{R}_u$ and the stationary reference system $\mathcal{R}_0$. Therefore, when another moving reference system $\mathcal{R}_{u'}$ appears in $\mathcal{R}_u$, $\mathcal{R}_u$ can, in turn, be treated as a stationary reference system, which is a useful feature. This reveals that any reference system that satisfies the conditions given in 3-dimensional **HYPO 1–3** can be regarded as a stationary reference system, regardless of whether it is an absolutely stationary reference system. At the same time, any $\mathcal{R}_u$ randomly generated in $\mathcal{R}_0$ (when u is fixed) is equivalent to $\mathcal{R}_u$ which can be regarded as a stationary reference frame, and is the same $\mathcal{R}_u$ without physical law difference. Therefore, any $\mathcal{R}_u$ (as an undifferentiated particle set) can be regarded as generated randomly by $\mathcal{R}_0$ with relative motion (the speed is u) to $\mathcal{R}_u$. In this way, $\mathcal{R}_0$ (as an undifferentiated particle set) can also be regarded as randomly generated by $\mathcal{R}_u$ with relative motion (the speed is u) to $\mathcal{R}_0$. Therefore, not only the speed slowing effect in $\mathcal{R}_u$ exists but also other special relativistic effects (such as time expansion and length contraction). In this article, we will not discuss more about the special relativistic effects based on this logic. The principle of the special relativity effect of moving particles in space is the statistical effect of randomly moving particles, which is exactly the statistical effect of randomly moving particles with velocity direction aggregation being dominant. When direction aggregation is dominant, this special relativistic effect manifests, while in general, the possible aggregation effects also include the situation in which position aggregation is dominant (it shows the influence of "gravitation"). Here, these two (aggregation) effects are collectively called the general relativistic effect. Their essence is the statistical effect of randomly moving particles, and they are equivalent (see Part 3 of Supplementary Information for the proof process). If the equation established in this article can capture the statistical effects of moving particles, then it can also describe the effects of relativity.

3.3.3 Establishment of the Classical Diffusion Equation

To comprehensively describe the physical model mentioned earlier, we should establish a four-parameter equation, including time, for the law governing the motion of each particle, i.e., $\wp(x, y, z, t)$. For a system with n particles, it is necessary to establish an equation with $3n + 1$ degrees of freedom in the same time dimension, where $n \to +\infty$. This is obviously extremely unrealistic. In this article, we take the second best approach. We do not expect to describe all of the motion characteristics of all particles; instead, we wish only to describe the laws of particle motion succinctly and practically, to establish an equation that does not fundamentally fail to capture any critical motion characteristics of particles and can be described (solved) in actuality to the greatest possible extent. To do so, it may be appropriate to approach the problem from the perspective of statistics, that is, to establish a mathematical model with certain statistical characteristics on the basis of the physical model.

Theoretically, infinitely many aggregations of any number of particles can be found in infinite 4-dimensional space-time, although the greater the difference between the degree of aggregation and the total average density in space-time, the lower the formation probability of the corresponding particles, and the more unstable they will be in the time dimension. However, it is difficult to describe this situation with a specific function. Therefore, this article does not seek functions that apply at the micro level or for uncertain cases but rather seeks statistical description functions that are relatively certain by expanding the considered scope to cover a sufficiently large range of cases. Regardless of how these particles move in 3-dimensional space, their trajectories are continuous, which will lead to diffusion (or agglomeration) behavior, which is the generalized diffusion of randomly moving particles. Here, each moving particle is regarded as a vector whose direction is the same as the movement direction of the particle and whose norm is equal to the movement speed. Therefore, the generalized diffusivity of randomly moving particles is equivalent to the generalized diffusivity of random vectors (in direction). Thus, the "random vectors" and "randomly moving particles (or velocities)" mentioned below have the same meaning. Considering particles of the same mass and speed, the generalized diffusivity of the corresponding random vectors is equivalent to the generalized diffusivity of random momenta (which are also vectors). It is considered that the scale of the "generalized diffusivity of vectors" is simply the scale that is most suitable for describing the invariant laws for randomly moving particles. More information will be lost if the scale is even slightly more macroscopic (e.g., the scale can be approximately described by real diffusion), and there will be no invariant statistical law to follow if the scale is even slightly more microscopic (for example, the scale described at the beginning of this paragraph). At this scale, the external behavior of the vectors in a tiny space cannot be considered isotropic. When the randomly moving particles are in the case of (being equivalent to) the position aggregation being dominant, according to the Maxwell distribution, the total vector in a certain domain always points in an uncertain direction, and the norm is directly proportional to $\sqrt{k}$, where k is the (equivalent) number of vectors (see Part 1 of the Supplementary Information for details). Although the direction of the total vector in a tiny space cannot be determined from the Maxwell distribution, we hope to use appropriate constraints to obtain the distribution rules governing the norm and direction of the total vector at any position in space.

First, we determine the constraints acting on spatial vectors. Let the density of the vector sum at some point in space be denoted by $\boldsymbol{\mathcal{X}}$, which is a function of position and time, namely, $\boldsymbol{\mathcal{X}}(x, y, z, t)$. It is defined as follows: At a certain time t, let $\boldsymbol{\mathcal{Y}}(\mathcal{V})$ be a function of the sum of all vectors in the closed domain $\mathcal{V}$ containing $\mathcal{P}(x, y, z)$; then, $\mathcal{X}(x,y,z,t)=\lim_{\mathcal{V}\to\mathcal{P}}\frac{\mathcal{Y}(\mathcal{V})}{\mathcal{V}}$) (in the following, $\mathcal{X}$ is also a function of the spatial coordinates (x, y, z) and the time coordinate t). The situation in which particle position aggregation is dominant will be studied in the following.

$\mathcal{X}$ is a statistical average vector. When position aggregation is dominant, the relationship between $\mathcal{X}$ and the number of vectors follows the Maxwell distribution. As illustrated in Fig. 1a, it is assumed that there are two microdomains $\mathcal{V}_A$ and $\mathcal{V}_B$ of the same size along the normal direction on both sides of the segmentation surface Φ. If the sum of all vectors in $\mathcal{V}_A$ is $\overrightarrow{OA}$ and the sum of all vectors in $\mathcal{V}_B$ is $\overrightarrow{OB}$, then their sum is $\overrightarrow{OC}$, and their difference is $\overrightarrow{AB}$. Let the sum and difference vectors intersect at point M (Fig. 1b). In view of the previous assumption that the domains $\mathcal{V}_A$ and $\mathcal{V}_B$ on both sides of Φ are equal, there is no need to consider statistical effects before the particles move. Due to the characteristic that the distribution of the velocity directions is homogeneous, both vectors must tend to approach their average value $\overrightarrow{OM}$, that is, both $\overrightarrow{OA}$ and $\overrightarrow{OB}$ will tend towards $\overrightarrow{OM}$. Accordingly, the rate of change in $\mathcal{X}$ along the normal direction at a particular point should be related to the time-dependent rate of change in $\mathcal{X}$. This time-dependent rate of change is also affected by another inherent factor (i.e., the velocity of the particles forming $\mathcal{X}$), the concrete value of which is temporally uncertain. Therefore, the above two rates of change should be directly proportional when the differences between particles caused by density (including position aggregation and direction aggregation) are neglected.

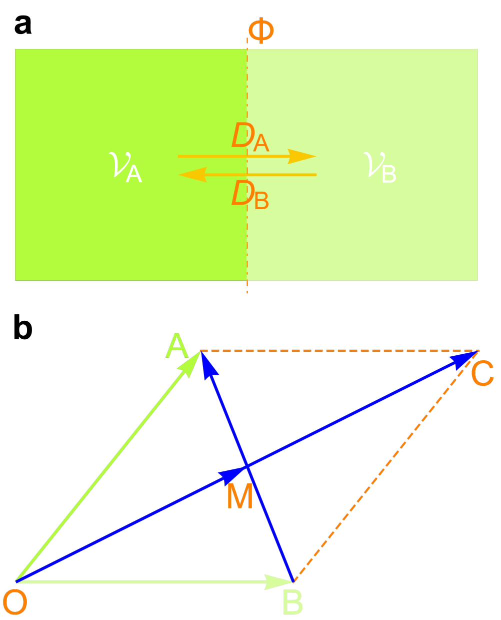

Figure 1 | Illustration of the principle of the generation of a mutual diffusion potential in microdomains $\mathcal{V}_A$ and $\mathcal{V}_B$.

In view of the similar calculus properties of vector and scalar, the derivation method for real diffusion is imitated here. If a domain $\mathcal{W}$ is enclosed by a closed surface Σ, then during the infinitesimal period $\mathrm{d}t$, the directional derivative $\frac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{N}}$ of $\boldsymbol{\mathcal{X}}$ along the normal direction of an infinitesimal area element dS on the surface Σ is directly proportional to the vector $\mathrm{d}\boldsymbol{\mathcal{X}}$ flowing through dS along the normal direction in the closed domain $\mathcal{W}$ enclosed by Σ (Fig. 2), under the assumption that the coefficient is a positive real number D.

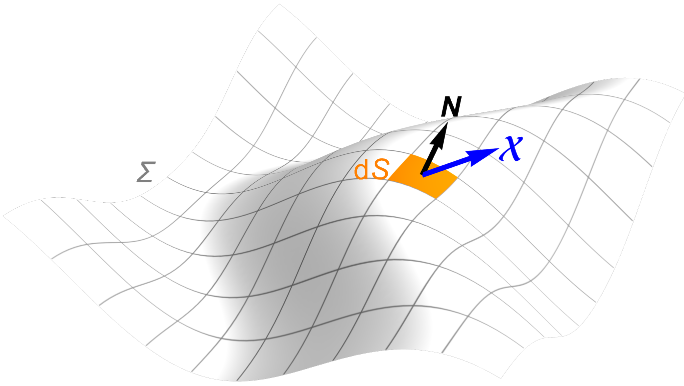

Figure 2 | Illustration of the diffusion of the vector sum density $\boldsymbol{\mathcal{X}}$.

From time t_1 to time t_2, when the influence of the vector density on D is not considered (i.e., the diffusion coefficient is the same at every position), the variation of the vector sum $\boldsymbol{\mathcal{A}}$ inside the closed surface Σ is

$$\delta \boldsymbol{\mathcal{A}} = \int_{t_1}^{t_2} \left(\oiint_{\Sigma} D \frac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{N}} \mathrm{d}S \right) \mathrm{d}t. \tag{11}$$

According to the Gauss formula, Eq. 11 can also be written in the form

$$\delta \boldsymbol{\mathcal{A}} = \int_{t_1}^{t_2} \left(\iiint_{\mathcal{W}} D \Delta \boldsymbol{\mathcal{X}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right) \mathrm{d}t, \tag{12}$$

where Δ is the Laplace operator, which describes the second derivative with respect to the position (x, y, z). The left-hand side of Eq. 11 (namely, $\delta \boldsymbol{\mathcal{A}}$) can also be written as

$$\delta \boldsymbol{\mathcal{A}} = \iiint_{\mathcal{W}} \left(\int_{t_1}^{t_2} \frac{\partial \boldsymbol{\mathcal{X}}}{\partial t} \mathrm{d}t \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{13}$$

By setting Eq. 13 equal to Eq. 12 and transforming the order of integration, we can obtain

$$\int_{t_1}^{t_2}\iiint\limits_{W}\frac{\partial \mathcal{X}}{\partial t}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t=\int_{t_1}^{t_2}\iiint\limits_{W}D\Delta\mathcal{X}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \tag{14}$$

Based on the observation that t_1, t_2 and the domain $\mathcal{W}$ are all arbitrary, the following equation can be written:

$$\frac{\partial \mathcal{X}}{\partial t}=D\Delta\mathcal{X}. \tag{15}$$

It is clear that the above conclusion still holds when $\mathcal{X}$ is formed by particles whose dominant velocity direction aggregation is equivalent to position aggregation being dominant.

To facilitate the task of vector decomposition in the following, a 3-dimensional vector needs to be converted into a plane vector. Next, we determine the constraints acting on plane vectors. Although the operation in Eq. 15 is performed using 3-dimensional vectors, when differential operations are performed on a spatial vector, the (sum or) difference operations are always performed at two points on the vectors that are separated by an infinitesimal distance; thus, all 3-dimensional vectors can exhibit only relative 2-dimensional characteristics. Consequently, by solving this differential equation, only 2-dimensional constraints can be obtained. Therefore, only the derivatives of plane vectors are needed to act as the derivatives of the 3-dimensional vectors (in this case, plane vectors can retain the important information, such as the norms of the vectors and the included angle between them). Moreover, the function of plane vectors obtained by solving the partial differential equation expressed in terms of plane vectors is unique and corresponds to the 3-dimensional vectors obtained from a differential equation of the same form. It is assumed that the function of plane vectors describing the density of the vectors or momenta is $\mathcal{M}(x, y, z, t)$, which corresponds to $\mathcal{X}$ at the point (x, y, z, t) (unless otherwise stated, in the following, $\mathcal{M}$ is a function of the spatial coordinates (x, y, z) and the time coordinate t). Thus, the abovementioned $\mathcal{X}$ can be replaced with $\mathcal{M}$. After this replacement, it is obvious that the norm of the plane vector will not change, but its direction will be reoriented. Finally, Eq. 15 can be written as

$$\left|\frac{\partial \mathcal{M}}{\partial t}\right|=D\left|\Delta\mathcal{M}\right|. \tag{16}$$

Now, let us determine the constraints on the direction of the plane vector $\mathcal{M}$. In view of the continuity of the trajectories of infinitesimal particles, since $\mathcal{M}$ is also characterized in terms of the statistical properties of an enormous number of particles, it should also be smooth. According to the theory of plane curves, the first and second derivatives of a plane vector in any direction in space are vertical. If an equation relating these derivatives is established following the above derivative relationship (Eq. 16), the direction needs to be adjusted to be consistent; then, this relationship can be written in the form

$$\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = \boldsymbol{i} D \Delta \boldsymbol{\mathcal{M}}. \tag{17}$$

where $\boldsymbol{i}$ is the imaginary unit. By multiplying both sides of Eq. 17 by $\boldsymbol{i}$, the form of the Schrödinger equation (without an external field) can be obtained:

$$\boldsymbol{i}\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = -D \Delta \boldsymbol{\mathcal{M}}. \tag{18}$$

Eq. 18 describes the distribution of a moving particle swarm (including the direction of movement) in space following the same diffusion coefficient; in other words, it is the classical diffusion equation. However, when the particle swarm is moving faster (velocity direction aggregation is dominant) or more particles are aggregating in a certain microdomain (position aggregation is dominant), the effect on diffusion is not clear. To more comprehensively describe this kind of diffusion process (which is called generalized diffusion), further analysis is needed.

3.3.4 Construction of the Generalized Diffusion Equation

To construct the generalized diffusion equation, we need to take into account many aspects, including whether the generalized diffusion coefficient *Đ* should vary and how to describe it to include the effects of general relativity (gravitation) and special relativity.

The classical view is that regardless of how large the target norms of vectors are, they follow a diffusion equation with the same diffusion coefficient (the Schrödinger equation). However, this article adopts an alternative viewpoint: *Đ* should vary with the value of the target vector. As derived above, when the influence of the vector sum density (including the aggregation densities of position and direction; the same is also considered below) on D is not considered, the diffusion for the vectors follows the Schrödinger equation. However, when the vector sum density is large, the effect on D cannot be ignored. Suppose that, as illustrated in Fig. 1a, the vector sum density in the microdomain $\mathcal{V}_A$ is greater than that in $\mathcal{V}_B$. If both microdomains exist in the same background field, there is a cost for the higher density in $\mathcal{V}_A$. If this high density is maintained at the next moment (in terms of probability, more uncertainty is introduced into the unit volume), which will inevitably affect the (average) movement speed of the particles, the overall movement speed of the particles in $\mathcal{V}_A$ will decrease (Section **3.3.2**). As mentioned above (or in Eq. 31 below), the particle speed is what determines D; therefore, the law governing the diffusion rate towards the right (D_A) is not the same as the law governing the diffusion rate in $\mathcal{V}_B$ towards the left (D_B) (under the assumption that $Đ$ is a combination of D_A and D_B). Therefore, it is necessary for the generalized diffusion coefficient to vary in time with the vector sum density to reflect this inequality.

In view of the above considerations, choosing the appropriate quantitative function to describe this phenomenon (with different laws) is the main problem to be solved in this article. First, the momentum vector in the microdomain is decomposed as follows.

3.3.4.1 Vector Decomposition

Let us determine the distribution function for a certain number of particles with equal probability (randomly) distributed in a certain domain, as follows: Suppose that the whole domain contains n particles in total. For convenience of description, the whole domain is also partitioned into n boxes of equal size. The gaps between boxes and the wall thickness are both 0. Now, let us determine the probability of k ($k \in \mathbb{N}$; the same also holds below) particles in a local area containing $\mathcal{M}$ boxes (suppose that the particles are small enough to fall into the box, not the wall). In view of the statement described above, the probability of particles existing in each domain is the same. Accordingly, the total number of possible cases describing how n particles can be randomly distributed among n boxes is n^n, there are $\binom{n}{k}$ total ways that k particles can be randomly chosen from among n particles, there are $\mathcal{M}^k$ total ways in which the k chosen particles can be randomly distributed among $\mathcal{M}$ boxes, and there are $(n-\mathcal{M})^{n-k}$ total ways in which the remaining n – k particles can be randomly distributed among the remaining $n - \mathcal{M}$ boxes. Therefore, the probability $P(\mathcal{M}, k)$ of k particles existing in $\mathcal{M}$ boxes can be expressed as

$$P(\mathcal{M},k)=\frac{\binom{n}{k}\mathcal{M}^k(n-\mathcal{M})^{n-k}}{n^n}. \tag{19}$$

Suppose that the number n of particles in the whole domain is infinite; then, by taking the limit of Eq. 19 as $n \to +\infty$, we find that

$$P(\mathcal{M},k)=\frac{e^{-\mathcal{M}}\mathcal{M}^k}{k!}, \tag{20}$$

again, where $\mathcal{M}$ denotes the number of boxes comprising the local domain of interest (the size of the volume in 3-dimensional space), k denotes the number of particles in that domain of $\mathcal{M}$ boxes, and P denotes the probability that k particles exist in that domain. Eq. 20 is the (position-based) Poisson distribution.

It is considered that this is the most appropriate method of partitioning a whole domain (the domain can be the whole universe or simply a broad range including the objects of investigation) into uniform boxes with the same number as that of particles. In addition to reducing the parameters involved and facilitating discussion, the reasons are as follows: if the boxes are slightly larger, they will not ensure the accuracy of the following vector decomposition; if they are slightly smaller, they will not adequately reflect the grouping effect of the particles. Therefore, in this article, the whole domain is divided into a number of uniform boxes equal to the number of particles it contains, and this partitioning serves as the basis for all of the following discussions. In this article, the whole domain (environment) is called the T-domain, and the local domain (target) is called the S-domain; the set of all particles contained in the T-domain is called the T-particle swarm, and the subset of particles contained in the S-domain is called the S-particle swarm.

Next, we will investigate the equiprobability distribution of the static particle swarm in the abovementioned S-domain $\mathcal{V}$. In Eq. 20, $\mathcal{M}$ denotes the number of boxes (volume) spanned by some S-domain (which belonged to the domain in which the target particles are distributed). Put another way, when the T-domain is partitioned into uniform boxes following the above method, $\mathcal{M}$ can also denote the average relative density of the particles in the S-domain $\mathcal{V}$, where the reference density is the average density of the T-particle swarm in the T-domain. $\mathcal{M}$ represents the corresponding multiple of the average density, k denotes the number of particles in one box, and P is the probability of k particles existing in that box. Thus, the distribution of the S-particle swarm in $\mathcal{V}$ is a Poisson distribution with density intensity $\mathcal{M}$. Next, we will analyze the Poisson distribution formula given in Eq. 20. In fact, it is the proportion of each term determined by k (when $e^{\mathcal{M}}$ is expanded as a power series) to the value of $e^{\mathcal{M}}$. The meaning here is that it is also the proportion of the number of boxes containing k

particles each to the total number of boxes in $\mathcal{V}$ when the S-particle swarm of relative density $\mathcal{M}$ is distributed among the reference boxes determined by the above criteria and spanned by the S-domain $\mathcal{V}$ (supposing that the number of boxes spanned by $\mathcal{V}$ is sufficiently large). According to mathematical analysis, we can see that the power series expansion for this case is unique, and obviously, this ratio distribution is also unique. If the right-hand side of Eq. 20 is multiplied by *k*, the result, denoted by $R(\mathcal{M}, k)$, takes the following form:

$$R(\mathcal{M},k)=\frac{e^{-\mathcal{M}}\mathcal{M}^k}{(k-1)!}. \tag{21}$$

In this way, termwise addition (by *k*) based on this expression offers a possible form for the decomposition of $\mathcal{M}$ into infinite items. Because the power series expansion above is unique, this decomposition form of the containing power series is also unique. According to the previous statement of physical meaning, the meaning of Eq. 21 is the relative density contributed by the particles in the boxes that contain *k* particles each to the total relative density $\mathcal{M}$ (the average relative density in $\mathcal{V}$) after the particles of relative density $\mathcal{M}$ are dispersed among the (infinitely many) reference boxes spanned by $\mathcal{V}$ with equal probability. Multiplying Eq. 21 by the number of boxes contained in $\mathcal{V}$ yields the total number of particles in the boxes containing *k* particles each. Since the distribution of particles in this form is definite (following the Poisson distribution), from this point of view, the decomposition of the relative density $\mathcal{M}$ in this (containing power series) form is also unique.

If $\boldsymbol{\mathcal{M}}$ is a complex number (or plane vector), Eq. 21 can be written in vector form as follows:

$$R(\boldsymbol{\mathcal{M}},k)=\frac{e^{-\mathcal{M}}\boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{22}$$

The form obtained by dividing Eq. 22 by *k* is still the ratio of each term (complex) determined by *k* (when $e^{\boldsymbol{\mathcal{M}}}$ is expanded as a power series) to the complex of $e^{\boldsymbol{\mathcal{M}}}$. There is one more dimension here, and the power series expansion is still unique. Similarly, the termwise addition of Eq. 22 also provides a decomposition form for the vector $\boldsymbol{\mathcal{M}}$. This decomposition form of the containing power series is also unique.

Now, we study the distribution of the velocity of the moving S-particle swarm in the abovementioned S-domain $\mathcal{V}$. If the particles of the T-particle swarm are moving randomly in the T-domain, the distribution of the S-particle swarm in a time slice in a sufficiently small S-domain (when the particle speed is fast enough) can also be approximately regarded as the equiprobable distribution. At the human scale (it will be proved with self-consistency that, in fact, in any scale range), the number of S-particles in almost every "microdomain" of the universe can be regarded as approaching infinity; therefore, the distribution of the moving S-particle swarm in a certain microdomain $\mathcal{V}$ can be described by Eq. 20. The moving particles in each type of box partitioned by k in one S-domain $\mathcal{V}$ can form a component vector, and these components can be added together to form the total vector in $\mathcal{V}$. Once the total 3-dimensional vector $\boldsymbol{\mathcal{Y}}$ of the moving S-particle swarm in $\mathcal{V}$, which includes the specific number of particles, is determined (that is, the average speed u of the system is determined), the norm (mathematical expectations) of each component vector should be (approximately) directly proportional to the number of particles forming it when the number of particles is large (see Part 4 of the Supplementary Information for details). It should be noted that even for $k = 1$, the number of samples in $\mathcal{V}$ should be very large. Therefore, the ratios between the norms (mathematical expectations) of the component vectors in various boxes partitioned by k are uniquely determined by the form of (containing) the power series determined by Eq. 20. As the limiting value $\boldsymbol{\mathcal{X}}$ of the quotient of $\boldsymbol{\mathcal{Y}}$ and $\mathcal{V}$, it can still be considered as a sum of 3-dimensional vectors in the S-domain $\mathcal{V}$. Therefore, there is also a form of component vectors with the ratios of norms determined by Eq. 20 spanning various boxes partitioned by k. When the 3-dimensional component vectors (spanning various boxes partitioned by k) of the 3-dimensional vector $\boldsymbol{\mathcal{X}}$ are mapped to the 2-dimensional component vectors (spanning various boxes partitioned by k) of the plane vector $\boldsymbol{\mathcal{M}}$, it is obvious that there is also a corresponding 2-dimensional form of component vectors with the ratios of norms determined by Eq. 20, but the direction is not determined. From the abovementioned decomposition method of scalar $\mathcal{M}$ (Eq. 21), it can be seen that if the ratios of norms of the component vectors of $\boldsymbol{\mathcal{M}}$ follow the Poisson distribution, it is necessary to use a unique and specified form of containing power series (this is one of the necessary conditions. If the ratios between the norms of the component vectors are required to be directly proportional to the numbers of particles forming them at the same time, the other necessary condition is required, i.e., $u = 1$. See Part 4 of the Supplementary Information for details), that is, the method for calculating the norms of the component vectors determined by Eq. 22. At this time, the direction of each component vector is uniquely determined. Therefore, the plane mapping of the sum of all the vectors in the boxes containing the same number k of particles is the component vector determined by k in Eq. 22. When k takes all values in $\mathbb{N}$, the termwise sum of these terms is the unique decomposition of $\boldsymbol{\mathcal{M}}$ (spanning various boxes partitioned by k), namely,

$$\boldsymbol{\mathcal{M}} = \sum_{k=1}^{\infty} \frac{e^{-\boldsymbol{\mathcal{M}}} \boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{23}$$

As mentioned earlier: regardless of whether the moving particles are dominated by position or direction aggregation, as long as their vectors are equal, their influences on diffusion and relativistic effects are the same; therefore, the position or direction aggregation are equivalent (when it is equivalent to position aggregation, the velocity direction is uniform distribution; when it is equivalent to velocity direction aggregation, the position is uniform distribution. The equivalent velocity direction or position aggregation represented by equal momentum in this article are both this meaning). According to the conclusion in Part 4 of the Supplementary Information, the norm (mathematical expectations) of each component vector is the product of the number of particles forming it and the average speed of the system it located. The average velocity of the particles in each S-domain $\mathcal{V}$ is regarded as 1, so the number of particles is numerically equal to the magnitude of the momentum. Accordingly, the equivalent numbers of vectors distributed in various boxes are directly proportional to the norms of vectors, and it is also comparable (computable) between S-domains. As mentioned above, $\mathcal{M}$ represents the relative density of particles in the S-domain $\mathcal{V}$, which is a concept of multiples. It is obvious that $\boldsymbol{\mathcal{M}}$ should also be a relative vector. The essence of the determination of the number of reference boxes for scalar $\mathcal{M}$ is the maximum number of boxes that can be occupied by particles in the T-domain. Here, when the particles in the S-domain $\mathcal{V}$ (which contains n particles in total, and the velocity of each particle is c) are thought of as a system with an average velocity of 1, the maximum number of boxes occupied (after expansion) is nc, namely, the number of (equivalent) reference boxes of vector $\boldsymbol{\mathcal{M}}$, and the direction of vector $\boldsymbol{\mathcal{M}}$ is the same as that of the absolute sum of vectors located at that place. Therefore, $\boldsymbol{\mathcal{M}}$ in Section **3.3.3** should be exactly the relative vector sum density. As mentioned above, the sum and difference operations between two spatial vectors are performed in their shared plane. In this plane, they can be decomposed respectively into a sum of plane vectors, as described in Eq. 23. Therefore, the two sets of plane component vectors can also serve as their respective spatial component vectors to correspondingly perform sum, difference or derivative operations.

3.3.4.2 Description of Diffusion

Suppose that the standard deviation of the projection (treated as a random variable; the same is done below) of the velocities of the k equivalent particles forming a kth-order particle onto each equivalent coordinate axis is σ. As mentioned earlier, the speed of kth-order particles follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$ (in this case, the situation of direction aggregation being dominant has been equivalented to the situation of position aggregation being dominant; the diffusion coefficient is the inherent statistical effect in the system, and only the average speed needs to be calculated in accordance with its definition). Then, the average speed of kth-order particles is

$$\overline{v} = 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{\sqrt{k}}. \tag{24}$$

For k_1th- and k_2th-order particles, the ratio of their average speeds is

$$\frac{\overline{v}_1}{\overline{v}_2} = \frac{\sqrt{k_2}}{\sqrt{k_1}}. \tag{25}$$

Because the sizes, or masses, of all 1st-order particles are the same, if the masses of a k_1th-order particle and a k_2th-order particle are m_1 and m_2, respectively ($m \propto k$), then according to the relationship shown in Eq. 25, the ratio of their average speeds can also be written as

$$\frac{\overline{v}_1}{\overline{v}_2} = \frac{\sqrt{m_2}}{\sqrt{m_1}}. \tag{26}$$

See Part 1 of the Supplementary Information for the detailed calculation and derivation process. According to Eq. 26, for any-order particles, the product of the square root of mass and the average speed is a constant (suppose it is κ_1). Then, when the mass of a kth-order particle is m, its average speed is

$$\overline{v} = \frac{\kappa_1}{\sqrt{m}}. \tag{27}$$

The diffusion coefficient can be defined as follows: it is the mass or mole number of a substance that diffuses vertically through a unit of area along the diffusion direction per unit time and per unit concentration gradient. Therefore, it is believed that classical real diffusion is consistent with the essence of vector diffusion described here. According to the Einstein-Brown displacement equation, the diffusion coefficient is

$$D = \frac{\overline{x}^2}{2t}, \tag{28}$$

where $\overline{x}$ is the average displacement of kth-order particles along the direction of the x-axis. To replace the average displacement $\overline{x}$ in Eq. 28 with the average speed (namely, $\overline{V}$) of kth-order particles along the direction of the x-axis, this diffusion coefficient can be transformed into

$$D = \frac{\overline{V}^2}{2} t^1. \tag{29}$$

The unit of the diffusion coefficient D is $m^2 \cdot s^{-1}$. By combining Eq. 28 and Eq. 29 (where t^1 and the t implied in $\overline{V}^2$ are consistent, so $t^1 = 1$ s), the abovementioned diffusion coefficient can also be regarded as follows: it is the average area over which kth-order particles spread out on a plane per unit time. This average area is related to the speed of a single kth-order particle. If the (average) speed of a single kth-order particle is $\overline{v}$, then the statistical average speed of these particles in one direction is

$$\overline{V} = \frac{\overline{v}}{2}. \tag{30}$$

The kth-order particle swarm spreads in the plane at this rate. By substituting Eq. 30 into Eq. 29 and combining $t^1 = 1$ s into the coefficient, which we then denote by κ_2, we can obtain

$$D = \kappa_2 \overline{v}^2, \tag{31}$$

where κ_2 is a constant coefficient with units of seconds (s).

By substituting Eq. 27 into Eq. 31, the diffusion coefficient of a (kth-order) particle swarm of (average) mass m is obtained:

$$D = \kappa_2 \left(\frac{\kappa_1}{\sqrt{m}} \right)^2 = \frac{\kappa_1^2 \kappa_2}{m}. \tag{32}$$

The above equation (Eq. 32) can also be thought of as the apparent diffusion coefficient of particle(s) with mass m described by the 1st-order particle swarm (which forms a particle of mass m after collapse) without relativistic effects. Moreover, the specific form of this coefficient is given in the Schrödinger equation without relativistic effects (i.e., in the case of the apparent diffusion described by a 1st-order particle swarm). By comparing the diffusion coefficient in the Schrödinger equation with Eq. 32, the following relationship can be immediately obtained:

$$\kappa_1^{\ 2}\kappa_2 = \frac{\hbar}{2}, \tag{33}$$

where $\hbar$ is the reduced Planck's constant.

3.3.4.3 Construction of the Generalized Diffusion Equation

Previously, we adopted the assumption that there is no interaction between infinitesimal particles. Even if there are interactions (which are produced by statistical effects and held by this article) between particles of larger mass levels they form, there is also a continuous dynamic process of large particles disappearance and generation, meaning that in fact, there is no interaction. In addition, considering that the essence of these "interactions" is gravitation (that is, the statistical effects of moving particles; other types of interactions can be treated similarly), it is equivalent to the concept that there is no interaction between advanced particles of various mass levels (The following Section **3.3.5.4** will prove that this view is self-consistent). Accordingly, in a time slice of a microdomain, the decomposition of the vector given by Eq. 23 must be exhibited, and all boxes containing the same number of particles in different microdomains containing different densities of vectors are equivalent. This is because there should be no differences between boxes of the same type (i.e., containing the same number of particles) when (the whole domain is equivalent to a system with an average speed of 1 and) the Poisson distribution determines the numbers of boxes of different types in different microdomains of different vector densities. Although the moving particles are distributed in a time slice of the microdomains with the same probability, when the overall behavior of k equivalent particles is counted, their equivalent average speed will inevitably slow down. Therefore, when more than the average number of equivalent particles appears in a limited domain, the overall speed of the particles in the domain will slow down or be significantly affected by statistical effects. The particles in various boxes partitioned by k move at their average equivalent speed and the centroids of boxes containing k equivalent particles each are, on average, located at the center of each box. Among all boxes of the same type (i.e., containing k equivalent particles), the average equivalent speed of each kth-order equivalent particle is the same and must conform to the diffusion form of the Schrödinger equation determined by the diffusion coefficient for particles of this type. Therefore, according to the particle numbers k in the previously partitioned boxes, from 1 to ∞, we study the corresponding term $R(\mathcal{M}, k)$, which is the component vector of $\mathcal{M}$. First, we investigate the diffusion of individual terms, and then, we add them together to characterize the overall slowing down behavior of diffusion.

Here, all the particles in each box containing k particles are regarded as forming a kth-order particle of a larger mass level, and together, all kth-order particles in all boxes containing k particles in microdomain $\mathcal{V}$ are called the kth-order particle swarm in that microdomain. Based on the above discussion, it can be considered that the average equivalent speed of each (kth-order) particle in the kth-order particle swarm is the same, and all of them have the same diffusion coefficient. According to the relationship given in Eq. 32 (the diffusion coefficient is inversely proportional to the mass of a kth-order particle, or the number of 1st-order particles forming a kth-order particle), if the diffusion coefficient of a 1st-order particle swarm is D_1, then the diffusion coefficient of a kth-order particle swarm is

$$D_k = D_1 \cdot \frac{1}{k}, \tag{34}$$

where $\frac{1}{k}$ is called the diffusion coefficient factor.

When it is not necessary to consider the influence of the deceleration effect of the statistical speed due to particle (position or direction) aggregation on diffusion, the diffusion behavior of interest is that of a 1st-order particle swarm, which is consistent with the description of diffusion given by the Schrödinger equation. Therefore, the diffusion coefficient is

$$D_1 = -\frac{\hbar}{2m}. \tag{35}$$

The diffusion equation determined by this coefficient describes the dynamics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m on the basis of the apparent diffusion rate (after deceleration) determined by the 1st-order particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the 1st-order particles in the background field. When $k > 1$, according to the above discussion, the diffusion coefficient of a kth-order particle swarm can be obtained by substituting Eq. 35 into Eq. 34, namely,

$$D_k = -\frac{\hbar}{2m} \cdot \frac{1}{k}. \tag{36}$$

This is equivalent to the proportional decline in the apparent diffusion rate of a target object (or the aggregation after collapse) of mass m due to the slowdown in the speed of the kth-order particles forming the target object. The meaning of the diffusion equation determined by this diffusion coefficient is similar to the case for 1st-order particles as considered above, that is, the dynamics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m are described on the basis of the apparent diffusion rate (after deceleration) determined by the kth-order particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the kth-order particles in the background field.

By taking the second partial derivative of $R(\mathcal{M}, k)$ (this is the plane vector sum in the boxes containing k moving particles, namely, the kth-order particle swarm, which is one of the component vectors in the whole microdomain $\mathcal{V}$) with respect to position (x, y, z) and considering the intermediate variable $\mathcal{M}$, we obtain the following expression:

$$\frac{\partial^2 R(\mathcal{M},k)}{\partial \mathcal{M}^2} \cdot T^2(\mathcal{M}) + \frac{\partial R(\mathcal{M},k)}{\partial \mathcal{M}} \cdot \Delta \mathcal{M}, \tag{37}$$

where $T^2(\mathcal{M}) = \left(\frac{\partial \mathcal{M}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{M}}{\partial y}\right)^2 + \left(\frac{\partial \mathcal{M}}{\partial z}\right)^2$. It should be emphasized that the absolute sizes of the two (infinitesimal) microdomains $\mathcal{V}_1$ and $\mathcal{V}_2$, which are selected to compare their differences, are equal when calculating the derivative of the vector $\mathcal{M}$. After multiplying Eq. 37 by the diffusion coefficient for the particle swarm of each order (Eq. 36) and then adding the products for all orders together, the complete generalized diffusion expression (including coefficients) can be obtained as follows:

$$-\frac{\hbar}{2m} \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{\partial^2 R(\mathcal{M},k)}{\partial \mathcal{M}^2} \cdot T^2(\mathcal{M}) + \frac{1}{k} \cdot \frac{\partial R(\mathcal{M},k)}{\partial \mathcal{M}} \cdot \Delta \mathcal{M} \right]. \tag{38}$$

The diffusion calculated in this way is the generalized diffusion from the whole (infinitesimal) microdomain $\mathcal{V}_1$ to $\mathcal{V}_2$. Eq. 38 can be simplified as follows:

$$-\frac{\hbar e^{-\mathcal{M}}}{2m} \left[\Delta \mathcal{M} - T^2(\mathcal{M}) \right]. \tag{39}$$

By combining the left-hand side of Eq. 18 with Eq. 39, a complete expression for the generalized diffusion equation for vectors is obtained:

$$\boldsymbol{i}\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t}=-\frac{\hbar e^{-\mathcal{M}}}{2m}\left[\Delta\boldsymbol{\mathcal{M}}-T^{2}(\boldsymbol{\mathcal{M}})\right]. \tag{40}$$

Therefore, the expression for the generalized diffusion coefficient with relativistic effects (including gravitation) is

$$Đ=-\frac{\hbar e^{-\mathcal{M}}}{2m}. \tag{41}$$

The diffusion coefficient here is not a constant but rather a natural exponential function that varies with the relative vector density of moving particles. Hence, the generalized diffusion equation and the generalized diffusion coefficient *Đ* for vectors have been determined. The norms of the spatial equivalent vectors in a microdomain can be determined in accordance with the Maxwell distribution, while the norms and directions of the spatial equivalent vectors in the complex plane can be determined in accordance with Eq. 40. Thus, the basic effective information for a spatial (moving) particle swarm has been derived.

The slowing down of diffusion based on spatial position is the only manifestation of the statistical effect of position aggregation being dominant in diffusion. Obviously, the statistical (gravitational) effect of particles can be reflected according to the treatment method in Eq. 38 when position aggregation is dominant. As mentioned above, all relativistic effects include statistical effects of position and direction aggregation. For the case of particle velocity direction aggregation being dominant, because it can be equivalent to the situation of position aggregation being dominant, the statistical effect is also transferred accordingly; that is, the phenomenon of diffusion becoming slow based on spatial position is the only embodiment. In summary, the statistical effects of these moving particles can be incorporated into Eq. 38 by multiplying the second derivatives of the component vectors (after comparative treatment) by different diffusion coefficients according to the classification standard based on k and summing the results. By contrast, equations that are subject to the constraints of Lorentz covariance (such as the Dirac equation and quantum field theory) are not sufficient to reflect all relativistic effects.

3.3.5 Further Study of Eq. 40

3.3.5.1 The Relationship with the Schrödinger Equation

By expanding the right-hand side of Eq. 40 using the power-series representation of $e^{-\mathcal{M}}$, we can obtain the following equation:

$$\begin{aligned} \boldsymbol{i}\frac{\partial \mathcal{M}}{\partial t} &= -\frac{\hbar}{2m}\left(1-\mathcal{M}+\frac{\mathcal{M}^2}{2}+\cdots\right)\left[\Delta\mathcal{M}-T^2(\mathcal{M})\right], \\ &= -\frac{\hbar}{2m}\left[\Delta\mathcal{M}-T^2(\mathcal{M})-\mathcal{M}\cdot\Delta\mathcal{M}+\mathcal{M}\cdot T^2(\mathcal{M})+\cdots\right]. \end{aligned} \tag{42}$$

When only the first term to the right of the equals sign in the second line of Eq. 42 is considered, this equation has the form of the Schrödinger equation without an external field. Thus, it can be concluded that Eq. 40 is the result of adding several corrections to the Schrödinger equation. Obviously, the form of the Schrödinger equation does not contain a relativistic effect, so the rest of the equation is caused by a relativistic effect. When the norm of the wave function $|\mathcal{M}| \to 0$, obviously, $T^2(\mathcal{M})$ is an infinitesimal of higher order than $\Delta\mathcal{M}$ (this is similar to the case of the sine and cosine wave functions when the velocity is small but the acceleration is large); moreover, the terms after $-T^2(\mathcal{M})$ to the right of the equals sign in the second line of Eq. 42 are all the products with $\mathcal{M}$ or the higher power of $\mathcal{M}$, and it is obviously they are also the infinitesimal of higher order than $\Delta\mathcal{M}$. Therefore, when $|\mathcal{M}|$ is sufficiently small, Eq. 40 can be approximated to take the form of the Schrödinger equation without an external field; however, when $|\mathcal{M}|$ is larger, the relativistic effect (the statistical effect of moving particles) in Eq. 40 is nonnegligible, and this equation cannot be replaced by the Schrödinger equation.

3.3.5.2 Nondispersive Particle Swarm

Creation and annihilation operators for particles are included in quantum field theory, but such descriptions are rigid. By contrast, the equation (Eq. 40) presented in this article naturally contains the processes of the appearance and disappearance of particles and can even give their half-lives (we will not study this problem in detail here). Eq. 40 is the equation describing the generalized diffusion of a randomly moving particle swarm. When

$$\Delta\mathcal{M}-T^2(\mathcal{M})=0, \tag{43}$$

$\mathcal{M}$ does not vary with time t, and a particle swarm that meets this condition is a nondispersive particle swarm. Such a particle swarm can also be regarded as a particle of a higher mass level, which is composed of a set of particles of a lower mass level that obey statistical laws.

To investigate the shape of a nondispersive particle swarm in detail, it is assumed that $\mathcal{M}$ is a function only of position (x, y, z) and that the position aggregation of moving particles is dominant. In 3-dimensional space, the following initial conditions are specified for Eq. 43:

$$\begin{cases} \mathcal{M}(0,0,0) = 1 + 2\boldsymbol{i}, \\ \mathcal{M}(x,y,z) = 0, x^2 + y^2 + z^2 = 4^2. \end{cases} \tag{44}$$

To numerically solve the simultaneous equations given in Eq. 43 and Eq. 44 (see the description of the process of generating Fig. 3 in Part 8 of the Supplementary Information for the detailed Mathematica code for the solution process), the distribution of mass density ($|\mathcal{M}|^2$) can be obtained, as illustrated in Fig. 3.

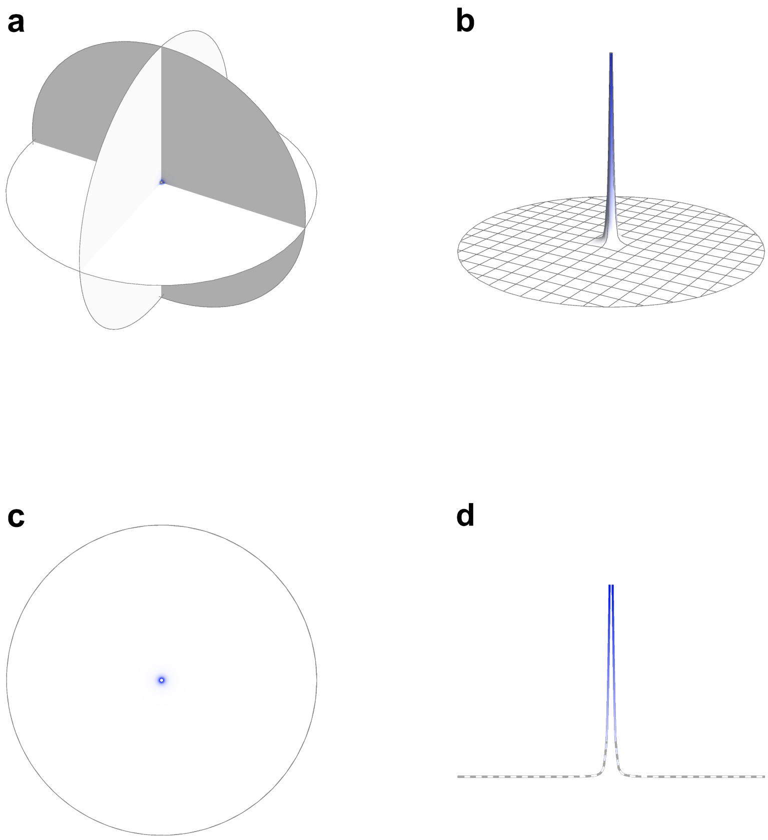

Figure 3 | Distribution of mass density for a particle swarm meeting the conditions given by Eq. 43 and Eq. 44 (shown from various perspectives): **a**, 3-dimensional density distribution; **b**, 2-dimensional density distribution at $z = 0$; **c**, 2-dimensional density distribution on the plane at $z = 0$; **d**, 1-dimensional density

distribution at $y = 0$ and $z = 0$. For convenience of comparison, the three (two) coordinate axes in each figure are displayed at a scale of 1:1.

It can be seen from the figure that the mass of such a stable particle is almost entirely concentrated in a small spherical area near the center of a larger spherical region and that the rest of this region is very sparse (with a very low mass density), similar to the structure of an atom. This result further qualitatively shows that we can study not only the distribution of electrons but also the distribution of nuclear mass by solving Eq. 40.

It should be noted that for the boundary condition $\mathcal{M}(0,0,0) = 1 + 2\boldsymbol{i}$ given in Eq. 44, the value on the sphere described by $x^2 + y^2 + z^2 = 0.04^2$ is assigned to be 1 + 2 $\boldsymbol{i}$ in the solution process to approximate this condition. It can be inferred that when the radius of this (inner boundary) sphere approaches infinitesimal, the shape is still similar to what is shown in Fig. 4. Moreover, the equations for the 2-dimensional case under the same conditions are also solved in this article; see Part 5 of the Supplementary Information for details. Without affecting the discussion of the problem, only a small value ($\sqrt{5}$) as the norm of the initial wave function and a relatively larger radius (0.04) of the inner boundary sphere are taken for the initial conditions. If the norm is further increased or the radius of the inner boundary sphere is further reduced, there will be a more obvious contrast in mass density, but the difficulty of solving the equation and drawing the graph will also be greatly increased. In addition, the value of the function on the sphere with a radius of 4 is 0 with the above boundary conditions, which is also approximately consistent with the actual situation. In reality, the mass density environment around the research object is complex. Even if this complex environment is not considered, the object will exist in a background field with a nonzero mass density. In this case, the outer boundary condition should be a constant value close to 0 or a wave function with a norm close to 0 at infinity (when $\mathcal{M}$ is a function of position (x, y, z) and time t).

Based on the analysis of the above equation (Eq. 40), the formation mechanism for particles of a large mass level in the universe can be estimated as follows: As particles of a lower mass level in the universe undergo randomly fluctuating movements, if they meet the appropriate external conditions, they will have the chance to form many standing waves with the same distribution. When the external conditions change, these standing waves will undergo generalized diffusion over time. Some of them will disappear; some of them will form particle swarms that also essentially meet the above conditions. These particle swarms will become larger-mass-level particles that will decay extremely slowly (the decay rate depends on the value of $e^{-\mathcal{M}}$ in Eq. 40 and the conformity of the particle shape to the condition given in Eq. 43) and thus will continue to stably exist in the universe for a long time (if the external or boundary conditions do not change significantly). At different positions and under different external conditions, standing waves of different densities may form. Once the above conditions are essentially met, these standing waves can persist for a long time, thus forming more stable particles of different mass levels. Thus, it can be concluded that the concept of macroscopic mass is a characterization of a number of agglomerated lower-mass-level particles in a certain domain, while the concept of macroscopic energy is a characterization of a number of nonagglomerated lower-mass-level particles in a certain domain. Moreover, the boundary between these two concepts is extremely blurred. It should be noted that due to limitations of computing scale, it is impossible to simulate or watch the process of the generation of particles from uniformly distributed energy or other conditions described in this article; therefore, the above possible generation process is merely hypothesized, and its veracity remains to be investigated.

The above content only discusses the case of the particle position aggregation being dominant. For the case of the particle moving direction aggregation being dominant, the solution is the same. The specific discussion will be carried out in Section **3.3.5.4**.

3.3.5.3 Method of Acquiring the Initial Wave Function

Obviously, the initial conditions for the solution to Eq. 40 place constraints on the norm of the wave function. The following presents the method of acquiring the initial wave function when the position aggregation of lower-mass-level particles is dominant (mostly, in this case). To eliminate the classical diffusion coefficient *D* by solving the simultaneous equations given by Eq. 29, Eq. 32 and Eq. 33, we can obtain

$$\bar{V}^2 t^1 = \frac{\hbar}{m}. \tag{45}$$

Note that t^1 = 1 s in Eq. 45; we ignore it for now. By replacing *m* in Eq. 46 with the mass $\mathfrak{M}$ in a certain domain $\mathcal{V}$ and extracting the roots of both sides of the equation, we can obtain a quantitative expression for the average velocity $\bar{V}$ of the particle swarm in this domain after finding the norm of both sides of the equation:

$$\left|\bar{V}\right| = \sqrt{\frac{\hbar}{\mathfrak{M}}}. \tag{46}$$

From a statistical point of view, the norm of the vector sum in a certain domain is $\left|\boldsymbol{\mathcal{X}}\right| = \left|\bar{\boldsymbol{\mathcal{X}}}\right| \cdot k$ in a system with identical norms and identical probabilities of all directions in space, where $\left|\bar{\boldsymbol{\mathcal{X}}}\right|$ is the average contribution of each particle to the total norm of the vector sum in the domain and *k* is the number of vectors. If these random vectors are regarded as representing the random movements of small particles moving with the same speed in space, then the total momentum of the particle swarm in domain $\mathcal{V}$, or the sum of the total velocity in domain $\mathcal{V}$ from a statistical point of view, is

$$\left|V\right| = \left|\bar{V}\right| \cdot k, \tag{47}$$

where *k* is the number of particles in domain $\mathcal{V}$. By substituting Eq. 46 into Eq. 47 and replacing *k* with $\frac{\mathfrak{M}}{\mu}$, we obtain

$$\left|V\right| = \sqrt{\frac{\hbar}{\mathfrak{M}}} \cdot \frac{\mathfrak{M}}{\mu} = \frac{\sqrt{\hbar \mathfrak{M}}}{\mu}, \tag{48}$$

where μ is the mass of a single particle.

From the perspective of Max Born's interpretation of the wave function, after the wave function of a system is normalized (let it be denoted by ψ_1), the mass density of the wave function everywhere it reaches is expressed as follows:

$$\rho_m = \left|\psi_1\right|^2 \cdot m, \tag{49}$$

where *m* is the mass of the target object (the same as the *m* given in Eq. 40). In fact, even from the perspective of statistics in accordance with the logic of this article, the square of the speed or the square of the norm of the wave function is also directly proportional to the mass; see Part 1 of the Supplementary Information for details.

Since the wave function represents the velocity or velocity density per unit volume, if ψ_0 is used to denote the wave function at a certain point, then by substituting Eq. 49 into Eq. 48, the norm of the wave function at a certain point can be obtained as follows:

$$\left|\psi_0\right| = \frac{\sqrt{\hbar m}}{\mu} \cdot \left|\psi_1\right|. \tag{50}$$

In view of the discussion presented in Section **3.3.4.1**, a further operation on ψ_0 is needed to obtain the relative wave function $\mathcal{M}_0$ (ψ_0 is divided by the speed of a single particle and the number of particles per unit volume in background field, and $\mathcal{M}_0$ is assigned to the same direction as ψ_1). If the system is composed of particles at the photon level, $\mathcal{M}_0$ can be written as

$$\mathcal{M}_0 = \lambda \cdot \frac{\sqrt{\hbar m}}{c \cdot \bar{\rho}_{m,0}} \cdot \psi_1, \tag{51}$$

where c is the speed of light, $\bar{\rho}_{m,0}$ is the average mass density over a range larger than $\mathcal{V}$ (the background field) and is generally accepted to be $\bar{\rho}_{m,0} = 2 \times 10^{-28}\,\mathrm{kg \cdot m^{-3}}$, and λ is the unit coefficient, whose value is $1\,\mathrm{m^{-3} \cdot s^{-\frac{1}{2}}}$. The purpose of this coefficient is mainly to correct the dimensional difference caused by the conversion of the diffusion coefficient into a velocity and to compensate for the specification of the unit volume implied in the conversion relationship. The method described above is the acquisition method for the initial condition for Eq. 40.

In the case of a low mass density (such as the electron distribution outside the nucleus of an atom), the norm $|\mathcal{M}|$ of the wave function is extremely small in the initial condition obtained from Eq. 51 (electrostatic interaction is not considered in the initial condition; however, even if the electrostatic interaction with the nucleus were to be considered in the calculation process, the norm of the wave function would still be small, and the details will not be discussed in this article). As mentioned before, in this case, Eq. 40 is almost the same as the Schrödinger equation. That is, Eq. 40 will reduce to the Schrödinger equation when solving for the electron distribution outside the nucleus of an atom, while the case of the application of an external electromagnetic field to the atomic system needs to be investigated separately. It should be noted that, as mentioned above, when the target system (background field) is composed of particles at the photon level, c in Eq. 51 is equal to the speed of light, while if the target system is composed of particles at another mass level, c is equal to the speed of particles at that mass level. The background domain here can be either the whole universe or a smaller range that encompasses the research objects. Once the background domain is defined, the corresponding average mass density $\bar{\rho}_{m,0}$ of the background field can be determined. In addition, as seen from Eq. 40 and the acquisition method for the initial wave function, only when both the position aggregation and direction aggregation are at a maximum is $e^{-\mathcal{M}}$ infinitesimal and can a particle swarm ($|\mathcal{M}|^2$) that does not satisfy the condition in Eq. 43 be completely nondispersive. In other words, for a particle swarm for which only direction or position aggregation is dominant, diffusion cannot be completely prohibited when the shape of the particle swarm does not satisfy the condition described in Eq. 43.

3.3.5.4 Further Discussion

Although Section **3.3.5.3** only considers the case when the particle position aggregation is dominant, the way in which the initial wave function $\mathcal{M}_0$ is acquired from Eq. 51 still reflects the way in which the wave function at a point is calculated. Therefore, to judge whether the wave function $\mathcal{M}$ at a point changes with the selections of the reference system or the minimum reference particles, it is necessary only to examine whether the method of acquiring the initial wave function $\mathcal{M}_0$ has changed. In view of the discussion in Section **3.3.2**, in any stationary (inertial) reference system, the synchronous change in movement and time from which movement is measured, from which the obtained speed of light c and the velocity determining $\mathcal{M}$ are both constant; in addition, the speed of light is included in Eq. 51, and other parameters are not limited by the reference system. Therefore, in any reference system, as long as the conditions **HYPO 1–3** in 3-dimensional space are satisfied, Eq. 40, which is deduced in this article, and Eq. 51, which is the acquisition method for the initial wave function, are applicable. Moreover, let us consider the case of particles of different mass levels being thought of as the minimum (infinitesimal) reference particles in the same reference system. There is no need to consider the mass of an infinitesimal particle in the acquisition method for the initial wave function (Eq. 51); regardless of which mass-level particle is treated as the minimum reference particle, the synchronous change in movement and time from which movement is measured, from which the obtained speed of light c and the velocity determining $\mathcal{M}$ are both constant; in addition, the other parameters in the acquisition method (Eq. 51) for the initial wave function $\mathcal{M}_0$ are not limited by the selection of the minimum reference particle. Therefore, regardless of how large the particles are that are regarded as the minimum reference particles, Eq. 40 and Eq. 51 are still applicable. In summary, the gravitational effect between various particles can be regarded as a statistical effect of moving particles; it can be considered that there is no interaction between particles of any mass level. This is self-consistent with the hypothesis stated in **HYPO 3**. In this way, particles can be constructed step by step, and particles of a high mass level can form particles of a higher mass level under appropriate conditions. The whole universe is quantized regardless of the mass level, and each mass level is also equivalent. This is self-consistent with the statement that "the substance in the world is quantized", which is the axiomatic inference (or hypothesis) derived from **AXIO 2**.

When direction aggregation is dominant, the form of Eq. 43 allows the velocities of some particles to be extremely fast, while the velocities of other particles that are not far from them decrease rapidly. Particles with a very fast velocity can also have a higher mass density than their surroundings, and under certain conditions, the mass and velocity can mutually transform (as long as the condition of Eq. 43 is met). The above conclusion is consistent with the hypothesis of "high-speed and random motion of particles in the universe" mentioned above.

To summarize the results stated above, in any reference system that satisfies the conditions of **HYPO 1**–**3** in 3-dimensional space, no matter what the mass level of the basic (infinitesimal) reference particle considered in this article actually is, and no matter how slow the "absolute" movement speed of that particle, from the perspective of human understanding, the particle mass at this level is infinitesimal, and the speed is infinite (corresponding to the expansion of the self-consistent range). This conclusion gives legitimacy to the view that "photons have small or infinitesimal masses". At the same time, it also gives legitimacy to the vector decomposition in the (infinitesimal) microdomain $\mathcal{V}$ introduced in Section **3.3.4.1** and the viewpoint that "the absolute coordinate system needs to move along with the whole particle swarm". In this way, Eq. 40, which is derived in this article, and Eq. 51, which is the acquisition method for the initial wave function, can be applied not only in a local space but also in a broader space (or in various inertial reference systems), and they can also be applied not only in a low-mass-level particle system but also in a high-mass-level particle system (i.e., either low-mass-level particles or high mass-level particles can be treated as infinitesimal particles). Based on the above conclusions, we infer that Eq. 40 and the abovementioned physical model are completely equivalent.

3.4 A Simple Verification of the Mathematical Model

It can be seen from the above discussion that Eq. 40 can completely describe all objects and phenomena in nature and that the situation described by Eq. 40 is logically self-consistent with the physical model (hypotheses) given at the beginning of this article; however, its reliability in real situations should be further tested. In this article, the description of the time-dependent diffusion of a 1-dimensional Gaussian wave packet without an external field with even parity along the *x*-axis and the initial condition e^{-2x^2} is solved for comparison with known theories to guide further discussion. For convenience of operations, we adopt natural units (i.e., $\hbar = c = 1$) and set $m = 1$ eV for all evaluations in this section, while the International System of Units is still adopted in other sections.

As mentioned above, to correctly solve Eq. 40, it is necessary to give the equation an initial condition with an appropriate norm in accordance with Eq. 51, which is different from solving the Schrödinger equation. In the following, the average electron mass density outside the nucleus of the hydrogen atom is taken as a reference to determine the norm of the wave function for the initial condition of the Gaussian wave packet e^{-2x^2} in the case of time-dependent diffusion. It is assumed that these two kinds of problems are essentially the same; both of them concern the movements of particles at the photon level. Let $m = 9.109\ 389\ 7(54) \times 10^{-31}$ kg, which is the electron mass; then, the coefficient pre the normalized wave function ψ_1 in Eq. 51 can be evaluated to be approximately 1.63×10^{-13}. The normalized norm of the above Gaussian wave-packet is $\sqrt{\frac{2}{\pi}}$. Therefore, the approximate value $\mathcal{M}_0(x,0) = 10^{-13}e^{-2x^2}$ of the same order of magnitude can be taken as the initial condition without affecting the discussion of the problem (after verification, when the coefficient of e^{-2x^2} is less than 10^{-3}, the maximum relative deviation between the contours for the wave packet obtained using these two methods is less than 1.14% in all ranges; see Part 6 of the Supplementary Information for details). For comparison, the case of a larger norm in the initial conditions (such as $\mathcal{M}_0(x,0) = e^{-2x^2}$) is also evaluated. At the same time, the Schrödinger and Dirac equations are used to solve for the description of the time-dependent diffusion of this wave packet. For the Dirac equation, the case in which the two components of the wave function are equal (i.e., $\chi_1(x,0) = \chi_2(x,0) = \frac{\sqrt{2}}{2}e^{-2x^2}$) is taken as the initial condition here (see the description of the process of generating Fig. 4 in Part 8 of the Supplementary Information for the detailed Mathematica code for the solution process).

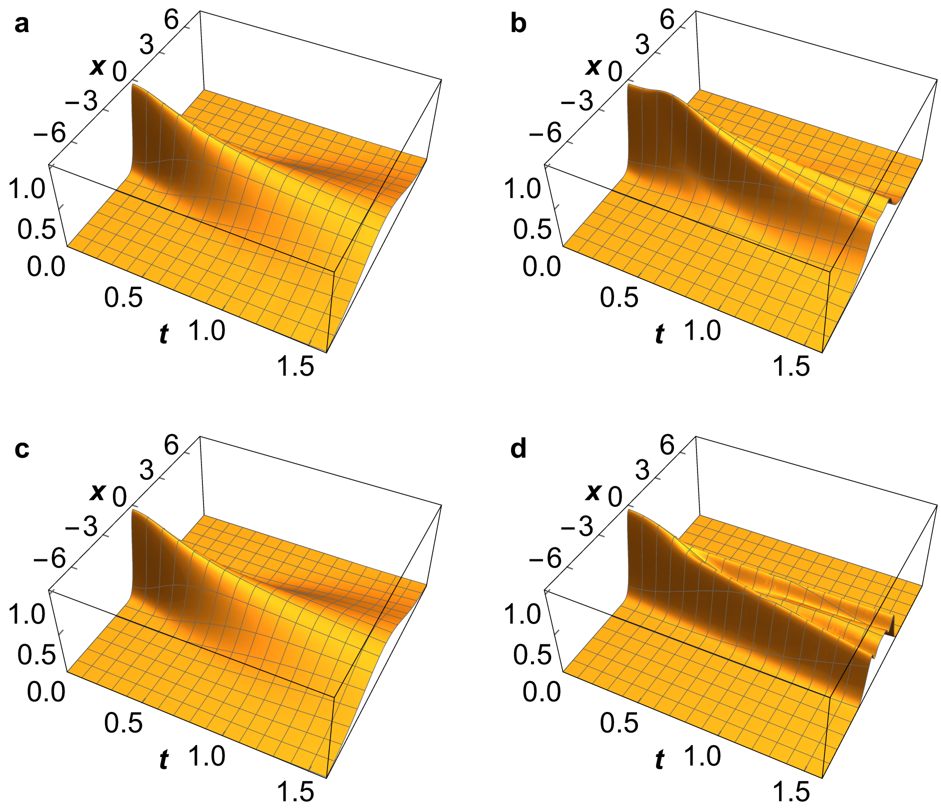

Figure 4 | Illustrations of the 1-dimensional time-dependent diffusion of the Gaussian wave packet e^{-2x^2} as obtained using various methods in natural units. **a**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x,0)=10^{-13}e^{-2x^2}$. The norm has been magnified ($\times 10^{13}$) to facilitate the shape comparison. **b**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x,0)=e^{-2x^2}$. **c**, Computation result of the Schrödinger equation. **d**, Computation result of the Dirac equation.

As illustrated in Fig. 4, there is almost no difference between the visualization of the time-dependent diffusion of the wave packet obtained from Eq. 40 under an appropriate initial condition $\mathcal{M}_0(x,0)=10^{-13}e^{-2x^2}$ (Fig. 4a) and that obtained from the Schrödinger equation (Fig. 4c) (note: for convenience, the norms of wave functions, not the squares of the norms, are discussed in this section). This small difference is illustrated in greater detail in Fig. 5 by presenting the standard deviations of the norms at different diffusion times, from which it can be seen that the profiles of the Gaussian wave packets predicted by the two methods almost completely coincide at each time point. Thus, it is further verified that the equation given in this article well approximates the Schrödinger equation (at least for the problem of a Gaussian wave packet) in a domain with an extremely sparse mass density (such as the distribution of electrons outside the nucleus of an atom, excluding the influence of the electric field of the nucleus), which is consistent with the conclusion presented in Section **3.3.5.1** above.

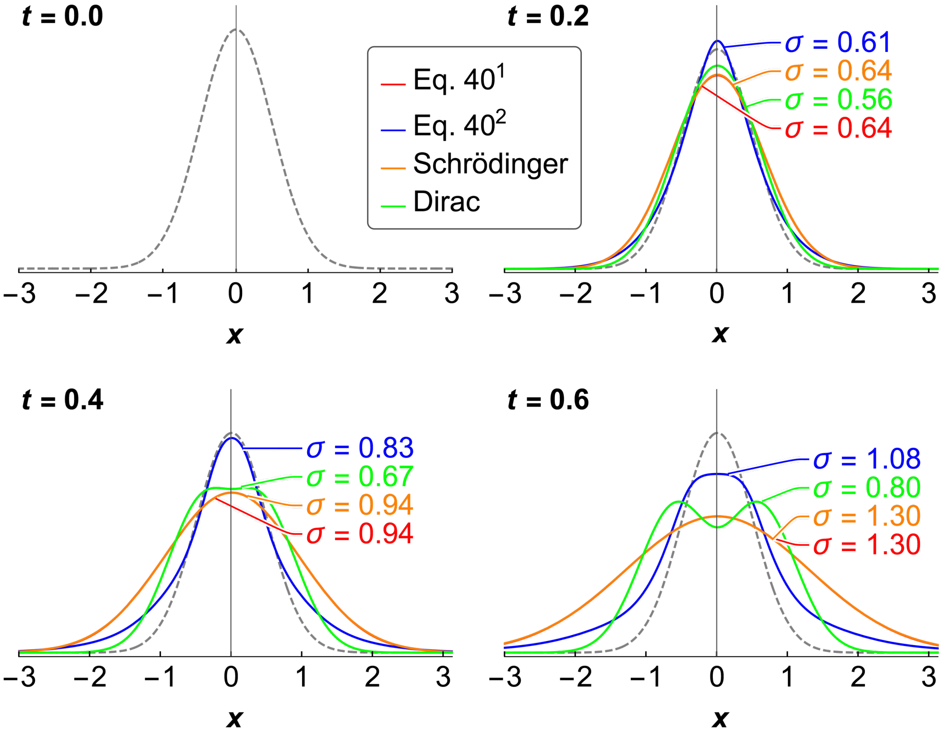

Figure 5 | Visualizations of the time-dependent diffusion trend of a Gaussian wave packet (norm) as predicted using four methods (for Eq. 40[1], the initial condition is $\mathcal{M}_0(x,0)=10^{-13}e^{-2x^2}$; for Eq. 40[2], the initial condition is $\mathcal{M}_0(x,0)=e^{-2x^2}$) in natural units at various moments in time (t = 0.0, 0.2, 0.4, and 0.6 eV^{-1}).

If the norm of the wave function in the initial condition is large (such as $\mathcal{M}_0(x,0)=e^{-2x^2}$), the profile of the wave packet will show an obvious bulge or particle (position or direction) aggregation near t = 0.3 eV^{-1} (Fig. 4b). Because of self-aggregation, the profile obtained from Eq. 40 is steeper along the direction of the x-axis, which can be clearly seen from the standard deviation of the norm of the Gaussian wave packet in Fig. 5 at the three nonzero times. Under such initial conditions, the diffusion rate predicted by Eq. 40 is not as fast as that predicted by the Schrödinger equation, and the main peak in the profile does not tend to quickly dissipate; this is closer to the situation described by the Dirac equation (Fig. 4d). It can be speculated that this behavior is mainly caused by the gravitation of the wave packet itself. After t = 1 eV^{-1}, the main peak begins to split into two peaks (for more obvious splitting, see the case in which the coefficient of e^{-2x^2} is 1.4 in Part 7 of the Supplementary Information); in the case described by the Dirac equation, strong splitting occurs after t = 0.5 eV^{-1} (the main peak splits into two secondary peaks, and then each secondary peak splits into two smaller peaks). This phenomenon is considered to be caused by the fact that the gravitation of the wave packet itself is not considered in the Dirac equation and the corresponding correction to the real result is excessive. This is also confirmed by the standard deviation profiles illustrated in Fig. 5.

To study the influence of the norm in the initial condition on the diffusion of the wave packet in greater detail, we also compare the shapes of the key parts of the profiles (the time-dependent trend of the norm for the wave packet at $x = 0$ and the wave packet at the maximum value of the norm) after assigning various initial conditions ($10^{-13}e^{-2x^2}$, e^{-2x^2}, $1.2e^{-2x^2}$ and $1.4e^{-2x^2}$) for Eq. 40 (see the description of the process of generating Fig. 6 in Part 8 of the Supplementary Information for the detailed Mathematica code for the solution process); the results are illustrated in Fig. 6. It can be seen from this figure that when different initial conditions are specified, with norms ranging from small to large, the wave-packet diffusion profiles predicted by Eq. 40 (at $x = 0$) are initially consistent with those predicted by the Schrödinger equation and then gradually tend to continue to agglomerate near 0.3 eV^{-1} (the profile gradually begins to bulge); the corresponding trend is shown in Fig. 6a. In addition, Fig. 6b shows the shape of the wave packet at the highest point (see Part 7 of the Supplementary Information for the full spectrum waveforms for the initial conditions of $\mathcal{M}_0(x,0) = 1.2e^{-2x^2}$ and $\mathcal{M}_0(x,0) = 1.4e^{-2x^2}$ and the corresponding comparisons with the Dirac equation). As the initial norm gradually increases, the wave packet initially will gradually shrink, and when the norm reaches the maximum, the waveform will gradually become steeper and steeper and (presumably) will gradually approach that of the function satisfying Eq. 43. It can also be seen from this trend that as the mass density of the wave packet increases, the attenuation speed of the wave packet becomes slower.

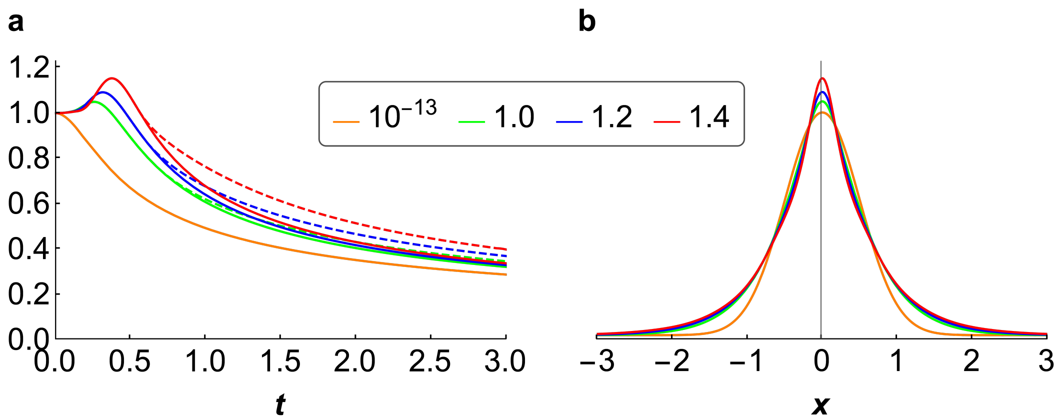

Figure 6 | Comparison of the shapes of the Gaussian wave packets obtained when assuming different initial conditions $\mathcal{M}_0 = (10, 1.0, 1.2, 1.4)\,e^{-2x^2}$ in natural units. **a**,

Time-dependent trends of the norm for the wave packet at $x = 0$ (solid line) and for the wave packet at the maximum value of the norm (dashed line of the same color). **b**, Profiles at the maximum values of the norms. The numbers in the legend are the coefficients of e^{-2x^2} corresponding to the different initial conditions. The initial norm at $t = 0$ eV^{-1} in each case is normalized to facilitate the shape comparison.

4. Conclusions

In this article, a physical model of the whole universe has been constructed from the most basic philosophical paradoxes. Based on this model, a mathematical equation has been established to describe the generalized diffusion behavior of moving particles, and its simple verification is also carried out. For the first time, relativistic effects have been interpreted as statistical effects of moving particles, based on the understanding that the higher the degree of aggregation of particles is (in terms of either position or movement direction), the greater their average velocity in other directions is consumed. Thus, the gravitational force and (special) relativistic effects can be actually integrated into the derived equation (achieved by selecting an initial wave function with a specific norm when solving it), thus avoiding the problem of nonrenormalizability when gravitation is introduced into quantum mechanics. Further analysis has shown that the gravitation between objects is also caused by a statistical effect of randomly moving particles. These particles can also form stable nondispersive particle swarms, which, as larger-mass-level particles, can further unite into stable nondispersive particle swarms. Regardless of the mass level of the particles that are regarded as infinitesimal particles and regardless of how slow the speed is regarded as an infinite speed, the equations derived in this article are equivalent at the scale of human understanding. On the one hand, based on the hypotheses stated in **HYPO 1–3** (physical model), this article has deduced the form of the Schrödinger equation and the conclusions of special relativity, thus further confirming the rationality of these hypotheses concerning the universe. On the other hand, based on these assumptions, the derived equation contains the conditions for the generation of stable particles, which, in turn, form a logical self-consistency with the previous assumptions. Therefore, the basic physical model of the universe established in this article is a relatively reliable and complete logical model, and the universe is likely to be a product of the movement of noninteracting random particles and to obey the mathematical equation given in Eq. 40.

Based on this physical model, we can answer the questions raised at the beginning of this article. The universe is both large and small, and its size is only a relative logical concept. From this relative point of view, the universe is boundless (for human beings). The current appearance of the universe is only one stage of its evolution, and this evolution is a process without beginning or end. The constant random motion or generalized diffusion of particle swarms is the mechanism by which it operates, and there is no beginning or end point of this diffusive movement (although there may be a beginning and end in local space). The energy in the universe cannot be designated as existing or not; it is merely a relative concept arising from the movement of infinitesimal particles. If we observe the group behavior of these particles, their average speed will decrease, giving rise to the concepts of time, space, speed and energy. Therefore, these concepts (including force) are all statistical effects that arise when observing these moving particles from different angles. Energy will never be exhausted, nor will it increase or decrease. According to this view, the total entropy in the whole universe will also not increase or decrease.

Acknowledgments I thank the engineers at *Wolfram Inc*. for technical support.

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Appendix:

Supplementary Information
(Mathematica v12.1.1.0 code of TraditionalForm)

Part 1. The Square of the Norm of the Average Velocity is Proportional to the Number of Vectors

Definition: Particles with a higher mass level composed of *k* particles are called *k*th-order particles. Then, the velocity of a *k*th-order particle is the velocity of the overall center of mass of the *k* particles, which is the average of the velocity vectors of all these particles.
Assumption: Each particle is moving at the same speed and in a random direction in space.
Thus, the projection of the velocity vector of a *k*th-order particle onto one of the three equivalent coordinate axes of the 3-dimensional Cartesian coordinate system is the mean value of the projection (onto the same axis) of the velocity vectors of the 1st-order particles forming the *k*th-order particle, which follow the same distribution; therefore, it approximately follows a normal distribution (central limit theorem).
There are three equivalent (approximate) normal distributions, one on each of the three axes, which are not completely independent. However, James Clerk Maxwell and Ludwig Boltzmann proved that these distribution can, in fact, be equivalently treated as completely independent. This is because randomly selecting a vector is equivalent to randomly determining a three-axis coordinate; moreover, the problem of the momentum transfer of gas molecules participating in random collisions is also equivalent to the problem discussed in this article.

First, the probability density of the norm of the 3-dimensional vectors formed by three normal distribution $N(0, \sigma_2)$ components that are independent on three coordinate axes is calculated.

```
In[∘]:= Clear["Global`*"];
        𝒟 = Simplify[PDF[TransformedDistribution[x^2 + y^2 + z^2,
              {x, y, z} ≈ ProductDistribution[{NormalDistribution[0, σ2], 3}]], x], Assumptions → σ2 > 0];
        𝒟1 = PDF[TransformedDistribution[√x , x ≈ ProbabilityDistribution[𝒟, {x, 0, +∞}]], x]
```

Out[∘]=
$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}}\, x^2 e^{-\frac{x^2}{2\sigma_2^2}}}{\sigma_2^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

Then, we find the probability density of the Maxwell distribution with scale parameter σ_2:

```
In[∘]:= 𝒟2 = PDF[MaxwellDistribution[σ2], x]
```

Out[∘]=
$$\begin{cases} \frac{\sqrt{\frac{2}{\pi}}\, x^2 e^{-\frac{x^2}{2\sigma_2^2}}}{\sigma_2^3} & x > 0 \\ 0 & \text{True} \end{cases}$$

Therefore, these two probability densities are equal:

```
In[∘]:= 𝒟1 − 𝒟2
Out[∘]= 0
```

We verify the above conclusion (*c* is the speed of 1st-order particle; *n* is the number of vectors) (This code takes approximately 13 hours):

In[∘]:=
```
c = 1;
n = 1000;
m = 3 000 000;
dd = {};
ProgressIndicator[Dynamic[i], {1, m}]
For[i = 1, i < m, i++,
    𝓗 = RandomPoint[Sphere[{0, 0, 0}, c], n];
    𝓗𝓗 = Norm[Total /@ Transpose[𝓗]];
    dd = AppendTo[dd, 𝓗𝓗]];
𝒟 = SmoothKernelDistribution[dd, {"Adaptive", Automatic, Automatic}];
s1 = Plot[{PDF[𝒟, x], PDF[MaxwellDistribution[c/√3 √n], x]},
    {x, 0, 100 c}, PlotStyle → {{Red, Thickness → 0.0032}, {Blue, Thickness → 0.0032}},
    Frame → {{True, False}, {True, False}}, FrameLabel → {"Momentum", "Probability Density"},
    FrameStyle → Directive[Black, Thickness → 0.0017],
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14],
    Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.0032]], Directive[Red, Thickness[0.0032]]},
          {Style["Theoretical", FontFamily → "Arial", FontSize → 14],
            Style["Simulated", FontFamily → "Arial", FontSize → 14]}, LegendFunction →
            (Framed[#, RoundingRadius → 4, FrameStyle → GrayLevel[0.6]] &)], Scaled[{0.732, 0.644}]]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S1.png",
    s1, Background → None, ImageResolution → 600];
```

Out[∘]=

Out[∘]=

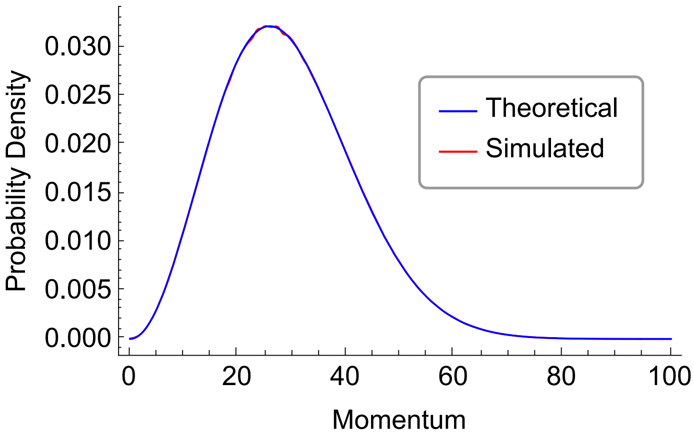

Figure S1. Probability density of momentum norm formed by 1000 randomly moving particles with c = 1 (theoretical and simulated results).

Accordingly, the norm of the 3-dimensional vectors formed by three normal distribution $N(0, \sigma_2)$ components which are independent on three coordinate axes follows the Maxwell distribution with the scale parameter σ_2.

Suppose that the standard deviation of the projection of the velocity of any one of the k equivalent particles forming a kth-order particle onto each equivalent coordinate axis is σ. Then, the standard deviation of the projection of the velocity of a kth-order particle onto each equivalent coordinate axis (i.e., the mean value of the projection of the velocity of 1st-order particle) is $\frac{\sigma}{\sqrt{k}}$, namely, the projection onto each coordinate axis (approximate) follows a normal distribution with a mean value of 0 and a standard deviation of $\frac{\sigma}{\sqrt{k}}$. As a result, the speed of kth-order particles follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$.

Then, the average velocity of the kth-order particles is

In[∘]:= $\overline{v}$ **= Mean[MaxwellDistribution[** $\frac{\sigma}{\sqrt{k}}$ **]]**

Out[∘]= $\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k}}$

For the kth-order particles in different reference frames ($\mathcal{R}_u$ and $\mathcal{R}_0$) and with different standard deviations (σ_u and σ_0), the ratio of their average velocity $\overline{v}_u / \overline{v}_0$ =

In[∘]:= $\frac{2\sqrt{\frac{2}{\pi}}\,\sigma_u}{\sqrt{k}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\sigma_0}{\sqrt{k}}$

Out[∘]= $\frac{\sigma_u}{\sigma_0}$

Therefore, the ratio of σ_u to σ_0 is the ratio between the average speeds of particles of higher mass levels in $\mathcal{R}_u$ and $\mathcal{R}_0$.

For k_1th- and k_2th-order particles, the ratio of their average velocity $\overline{v}_1 / \overline{v}_2$ =

In[∘]:= $\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_1}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_2}}$

Out[∘]= $\frac{\sqrt{k_2}}{\sqrt{k_1}}$

And because: $m_1 = \mu\, k_1$ and $m_2 = \mu\, k_2$, where μ is the scale factor or the mass of 1st-order particle. $\overline{v}_1 / \overline{v}_2$ is also equal to

In[∘]:= **Simplify[** $\frac{\sqrt{\frac{m_2}{\mu}}}{\sqrt{\frac{m_1}{\mu}}}$ **, Assumptions → $\mu > 0$]**

Out[∘]= $\frac{\sqrt{m_2}}{\sqrt{m_1}}$

Therefore, the square of the average velocity of particles is directly proportional to the mass of particles or the number of 1st-order particles forming it.

Part 2. Special Relativistic Effects on Infinitesimal Particles

Correspondence:

The mixed distribution of $\mathcal{D}_1$and $\mathcal{D}_2$ is represented by $\mathcal{D}$12;

The mixed distribution of $\mathcal{D}_3$and $\mathcal{D}_4$ is represented by $\mathcal{D}$34;

The rest of the symbols are consistent with those in the main text.

```
In[∘]:= Clear["Global`*"];
𝒟 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
      {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{-1, 1}]}];
𝒟1 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
      {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{u/c, 1}]}];
𝒟2 = TransformedDistribution[c Cos[θ] Sin[ArcCos[η]],
      {θ ≈ UniformDistribution[{-π, π}], η ≈ UniformDistribution[{-1, u/c}]}];
𝒟3 = TruncatedDistribution[{u, c}, UniformDistribution[{-c, c}]];
𝒟4 = TruncatedDistribution[{-c, u}, UniformDistribution[{-c, c}]];
𝒟34 = MixtureDistribution[{w, 1 - w}, {𝒟3, 𝒟4}];
Simplify[Mean[𝒟34], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{1}{2}(c(2w-1)+u)$

Let the mean value expression be $\frac{1}{2}(c(2w-1)+u) = u$, and then we find the weight w

```
In[∘]:= Reduce[1/2 (c (2 w - 1) + u) == u, w]
```

Out[∘]= $(u = 0 \wedge c = 0) \vee \left(c \neq 0 \wedge w = \frac{c+u}{2c}\right)$

Then, the mixed distribution $\mathcal{D}12$ consisting of $\mathcal{D}_1$ and $\mathcal{D}_2$ can be calculated in accordance with this weight w. The analytical form of $\mathcal{D}12$ cannot be given by Mathematica. Therefore, the standard deviation of $\mathcal{D}12$ is calculated directly (This code takes approximately 72 seconds).

```
In[∘]:= w = (c + u)/(2 c);
𝒟12 = MixtureDistribution[{w, 1 - w}, {𝒟1, 𝒟2}];
σu = Simplify[StandardDeviation[𝒟12], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{\sqrt{3}}$

The standard deviation of $\mathcal{D}34$ is the same.

```
In[∘]:= Simplify[StandardDeviation[𝒟34], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{\sqrt{3}}$

Then, the ratio between σ_u and the velocity components on the x-axis of the particles in $\mathcal{R}_0$ can be obtained.

```
In[∘]:= Simplify[σu / StandardDeviation[𝒟], Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2-u^2}}{c}$

The same factor can also be obtained by evaluating the ratio of the standard deviation of $\mathcal{D}34$ to the standard deviation of the velocity components on the z-axis in $\mathcal{R}_0$.

In[∘]:=
```
Simplify[StandardDeviation[𝒟34] / StandardDeviation[UniformDistribution[{−c, c}]],
  Assumptions → 0 < u < c]
```

Out[∘]= $\frac{\sqrt{c^2 - u^2}}{c}$

When $c = 10$ and $u = 6$, the distribution of $\mathcal{D}12$ on x- or y-axes is like this (This code takes approximately 350 seconds):

In[∘]:=
```
c = 10;
u = 6;
data = RandomVariate[𝒟12, 300 000 000];
𝒟0 = SmoothKernelDistribution[data, {"Adaptive", Automatic, Automatic}];
s2 = Plot[PDF[𝒟0, x], {x, −10, 10}, PlotRange → Full, PlotStyle → {Blue, Thickness → 0.0032},
    AxesLabel → {HoldForm[Speed], HoldForm[Probability Density]},
    AxesStyle → Directive[Black, Thickness → 0.001],
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S2.png",
    s2, Background → None, ImageResolution → 600];
```

Out[∘]=

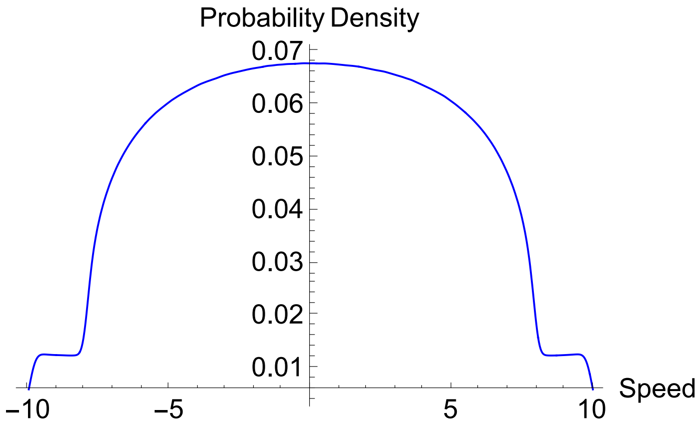

Figure S2. Simulated probability density of the mixed distribution $\mathcal{D}12$ when $c = 10$ and $u = 6$.

Part 3. Position aggregation being dominant is equivalent to velocity direction aggregation being dominant

Suppose that the standard deviation of the projection of the velocity of the 1st-order particle swarm onto each equivalent coordinate axis is σ. Then, the Maxwell speed density function related to mass is as follows:

In[∘]:=
```
𝒟 = MaxwellDistribution[σ/√k];
PDF[𝒟, x]
```

Out[∘]= $\begin{cases} \frac{\sqrt{\frac{2}{\pi}}\, k^{3/2}\, x^2\, e^{-\frac{k x^2}{2 \sigma^2}}}{\sigma^3} & x > 0 \\ 0 & \text{True} \end{cases}$

In the main text, $\mathcal{Y}$ is generally used to represent the magnitude of the momentum in a microdomain $\mathcal{V}$. However, due to the characteristics of Mathematica, we replace $\mathcal{Y}$ with Y and substitute it with x in the above formula, namely,

$$x = \frac{Y}{k};$$

Then, the probability density of the magnitude of the momentum (Y) is

In[∘]:= $\mathcal{D}$ = TransformedDistribution$\left[k\,x,\ x \approx \text{ProbabilityDistribution}\left[\frac{\sqrt{\frac{2}{\pi}}\,k^{3/2}\,x^2\,e^{-\frac{k x^2}{2\sigma^2}}}{\sigma^3},\ \{x, -\infty, \infty\}\right]\right]$;

Simplify[PDF[$\mathcal{D}$, Y], Assumptions → $k > 0$]

Out[∘]= $$\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3}$$

Find the value of k at the maximum value of the above formula:

In[∘]:= Reduce$\left[D\left[\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3},\ \{k, 1\}\right] == 0,\ k\right]$

Out[∘]= $$(k \neq 0 \land \sigma \neq 0 \land Y = 0) \lor \left(\sigma \neq 0 \land Y \neq 0 \land k = \frac{Y^2}{3\,\sigma^2}\right)$$

And this formula is expanded by a Taylor series about the point $k = \frac{Y^2}{3\,\sigma^2}$ according to k.

In[∘]:= Series$\left[\frac{\sqrt{\frac{2}{\pi}}\,Y^2\,e^{-\frac{Y^2}{2 k \sigma^2}}}{k^{3/2}\,\sigma^3},\ \left\{k,\ \frac{Y^2}{3\,\sigma^2},\ 3\right\}\right]$

Out[∘]= $$\frac{3\,\sqrt{\frac{6}{\pi}}\,Y^2}{e^{3/2}\,\sigma^3\left(\frac{Y^2}{\sigma^2}\right)^{3/2}} - \frac{81\left(\sqrt{\frac{3}{2\pi}}\,\sigma\right)\left(k - \frac{Y^2}{3\sigma^2}\right)^2}{2\left(e^{3/2}\,Y^2\left(\frac{Y^2}{\sigma^2}\right)^{3/2}\right)} + \frac{81\,\sqrt{\frac{6}{\pi}}\,\sigma^3\left(k - \frac{Y^2}{3\sigma^2}\right)^3}{e^{3/2}\,Y^4\left(\frac{Y^2}{\sigma^2}\right)^{3/2}} + O\left(\left(k - \frac{Y^2}{3\,\sigma^2}\right)^4\right)$$

Therefore, this formula is a parabola with its opening facing downward about the point $k = \frac{Y^2}{3\,\sigma^2}$, which is symmetric! This result shows that the two aggregation effects are equivalent about the point $k = \frac{Y^2}{3\,\sigma^2}$.

Each mass level particle can be seen as being formed by particles of lower mass level. Regardless of how much mass aggregation or velocity direction aggregation the particles exhibit, it can be regarded as a slight one with a lower mass level. This is carried out step by step. Finally, the minimal deviation of the aggregation behavior of the position or velocity direction for infinitesimal particles can be achieved. The above results show that when the slightest aggregation behavior occurs, the difficulty of the two aggregation behaviors is equivalent. Therefore, the two aggregation behaviors can be replaced with each other for the statistical influence of the diffusion behavior on the infinitesimal particles. When the particles of higher mass level are investigated, their dynamic behaviors are affected by the dynamic behaviors of the lower-mass-level particles forming them (this is not contradictory to the viewpoint that each mass level particle can be treated equally, that is, when the behavior of particles of higher quality level is investigated, these particles can be regarded as free particles with a statistical effect, while the lower-mass-level particles forming them can be completely ignored. See Section **3.3.5.4** of the main text for more details. When the number of particles of higher mass level is small and does not meet the statistical conditions, their dynamic behavior should be investigated according to the dynamic behaviors of the particles of lower mass level. When the number of particles of higher mass level is large enough, their dynamic behaviors can be investigated separately, that is, they are not affected by the dynamic behaviors of lower-mass-level particles, and they show the minimal deviation of the aggregation behavior. However, when there are artificial regulations, even if the number of particles of higher mass level is large enough, they will still be affected by the dynamic behavior of particles of lower mass level), thus following the dynamic behaviors of the lower-mass-level particles. Therefore, the two aggregation behaviors are equivalent to that of the higher mass-level-particles. In summary, the two aggregation effects are interchangeable for any mass level particles and thus affect the diffusion behavior as a single statistical effect of the aggregation behavior.

Part 4. The Norm of the Component Vector is Proportional to the Number of Vectors Forming It

When the total vector value of a specified vector swarm is determined, the mean norms between different component vectors should be proportional to the number forming them. The following proves this viewpoint in detail.

It has been proven that the degree of slowdown on all three axes is the same in Part 2 of the Supplementary Information. Then, let $\mathcal{M}\mathrm{k}$ being the norm of momentum of k particles observed from $\mathcal{R}_u$, it follows Maxwell distribution with scale parameter $\frac{\sqrt{k}\ \sqrt{c^2-u^2}}{\sqrt{3}}$ when observing from $\mathcal{R}_u$. And when observing all of the moving particles in $\mathcal{R}_u$ from $\mathcal{R}_0$, all the randomly moving particles in $\mathcal{R}_u$ can be considered to have an additional velocity component u along the z-axis. Therefore, according to cosine theorem, the probability density of momentum norm formed by k particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ can be expressed as (This code takes approximately 54 seconds):

In[∘]:=
```
Clear["Global`*"];
𝒟 = TransformedDistribution[√((k u)^2 + ℳk^2 − 2 k u ℳk Cos[ArcCos[η]]),
    {ℳk ≈ MaxwellDistribution[(√k √(c^2 − u^2))/√3], η ≈ UniformDistribution[{−1, 1}]}];
FullSimplify[PDF[𝒟, x], Assumptions → c > 0 ∧ 0 < u < c]
```

Out[∘]=

$$\begin{cases} \dfrac{\sqrt{3}\, x \left(e^{\frac{6 u x}{c^2-u^2}}-1\right) e^{-\frac{3 (k u+x)^2}{2 k \left(c^2-u^2\right)}}}{k u \sqrt{2 \pi c^2 k-2 \pi k u^2}} & k>0 \land ((x>0 \land k u>x) \lor k u<x) \\ -\dfrac{\sqrt{6 \pi}\, \sqrt{c^2 k-u x}\, \left(5 u x-2 c^2 k\right) \operatorname{erf}\left(\frac{\sqrt{6}\, x}{\sqrt{c^2 k-u x}}\right)+4 x\, e^{\frac{6 x^2}{u x-c^2 k}} \left(c^2 (6 k+2)-u (2 u+3 x)\right)-8 x (c-u)(c+u)}{4 \sqrt{6 \pi}\, k^{5/2} u ((c-u)(c+u))^{3/2}} & k u=x \land k>0 \end{cases}$$

The meaningful part (first branch) is selected to be verified. Note that the sampling with the replacement method in the particle swarm with a mean speed of u can simulate all of the cases of the particle swarm with a mean speed of u. (The following code takes averagely 4.2 + 0.5 hours)

```
In[∘]:= c = 1;
n = 1 000 000;
ℋℋ = 0;
While[ℋℋ < 2700,
  ℋ = RandomPoint[Sphere[{0, 0, 0}, c], n];
  ℋℋ = Norm[Total /@ Transpose[ℋ]]];
m = 1 000 000;
dd = {};
ProgressIndicator[Dynamic[j], {1, m}]
For[j = 1, j < m, j++,
  ℋ0 = RandomChoice[ℋ, 0.3 n];
  ℋℋ0 = Norm[Total /@ Transpose[ℋ0]];
  dd = AppendTo[dd, ℋℋ0]];
𝒟 = SmoothKernelDistribution[dd, {"Adaptive", Automatic, Automatic}];
k = 0.3 n;
u = ℋℋ/n;
s3 = Plot[{PDF[𝒟, x], (Sqrt[3] x (e^(6 u x/(c^2 - u^2)) - 1) e^(-3 (k u + x)^2/(2 k (c^2 - u^2))))/(k u Sqrt[2 π c^2 k - 2 π k u^2])}, {x, 0, 2500},
  PlotStyle → {{Red, Thickness → 0.0032}, {Blue, Thickness → 0.0032}},
  Frame → {{True, False}, {True, False}}, FrameLabel → {"Momentum", "Probability Density"},
  FrameStyle → Directive[Black, Thickness → 0.0017],
  LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 14],
  Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.0032]], Directive[Red, Thickness[0.0032]]},
      {Style["Theoretical", FontFamily → "Arial", FontSize → 14],
       Style["Simulated", FontFamily → "Arial", FontSize → 14]}, LegendFunction →
       (Framed[#, RoundingRadius → 4, FrameStyle → GrayLevel[0.6]] &)], Scaled[{0.756, 0.644}]]]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S3.png",
  s3, Background → None, ImageResolution → 600];

Out[∘]=

Out[∘]=
```

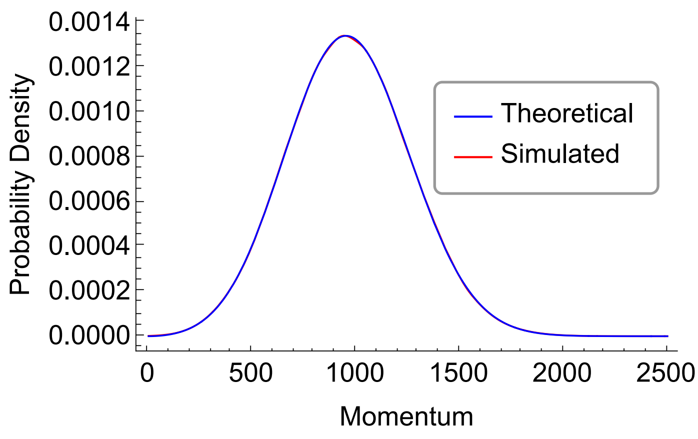

Figure S3. Probability density of momentum norm formed by 3×10^5 particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ when $c = 1$ and u is a fixed value.

In view of the above conclusions, we find the mean value of this distribution (This code takes approximately 150 seconds).

In[∘]:= $\overline{\mathcal{Y}}_k$ = **FullSimplify**[

Mean[**ProbabilityDistribution**[$\dfrac{\sqrt{3}\,x\left(e^{\frac{6ux}{c^2-u^2}}-1\right)e^{-\frac{3(ku+x)^2}{2k(c^2-u^2)}}}{k\,u\,\sqrt{2\pi c^2 k-2\pi k u^2}}$, {x, 0, +∞}]], **Assumptions** → c > u > 0 ∧ k > 0]

Out[∘]= $$\frac{\left(c^2+(3k-1)u^2\right)\operatorname{erf}\left(\frac{\sqrt{\frac{3}{2}}\,k\,u}{\sqrt{k(c-u)(c+u)}}\right)+\sqrt{\frac{6}{\pi}}\,u\,e^{\frac{3ku^2}{2(u^2-c^2)}}\sqrt{k(c-u)(c+u)}}{3u}$$

We find the limit of the ratio of this mean value $\overline{\mathcal{Y}}_k$ and k when k approaches +∞.

Simplify[**Limit**[$\frac{\overline{\mathcal{Y}}_k}{k}$, k → +∞], **Assumptions** → u > 0]

Out[∘]= $$\begin{cases} -u & \arg\left(c^2-u^2\right)\geq\pi \\ u & \text{True} \end{cases}$$

The second brunch is meaningful. Therefore, when k is a large number, the norm of the mean value $\overline{\boldsymbol{\mathcal{Y}}}_k$ is directly proportional to the number k forming $\overline{\boldsymbol{\mathcal{Y}}}_k$, namely $\overline{\mathcal{Y}}_k = k\cdot u$.

Eq. 21 in the main text determines the proportion of particle number distributed in various boxes partitioned by k, and these particles are distributed in each box of $\mathcal{V}$ with equal probability. That is, the particles are randomly extracted from the micro domain $\mathcal{V}$ to be distributed in each box. When the number of extractions is large enough, the norm of each component vector partitioned by k should be directly proportional to the number of particles according to the probability and the scale factor is u.

In view of the fact that the unique expansion of scalar $\mathcal{M}$ in the form of including power series is

$$\mathcal{M}=\sum_{k=1}^{\infty}\frac{e^{-\mathcal{M}}\,\mathcal{M}^k}{(k-1)!}$$

If the corresponding terms marked by k is directly proportional between the expansion of the norm $|\boldsymbol{\mathcal{M}}|$ of vector $\boldsymbol{\mathcal{M}}$ and the expansion of the scalar $\mathcal{M}$ representing the number of particles or let the numbers of particles be proportional to the norms of vectors they form, the number $\mathcal{M}$ of particles must be equal to the norm $|\boldsymbol{\mathcal{M}}|$ of the vector $\boldsymbol{\mathcal{M}}$ they form besides they are required to obey Poisson distribution. According to the above conclusion $\overline{\mathcal{Y}}_k = k\cdot u$, so the average speed $u = 1$ is needed in the system.

Next, we verify the standard deviations of this distribution in the three axes (This code takes averagely 291 seconds).

```
In[∘]:= c = 1;
n = 10 000 000;
ℋℋ = 0;
While[ℋℋ < 6600,
  ℋ = RandomPoint[Sphere[{0, 0, 0}, c], n];
  ℋℋ = Norm[Total /@ Transpose[ℋ]]];
ℋ0 = RandomChoice[ℋ, 0.3 n];
ℋℋx = StandardDeviation[Transpose[ℋ0][[1]]]
ℋℋy = StandardDeviation[Transpose[ℋ0][[2]]]
ℋℋz = StandardDeviation[Transpose[ℋ0][[3]]]
Out[∘]= 0.57735
Out[∘]= 0.577374
Out[∘]= 0.577327
```

The standard deviation in theory is:

$$u = \frac{\mathcal{HH}}{n};$$

$$\frac{\sqrt{c^2 - u^2}}{\sqrt{3}}$$

```
Out[∘]= 0.57735
```

This result also verifies that the conclusions in Part 2 and Part 3 are both correct.

Part 5. The 2-Dimensional Situation Under the Same Conditions

This code takes approximately 22 seconds.

In[∘]:=
```
Clear["Global`*"];
Needs["NDSolve`FEM`"];
Ω = ImplicitRegion[16/10000 ≤ x^2 + y^2 ≤ 16, {x, y}];
mesh = ToElementMesh[Ω, MeshRefinementFunction →
    Function[{vertices, area}, area > 3/100000 (1/10 + 80 Norm[Mean[vertices]])]];

uif = NDSolveValue[{∂²u(x, y)/∂x² + ∂²u(x, y)/∂y² − (∂u(x, y)/∂x)^2 − (∂u(x, y)/∂y)^2 == 0,
    DirichletCondition[u[x, y] == 1 + 2 i, x^2 + y^2 == 16/10000],
    DirichletCondition[u[x, y] == 0, x^2 + y^2 == 16]}, u, {x, y} ∈ mesh];
s4 = Plot3D[(Abs[uif[x, y]])^2, {x, y} ∈ mesh, PlotRange → {0, 5}, ColorFunction → (Hue[0.65, #3] &),
   MeshStyle → GrayLevel[0.4], AxesLabel → {Style["x", 15, FontFamily → "Arial", Black, Italic, Bold],
     Style["y", 15, FontFamily → "Arial", Black, Italic, Bold], Rotate["Density", π/2]},
   AxesStyle → Directive[Black, FontFamily → "Arial", FontSize → 15],
   BoxStyle → Directive[Black, Thickness → 0.002], BoxRatios → Automatic, ViewPoint → {15, −26, 16}];

ImageResize[s4, 700]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S4.png",
   s4, Background → None, ImageResolution → 600];
```

Out[∘]=

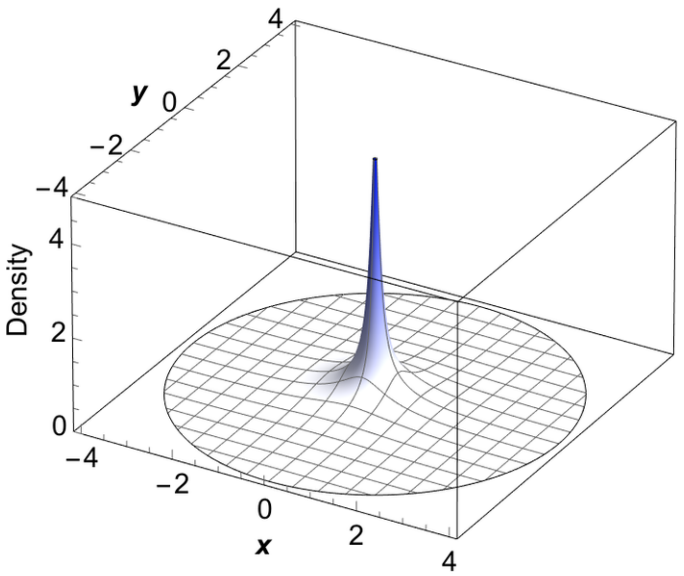

Figure S4. Distribution of mass density of a particle swarm meeting the conditions $\mathcal{M}(0, 0) = 1 + 2i \wedge (\mathcal{M}(x, y) = 0 \wedge x^2 + y^2 = 4^2)$.

It can be seen from Figure S4 that it is a circular symmetrical structure.

Part 6. Differences Between the Two Solving Methods (Schrödinger Equation and Eq. 40)

This code takes approximately 45 hours.

```
In[•]:= Clear["Global`*"];
usol = DSolveValue[{i ∂ψ(x, t)/∂t == -1/2 ∂²ψ(x, t)/∂x², ψ(x, 0) == e^(-2 x²)}, ψ, {x, t}];
F[x_] := e^(-x); L = 20;
vsol = NDSolveValue[{i ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²),
     M(x, 0) == 10^-2 e^(-2 x²), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 40];
s5 = Plot3D[Abs[usol[x, t]] - 10² Abs[vsol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60,
   MaxRecursion → 3, PlotRange → {{0, 1.6}, {-8, 8}, {-0.002, 0.003}},
   MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
   AxesLabel → {Style["t ", 15, FontFamily → "Arial", Black, Italic, Bold],
     Style["x", 15, FontFamily → "Arial", Black, Italic, Bold], Rotate["Deviation                ", π/2]},
   AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
   Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
       {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
       {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
       {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
     {{-0.002, -0.002, {0.012, 0}, Thickness → 0.003}, {-0.001, -0.001, {0.012, 0}, Thickness → 0.003},
       {0, "0.000", {0.012, 0}, Thickness → 0.003}, {0.001, 0.001, {0.012, 0}, Thickness → 0.003},
       {0.002, "0.002", {0.012, 0}, Thickness → 0.003}, {0.003, "0.003", {0.012, 0}, Thickness → 0.003}}},
   LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 15], ViewPoint → {1, -2, 2.1}];
FindMaxValue[{(Abs[usol[x, t]] - 10² Abs[vsol[x, t]]), x > 0, t > 0}, {x, t}, WorkingPrecision → 34] /
 (Abs[usol[x, t]] /. Last[
     FindMaximum[{Abs[usol[x, t]] - 10² Abs[vsol[x, t]], x > 0, t > 0}, {x, t}, WorkingPrecision → 34]])

ImageResize[s5, 700]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S5.png",
  s5, Background → None, ImageResolution → 600];

Out[•]= 0.01137609304650582034220637885507277
```

Out[•]=

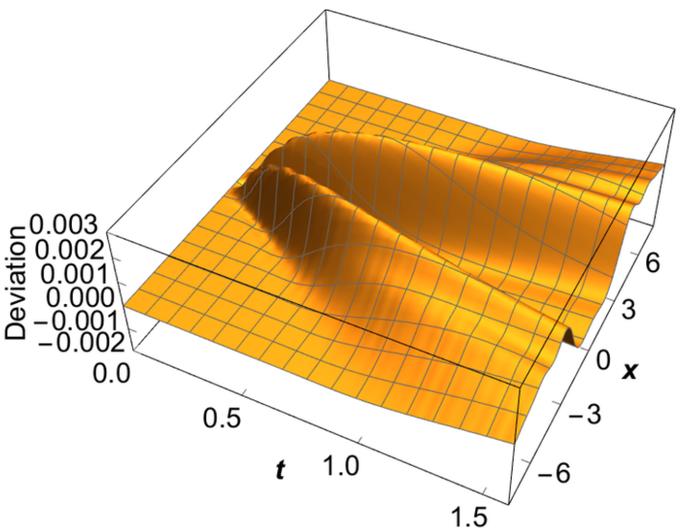

Figure S5. Deviation of the contours computed by the Schrödinger equation and Eq. 40 with the initial wave function being $10^{-2}\, e^{-2x^2}$.

Part 7. Another Comparison When the Initial Wave Function Is $1.4\, e^{-2x^2}$

This code takes approximately 3 hours.

```
In[•]:= Clear["Global`*"];
Off[NDSolveValue::eerr];
L = 20;
F[x_] := e^(-x);
usol = NDSolveValue[{i ∂M(x,t)/∂t == -1/2 F[M(x,t)] (∂²M(x,t)/∂x² - (∂M(x,t)/∂x)²),
    M(x,0) == 6/5 e^(-2 x²), M(-L,t) == M(L,t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 22];
vsol = NDSolveValue[{i ∂M(x,t)/∂t == -1/2 F[M(x,t)] (∂²M(x,t)/∂x² - (∂M(x,t)/∂x)²), M(x,0) == 7/5 e^(-2 x²),
    M(-L,t) == M(L,t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 26];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = (1 0; 0 -1); σ1 = (0 1; 1 0);
xsol = NDSolve[{i D[χ[x, t], t] == -σ1.χ(x, t) - i σ3.D[χ[x, t], x], u[x, 0] == √2/2 e^(-2 x²), v[x, 0] == √2/2 e^(-2 x²),
    u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot3D[5/6 Abs[usol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}}, MeshStyle → GrayLevel[0.4],
    BoundaryStyle → GrayLevel[0.4], AxesLabel → {Style["t ", 22, FontFamily → "Arial",
        Black, Italic, Bold], Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
        {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
        {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, -2, 2.1},
    Epilog → Text[Style["a", 22, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G2 = Plot3D[5/7 Abs[vsol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}}, MeshStyle → GrayLevel[0.4],
    BoundaryStyle → GrayLevel[0.4], AxesLabel → {Style["t ", 22, FontFamily → "Arial",
        Black, Italic, Bold], Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
        {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
        {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, -2, 2.1},
    Epilog → Text[Style["b", 22, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G3 = Plot3D[Norm[Evaluate[{u[x, t], v[x, t]} /. xsol]], {t, 0, 1.6}, {x, -8, 8},
    PlotPoints → 60, MaxRecursion → 3, PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.23}},
    MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
```

```
    AxesLabel → {Style["t ", 22, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 22, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
          {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 21], ViewPoint → {1, −2, 2.1},
    Epilog → Text[Style["c", 22, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G4 = Plot[{5/6 Norm[usol[x, 1]], 5/7 Norm[vsol[x, 1]], Norm[Evaluate[{u[x, 1], v[x, 1]} /. xsol]]}, {x, −3.8, 3.8},
    PlotStyle → {{Red, Thickness → 0.005}, {Blue, Thickness → 0.005}, {Black, Thickness → 0.005}},
    PlotRange → {{−3, 3}, {−0.02, 0.9}}, Frame → {{False, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0012], FrameTicks →
      {{{−3, −3, {0.01, 0}, Thickness → 0.003}, {−2, −2, {0.01, 0}, Thickness → 0.003}, {−1, −1, {0.01, 0},
            Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
          {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22], PlotLegends →
      Placed[LineLegend[{Directive[Red, Thickness[0.0036]], Directive[Blue, Thickness[0.0036]], ,
            Directive[Black, Thickness[0.0036]]}, {"1.2", "1.4", "Dirac"}], {0.873, 0.72}]];
s6 = GraphicsGrid[{{G1, G2}, {G3, G4}}, Alignment → Bottom, ImageSize → 700,
    Epilog → Text[Style["d", 22, FontFamily → "Arial", Bold], Scaled[{0.6053, 0.7776}]]];

ImageResize[s6, 1000]
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure S6.png",
   s6, Background → None, ImageResolution → 600];
```

Out[∘]=

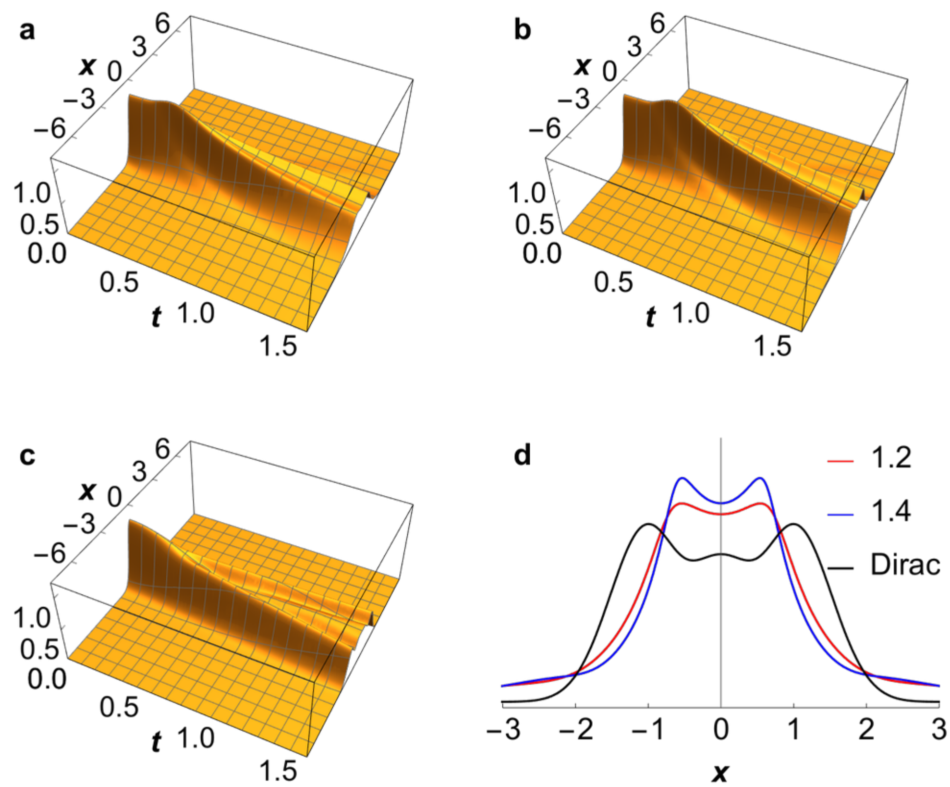

Figure S6. Illustrations of the 1-dimensional time-dependent diffusion of the Gaussian wave packet e^{-2x^2} as obtained using various methods in natural units. **a**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x, t) = 1.2\, e^{-2x^2}$. The norm has been reduced ($\times \frac{5}{6}$) to facilitate the shape comparison. **b**, Computation result of Eq. 40 when the initial condition is $\mathcal{M}_0(x, t) = 1.4\, e^{-2x^2}$. The norm has been reduced ($\times \frac{5}{7}$) to facilitate the shape comparison. **c**, Computation result of the Dirac equation. **d**, Comparison between them at $t = 1$.

Part 8. Figures Used in This Article

Figure1

```
In[∘]:= Clear["Global`*"];
aa = Graphics[{{RGBColor[178/255, 252/255, 61/255], Rectangle[{0, 0}, {1, 1}]},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Rectangle[{1, 0}, {2, 1}]},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0.7, 0.54}, {1.3, 0.54}}]}},
    {RGBColor[250/255, 200/255, 0], Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.3, 0.46}, {0.7, 0.46}}]}},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Arrowheads[0.06],
     {Thickness[0.006], Arrow[{{0, −1.3}, {1.2, −1.3}}]}},
    {RGBColor[178/255, 252/255, 61/255], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, −1.3}, {0.8, −0.3}}]}},
    {Orange, {Thickness[0.002], DotDashed, Line[{{1, −0.05}, {1, 1.05}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{0.8, −0.3}, {2, −0.3}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{1.2, −1.3}, {2, −0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.2, −1.3}, {0.8, −0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, −1.3}, {2, −0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, −1.3}, {1, −0.8}}]}},
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {0.45, 0.5}],
    Text[Style["A", 17, FontFamily → "Arial", White], {0.513, 0.456}],
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {1.55, 0.5}],
    Text[Style["B", 17, FontFamily → "Arial", White], {1.616, 0.455}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.63}],
    Text[Style["A", 17, FontFamily → "Arial", Orange], {1.063, 0.59}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.38}],
    Text[Style["B", 17, FontFamily → "Arial", Orange], {1.065, 0.34}],
    Text[Style["Φ", 24, FontFamily → "Arial", Orange], {1.06, 1.08}],
    Text[Style["O", 24, FontFamily → "Arial", Orange], {0, −1.39}],
    Text[Style["B", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255, 0.5]], {1.2, −1.39}],
    Text[Style["A", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255]], {0.7, −0.28}],
    Text[Style["C", 24, FontFamily → "Arial", Orange], {2.02, −0.4}],
    Text[Style["M", 24, FontFamily → "Arial", Orange], {0.972, −0.93}],
    Inset[Style["a", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, 1.12}],
    Inset[Style["b", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, −0.2}]}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 1.png",
   aa, Background → None, ImageResolution → 600];
```

```
######Figure1############################################
######Figure2############################################

In[•]:= Clear["Global`*"];
text = Graphics[{Gray, Line[{{1, 0}, {1, 10}}], Line[{{2, 0}, {2, 10}}],
      Line[{{3, 0}, {3, 10}}], Line[{{4, 0}, {4, 10}}], Line[{{5, 0}, {5, 10}}],
      Line[{{6, 0}, {6, 10}}], Line[{{7, 0}, {7, 10}}], Line[{{8, 0}, {8, 10}}], Line[{{9, 0}, {9, 10}}],
      Line[{{0, 1}, {10, 1}}], Line[{{0, 2}, {10, 2}}], Line[{{0, 3}, {10, 3}}], Line[{{0, 4}, {10, 4}}],
      Line[{{0, 5}, {10, 5}}], Line[{{0, 6}, {10, 6}}], Line[{{0, 7}, {10, 7}}], Line[{{0, 8}, {10, 8}}],
      Line[{{0, 9}, {10, 9}}], Orange, Rectangle[{6, 4}, {7, 5}]}, PlotRangePadding → 1/1000];
bb = Show[{Plot3D[Sin[x + Cos[y]], {x, −3, 3}, {y, −3, 3}, PlotPoints → 60, MaxRecursion → 3,
       PlotStyle → Texture[text], Mesh → None, Lighting → "Neutral",
       PlotLabels → Placed[Style["dS", 14, FontFamily → "Arial", Orange],
          {0.9, −0.9}, Background → None], BoundaryStyle → None, Boxed → False,
       Axes → None, ViewPoint → {1,  −1.9,  1.4}], Graphics3D[{{Thickness[0.007], Black,
          Arrow[{{0, 0, 0}, {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
                −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1}} +
              {{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {0.88, −0.3, Sin[0.88 + Cos[−0.3]]}}]},
        {Text[Style["N", 14, FontFamily → "Arial", Bold, Italic, Black],
           {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
              −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1} +
             {0.88, −0.3, Sin[0.88 + Cos[−0.3]]} + {0.02, 0.03, 0.23}]},
        {Thickness[0.007], Blue, Arrow[{{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {1.88, −0.5, 2}}]},
        {Text[Style["𝒳", 14, FontFamily → "Arial", Bold, Italic, Blue], {2.01, −0.5, 2.01}]},
        {Text[Style["Σ", 14, FontFamily → "Arial", Italic, Gray], {−2.14, −1.5, 0.7}]}}]}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png",
   bb, Background → None, ImageResolution → 600];
bfg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png"];
bfg = ImageTake[bfg, {290, 1870}, {110, 2920}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 2.png", bfg];
######Figure2############################################
######Figure3############################################
```

This code takes approximately 227 seconds.

```
In[•]:= Clear["Global`*"];
Needs["NDSolve`FEM`"];
Ω = ImplicitRegion[16/10000 ≤ x^2 + y^2 + z^2 ≤ 16, {x, y, z}];
nr = 50; nφ = 50; nθ = 50;
coordinates = Flatten[ Table[{r Sin[φ] Cos[θ], r Sin[φ] Sin[θ], r Cos[φ]}, {r, 4/100, 4, (4 - 4/100)/(nr - 1)},
    {φ, 1/10000, π - 1/10000, (π - 2/10000)/(nφ - 1)}, {θ, 0, 2 π, 2 π/(nθ - 1)}], 2];
incidents = Flatten[Table[Block[{p1 = (j - 1)*nr + i, p2 = j*nr + i, p3 = p2 + 1, p4 = p1 + 1, p5, p6, p7, p8},
      {p5, p6, p7, p8} = {p1, p2, p3, p4} + k*nr*nφ;
      {p1, p2, p3, p4} += (k - 1)*nr*nφ;
      {p1, p2, p3, p4, p5, p6, p7, p8}], {i, 1, nr - 1}, {j, 1, nφ - 1}, {k, 1, nθ - 1}], 2];
mesh =
  ToElementMesh["Coordinates" → coordinates, "MeshElements" → {HexahedronElement[incidents]}];
uif = NDSolveValue[{∂²M(x, y, z)/∂x² + ∂²M(x, y, z)/∂y² + ∂²M(x, y, z)/∂z² - (∂M(x, y, z)/∂x)^2 - (∂M(x, y, z)/∂y)^2 -
     (∂M(x, y, z)/∂z)^2 == 0, DirichletCondition[M[x, y, z] == 1 + 2 i, x^2 + y^2 + z^2 == 16/10000],
    DirichletCondition[M[x, y, z] == 0, x^2 + y^2 + z^2 == 16]}, M, {x, y, z} ∈ mesh];
G1 = SliceDensityPlot3D[(Norm[uif[x, y, z]])^2, "CenterPlanes", {x, -4, 4}, {y, -4, 4},
   {z, -4, 4}, Boxed → False, Axes → None, ColorFunction → (Hue[0.65, #1] &),
   BoundaryStyle → Directive[Thickness[0.001], Gray], PlotRange → {0, 5}, PlotPoints → 100,
   Epilog → Text[Style["a", 23, FontFamily → "Arial", Bold, Black], {0.2478, 0.93}, {0.5, 4}]];
G2 = Plot3D[(Norm[uif[x, y, 0]])^2, {x, -4, 4}, {y, -4, 4},
   PlotRange → {{-4.52, 4.52}, {-4.52, 4.52}, {-1.7, 5}}, MeshStyle →
    {Directive[GrayLevel[0.4], Thickness[0.001]], Directive[GrayLevel[0.4], Thickness[0.001]]},
   BoundaryStyle → Directive[Thickness[0.001], Gray], Boxed → False, Axes → None,
   ColorFunction → (Hue[0.65, #3] &), BoxRatios → Automatic, MeshStyle → Gray,
   ImageSize → {360, 360}, PlotPoints → 30, ViewPoint → {1.2, -2, 0.7},
   Epilog → Text[Style["b", 23, FontFamily → "Arial", Bold, Black], {0.2478, 0.93}, {0.5, 4}]];
G3 = DensityPlot[(Norm[uif[x, y, 0]])^2, {x, -4, 4}, {y, -4, 4}, PlotRange → {{-7.2, 7.2}, {-7.2, 7.2}, {0, 5}},
   ColorFunction → (Hue[0.65, #1] &), Frame → False, PlotPoints → 180,
   Epilog → {Text[Style["c", 23, FontFamily → "Arial", Bold, Black], {-3.75, 4.24}],
      {Directive[Thickness[0.001], Gray], Circle[{0, 0}, 4]}}];
G4 = Plot[(Norm[uif[x, 0, 0]])^2, {x, -4, 4}, PlotRange → {{-7.3, 7.3}, {-0.95, 6}},
   ColorFunction → (Hue[0.65, #2] &), PlotPoints → 180, PlotStyle → {Thickness → 0.0036},
   Frame → False, Axes → None, AspectRatio → Automatic];
G4 = Show[{G4, Plot[(Norm[uif[x, 0, 0]])^2, {x, -4, 4}, PlotRange → {{-7.3, 7.3}, {-0.95, 0.9}},
      PlotStyle → Directive[GrayLevel[0.66], Thickness → 0.0036, Dashed],
      Frame → False, Axes → None, AspectRatio → Automatic]}];
cc = GraphicsGrid[{{G1, G2}, {G3, G4}}, Spacings → {-70, -70}, ImageSize → 700,
   Epilog → Text[Style["d", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.6684, 0.3371}]]];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png",
  cc, Background → None, ImageResolution → 500];
cfg =
  Import[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png"];
cfg = ImageTake[cfg, {570, 4670}, {560, 4300}];
Export[
  "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 3.png", cfg];
```

Figure3

Figure4

This code takes approximately 12 minutes.

```
In[∘]:= Clear["Global`*"];
L = 20;
F[x_] := e^-x;
usol = NDSolveValue[{i ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²),
    M(x, 0) == 10^-13 e^(-2 x²), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 66];
F[x_] := e^-x;
vsol = NDSolveValue[{i ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)²),
    M(x, 0) == e^(-2 x²), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 20];
wsol = NDSolveValue[{i ∂ψ(x, t)/∂t == -1/2 ∂²ψ(x, t)/∂x², ψ(x, 0) == e^(-2 x²), ψ(-L, t) == ψ(L, t)},
   ψ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 18];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = (1 0; 0 -1); σ1 = (0 1; 1 0);
xsol = NDSolve[{i D[χ[x, t], t] == -σ1.χ(x, t) - i σ3.D[χ[x, t], x], u[x, 0] == √2/2 e^(-2 x²), v[x, 0] == √2/2 e^(-2 x²),
    u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot3D[10^13 Abs[usol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60,
   MaxRecursion → 3, PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.05}}, MeshStyle →
    {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
   BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
   AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
      Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
   AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
   Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
       {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
       {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
       {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
     {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
   LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1, -2, 2.1},
   Epilog → Text[Style["a", 20, FontFamily → "Arial", Bold, Black], {-0.09, 0.88}, {-1, 1}]];
G2 = Plot3D[Abs[vsol[x, t]], {t, 0, 1.6}, {x, -8, 8}, PlotPoints → 60, MaxRecursion → 3,
   PlotRange → {{0, 1.6}, {-8, 8}, {0, 1.05}}, MeshStyle →
    {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
   BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
   AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
      Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
   AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
   Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
       {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{-6, -6, {0.011, 0}, Thickness → 0.003},
       {-3, -3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
       {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
     {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
   LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1, -2, 2.1},
```

```
    Epilog → Text[Style["b", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G3 = Plot3D[Abs[wsol[x, t]], {t, 0, 1.6}, {x, −8, 8}, PlotPoints → 60, MaxRecursion → 3,
    PlotRange → {{0, 1.6}, {−8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
          {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1,  −2,  2.1},
    Epilog → Text[Style["c", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
G4 = Plot3D[Norm[Evaluate[{u[x, t], v[x, t]} /. xsol]], {t, 0, 1.6}, {x, −8, 8}, PlotPoints → 60,
    MaxRecursion → 3, PlotRange → {{0, 1.6}, {−8, 8}, {0, 1.05}}, MeshStyle →
      {Directive[GrayLevel[0.4], Thickness[0.0014]], Directive[GrayLevel[0.4], Thickness[0.0014]]},
    BoundaryStyle → Directive[GrayLevel[0.4], Thickness[0.0014]],
    AxesLabel → {Style["t ", 20, FontFamily → "Arial", Black, Italic, Bold],
        Style["x", 20, FontFamily → "Arial", Black, Italic, Bold], ""},
    AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
    Ticks → {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
          {1.5, 1.5, {0.01, 0}, Thickness → 0.003}}, {{−6, −6, {0.011, 0}, Thickness → 0.003},
          {−3, −3, {0.011, 0}, Thickness → 0.003}, {0, 0, {0.011, 0}, Thickness → 0.003},
          {3, 3, {0.011, 0}, Thickness → 0.003}, {6, 6, {0.011, 0}, Thickness → 0.003}},
        {{0, "0.0"}, {0.5, 0.5, {0.012, 0}, Thickness → 0.003}, {1, "1.0", {0.012, 0}, Thickness → 0.003}}},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1,  −2,  2.1},
    Epilog → Text[Style["d", 20, FontFamily → "Arial", Bold, Black], {−0.09, 0.88}, {−1, 1}]];
dd = GraphicsGrid[{{G1, G2}, {G3, G4}}, Spacings → {Scaled[−0.01], Scaled[−0.01]}, ImageSize → 700];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png",
   dd, Background → None, ImageResolution → 600];
dfg =
   Import[
     "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png"];
dfg = ImageTake[dfg, {220, 5009}, {120, 5620}];
Export[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 4.png", dfg];
```

Figure4

Figure5

This code takes approximately 11 minutes.

```
Clear["Global`*"];
L = 20;
F[x_] := ⅇ^(−x);
usol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == −1/2 F[M(x, t)] (∂²M(x, t)/∂x² − (∂M(x, t)/∂x)²),
      M(x, 0) == 10^(−13) ⅇ^(−2 x²), M(−L, t) == M(L, t)}, M, {x, −L, L}, {t, 0, 3}, WorkingPrecision → 66];
F[x_] := ⅇ^(−x);
vsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == −1/2 F[M(x, t)] (∂²M(x, t)/∂x² − (∂M(x, t)/∂x)²),
      M(x, 0) == ⅇ^(−2 x²), M(−L, t) == M(L, t)}, M, {x, −L, L}, {t, 0, 3}, WorkingPrecision → 20];
```

```
wsol = NDSolveValue[{i ∂ψ(x, t)/∂t == -1/2 ∂²ψ(x, t)/∂x², ψ(x, 0) == ⅇ^(-2 x²), ψ(-L, t) == ψ(L, t)},
    ψ, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 18];
χ[x, t] = {u[x, t], v[x, t]};
σ3 = ({{1, 0}, {0, -1}}); σ1 = ({{0, 1}, {1, 0}});
xsol = NDSolve[{i D[χ[x, t], t] == -σ1.χ(x, t) - i σ3.D[χ[x, t], x], u[x, 0] == √2/2 ⅇ^(-2 x²), v[x, 0] == √2/2 ⅇ^(-2 x²),
    u[L, t] == u[-L, t], v[L, t] == v[-L, t]}, {u, v}, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 14];
G1 = Plot[ⅇ^(-2 x²), {x, -3.8, 3.8}, PlotStyle → {Gray, Thickness → 0.005, Dashed},
    PlotRange → {{-3, 3}, {-0.02, 1.1}}, Frame → {{False, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
      {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
          Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
        {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22],
    Epilog → Text[Style["t = 0.0", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
datau = Table[{x, 10^13 Norm[usol[x, 0.2]]}, {x, -4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x²], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fu ⅆx)^-1 fu, {x, -∞, ∞}]], {3, 2}];
σv = NumberForm[√(N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[vsol[x, 0.2]] ⅆx)^-1] x² Norm[vsol[x, 0.2]] ⅆx]), {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.2]]}, {x, -4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x²], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{-∞}^{∞} fw ⅆx)^-1 fw, {x, -∞, ∞}]], {3, 2}];
σx = NumberForm[√(N[∫_{-10}^{10} N[(∫_{-10}^{10} Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]] ⅆx)^-1]
        x² Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]] ⅆx]), {3, 2}];
G2 = Plot[{ⅇ^(-2 x²), Callout[10^13 Norm[usol[x, 0.2]], StringForm["σ = ``", σu], {0.7, 0.64}, {-0.21, 0.83},
      CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
      Background → None], Callout[Norm[vsol[x, 0.2]], StringForm["σ = ``", σv],
      {0.7, 1.03}, {0.11, 1.01}, CalloutStyle → {Blue, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.2]], StringForm["σ = ``", σw], {0.7, 0.9}, {0.23, 0.84},
      CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
      Background → None], Callout[Norm[Evaluate[{u[x, 0.2], v[x, 0.2]} /. xsol]],
      StringForm["σ = ``", σx], {0.7, 0.77}, {0.42, 0.74}, CalloutStyle → {Green, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]}, {x, -3.8, 3.8},
    PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005}, {Blue, Thickness → 0.005},
      {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}}, PlotRange → {{-3, 3}, {-0.02, 1.1}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
    Frame → {{False, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
      {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
```

```
      Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
     {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}}},
  Epilog → Text[Style["t = 0.2", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G2 = Show[G2, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
datau = Table[{x, 10^13 Norm[usol[x, 0.4]]}, {x, −4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fu dx)^−1 fu, {x, −∞, ∞}]], {3, 2}];
σv = NumberForm[Sqrt[N[∫_{−10}^{10} N[(∫_{−10}^{10} Norm[vsol[x, 0.4]] dx)^−1] x^2 Norm[vsol[x, 0.4]] dx]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.4]]}, {x, −4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fw dx)^−1 fw, {x, −∞, ∞}]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.4]]}, {x, −4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fw dx)^−1 fw, {x, −∞, ∞}]], {3, 2}];
σx = NumberForm[Sqrt[N[∫_{−10}^{10} N[(∫_{−10}^{10} Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]] dx)^−1]
      x^2 Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]] dx]], {3, 2}];
G3 = Plot[{e^(−2 x^2), Callout[10^13 Norm[usol[x, 0.4]], StringForm["σ = ``", σu], {0.8, 0.54}, {−0.22, 0.7},
     CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
     Background → None], Callout[Norm[vsol[x, 0.4]], StringForm["σ = ``", σv],
     {0.8, 0.93}, {0.157, 0.93}, CalloutStyle → {Blue, None},
     LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.4]], StringForm["σ = ``", σw], {0.8, 0.67}, {0.23, 0.713},
     CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
     Background → None], Callout[Norm[Evaluate[{u[x, 0.4], v[x, 0.4]} /. xsol]],
     StringForm["σ = ``", σx], {0.8, 0.8}, {0.14, 0.756}, CalloutStyle → {Green, None},
     LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]},
  {x, −3.8, 3.8}, PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005},
     {Blue, Thickness → 0.005}, {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}},
  PlotRange → {{−3, 3}, {−0.02, 1.1}}, Frame → {{False, False}, {True, False}},
  FrameStyle → Directive[Black, Thickness → 0.002],
  AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
   {{{−3, −3, {0.01, 0}, Thickness → 0.003}, {−2, −2, {0.01, 0}, Thickness → 0.003}, {−1, −1, {0.01, 0},
      Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
     {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}}},
  FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
  Epilog → Text[Style["t = 0.4", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G3 = Show[G3, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
datau = Table[{x, 10^13 Norm[usol[x, 0.6]]}, {x, −4, 4}];
fu = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σu = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fu dx)^−1 fu, {x, −∞, ∞}]], {3, 2}];
σv = NumberForm[Sqrt[N[∫_{−10}^{10} N[(∫_{−10}^{10} Norm[vsol[x, 0.6]] dx)^−1] x^2 Norm[vsol[x, 0.6]] dx]], {3, 2}];
```

```
dataw = Table[{x, Norm[wsol[x, 0.6]]}, {x, −4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fw ⅆx)^−1 fw, {x, −∞, ∞}]], {3, 2}];
dataw = Table[{x, Norm[wsol[x, 0.6]]}, {x, −4, 4}];
fw = Normal[NonlinearModelFit[datau, a Exp[b x^2], {a, b}, x]];
σw = NumberForm[StandardDeviation[ProbabilityDistribution[(∫_{−∞}^{∞} fw ⅆx)^−1 fw, {x, −∞, ∞}]], {3, 2}];
σx = NumberForm[√(N[∫_{−10}^{10} N[(∫_{−10}^{10} Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]] ⅆx)^−1]
      x^2 Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]] ⅆx], {3, 2}];
G4 = Plot[{ⅇ^(−2 x^2), Callout[10^13 Norm[usol[x, 0.6]], StringForm["σ = ``", σu], {1.1, 0.49}, {0.79, 0.52},
      CalloutStyle → {Red, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Red},
      Background → None], Callout[Norm[vsol[x, 0.6]], StringForm["σ = ``", σv],
      {1.1, 0.88}, {0.157, 0.815}, CalloutStyle → {Blue, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Blue}, Background → None],
    Callout[Norm[wsol[x, 0.6]], StringForm["σ = ``", σw], {1.1, 0.62}, {0.79, 0.52},
      CalloutStyle → {Orange, None}, LabelStyle → {FontFamily → "Arial", FontSize → 22, Orange},
      Background → None], Callout[Norm[Evaluate[{u[x, 0.6], v[x, 0.6]} /. xsol]],
      StringForm["σ = ``", σx], {1.1, 0.75}, {0.64, 0.69}, CalloutStyle → {Green, None},
      LabelStyle → {FontFamily → "Arial", FontSize → 22, Green}, Background → None]},
   {x, −3.8, 3.8}, PlotStyle → {{Gray, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005},
      {Blue, Thickness → 0.005}, {Orange, Thickness → 0.005}, {Green, Thickness → 0.005}},
   PlotRange → {{−3, 3}, {−0.02, 1.1}}, Frame → {{False, False}, {True, False}},
   FrameStyle → Directive[Black, Thickness → 0.002],
   AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0016], FrameTicks →
    {{{−3, −3, {0.01, 0}, Thickness → 0.003}, {−2, −2, {0.01, 0}, Thickness → 0.003}, {−1, −1, {0.01, 0},
         Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
       {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
   FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
   Epilog → Text[Style["t = 0.6", 22, FontFamily → "Arial", Black, Bold], Scaled[{0.1, 0.96}]]];
G4 = Show[G4, LabelStyle → {FontFamily → "Arial", 22, GrayLevel[0]}];
ee = GraphicsGrid[{{G1, G2}, {G3, G4}}, ImageSize → 800, Spacings → {Scaled[−0.2], Scaled[0.16]},
   Epilog → Inset[LineLegend[{Directive[Red, Thickness[0.004]], Directive[Blue, Thickness[0.004]],
         Directive[Orange, Thickness[0.004]], Directive[Green, Thickness[0.004]]},
        {Style["Eq. 40^1", FontFamily → "Arial", FontSize → 22], Style["Eq. 40^2", FontFamily → "Arial",
          FontSize → 22], Style["Schrödinger", FontFamily → "Arial", FontSize → 22],
         Style["Dirac", FontFamily → "Arial", FontSize → 22]}, LegendFunction →
         (Framed[#, RoundingRadius → 5, FrameStyle → GrayLevel[0.3]] &)], Scaled[{0.5, 1.081}]]];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png",
  ee, Background → None, ImageResolution → 600];
efg =
  Import[
   "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png"];
efg = ImageTake[efg, {0, 4650}, {260, 6400}];
Export[
  "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 5.png", efg];
```

Figure5

Figure6

This code takes approximately 3 hours.

```
Clear["Global`*"];
Off[NDSolveValue::eerr];
L = 20;
F[x_] := ⅇ^(-x);
usol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2),
     M(x, 0) == 10^-13 ⅇ^(-2 x^2), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 66];
vsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2), M(x, 0) == ⅇ^(-2 x^2),
     M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 20];
wsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2),
     M(x, 0) == 6/5 ⅇ^(-2 x^2), M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 22];
xsol = NDSolveValue[{ⅈ ∂M(x, t)/∂t == -1/2 F[M(x, t)] (∂²M(x, t)/∂x² - (∂M(x, t)/∂x)^2), M(x, 0) == 7/5 ⅇ^(-2 x^2),
     M(-L, t) == M(L, t)}, M, {x, -L, L}, {t, 0, 3}, WorkingPrecision → 26];
G1 = Plot[{10^13 Norm[usol[0, t]], Norm[vsol[0, t]], FindMaxValue[Norm[vsol[x, t]], {x, 0, 3}],
     5/6 Norm[wsol[0, t]], FindMaxValue[5/6 Norm[wsol[x, t]], {x, 0, 3}],
     5/7 Norm[xsol[0, t]], FindMaxValue[5/7 Norm[xsol[x, t]], {x, 0, 3}]},
    {t, 0, 3}, PlotStyle → {{Orange, Thickness → 0.005}, {Green, Thickness → 0.005},
      {Green, Thickness → 0.005, Dashed}, {Blue, Thickness → 0.005},
      {Blue, Thickness → 0.005, Dashed}, {Red, Thickness → 0.005}, {Red, Thickness → 0.005, Dashed}},
    PlotRange → {{0, 3}, {0, 1.23}}, Frame → {{True, False}, {True, False}},
    FrameStyle → Directive[Black, Thickness → 0.002], FrameTicks →
     {{{0, "0.0"}, {0.5, 0.5, {0.01, 0}, Thickness → 0.003}, {1.0, "1.0", {0.01, 0}, Thickness → 0.003},
        {1.5, 1.5, {0.01, 0}, Thickness → 0.003}, {2.0, "2.0", {0.01, 0}, Thickness → 0.003},
        {2.5, "2.5", {0.01, 0}, Thickness → 0.003}, {3.0, "3.0", {0.01, 0}, Thickness → 0.003}},
      {{0, "0.0"}, {0.2, 0.2, {0.012, 0}, Thickness → 0.003}, {0.4, "0.4", {0.012, 0}, Thickness → 0.003},
        {0.6, "0.6", {0.012, 0}, Thickness → 0.003}, {0.8, "0.8", {0.012, 0}, Thickness → 0.003},
        {1.0, "1.0", {0.012, 0}, Thickness → 0.003}, {1.2, "1.2", {0.012, 0}, Thickness → 0.003}}},
    FrameLabel → {Style["t", 22, FontFamily → "Arial", Black, Italic, Bold]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22]];
xv = NArgMax[Norm[vsol[0, t]], {t, 0.1, 0.5}];
xw = NArgMax[Norm[wsol[0, t]], {t, 0.1, 0.5}];
xx = NArgMax[Norm[xsol[0, t]], {t, 0.1, 0.5}];
G2 = Plot[{10^13 Norm[usol[x, 0]], Norm[vsol[x, xv]], 5/6 Norm[wsol[x, xw]], 5/7 Norm[xsol[x, xx]]},
    {x, -3, 3}, PlotStyle → {{Orange, Thickness → 0.005}, {Green, Thickness → 0.005},
      {Blue, Thickness → 0.005}, {Red, Thickness → 0.005}}, PlotRange → {{-3, 3}, {-0.02, 1.23}},
    Frame → {{False, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    AxesStyle → Directive[GrayLevel[0.3], Thickness → 0.0012], FrameTicks →
     {{{-3, -3, {0.01, 0}, Thickness → 0.003}, {-2, -2, {0.01, 0}, Thickness → 0.003}, {-1, -1, {0.01, 0},
          Thickness → 0.003}, {0, 0, {0.01, 0}, Thickness → 0.003}, {1, 1, {0.01, 0}, Thickness → 0.003},
        {2, 2, {0.01, 0}, Thickness → 0.003}, {3, 3, {0.01, 0}, Thickness → 0.003}}, {{0, 0}}},
    FrameLabel → {Style["x", 22, FontFamily → "Arial", Black, Italic, Bold]},
```

```
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 22]];
ff = Labeled[GraphicsRow[{G1, G2}, ImageSize → 800, Spacings → Scaled[0.15],
      Epilog → Inset[LineLegend[{Directive[Orange, Thickness[0.004]], Directive[Green, Thickness[0.004]],
            Directive[Blue, Thickness[0.004]], Directive[Red, Thickness[0.004]]},
          {Style["10^-13 ", FontFamily → "Arial", FontSize → 22],
            Style["1.0 ", FontFamily → "Arial", FontSize → 22], Style["1.2 ", FontFamily → "Arial",
              FontSize → 22], Style["1.4", FontFamily → "Arial", FontSize → 22]},
          LegendFunction → (Framed[#, RoundingRadius → 5, FrameStyle → GrayLevel[0.3]] &),
          LegendLayout → "Row"], Scaled[{0.5, 0.8}]]],
    Text[Style["    a                                            b", 22, FontFamily → "Arial", Bold, Black]],
    {{Top, Left}},
    Spacings → {0, −0.3}];
Export["/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png",
  ff, Background → None, ImageResolution → 600];
ffg =
  Import[
    "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png"];
ffg = ImageTake[ffg, {0, 2500}, {180, 6500}];
Export[
  "/Users/gotall/Library/Mobile Documents/com~apple~CloudDocs/SPaper/Figures/Figure 6.png", ffg];

## ## ## Figure6 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```