Maxwell's Equations: A Brief Exploration

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Abstract

We perform a brief analysis of Maxwell's equations in this writing to bring out a bewildering aspect that an infinitude of boundary conditions are possible for any given global source distribution of charges and currents. This concept is distinct from an infinitude of boundary conditions resulting from different source distributions. For a given source distribution in a finite [or semi infinite]enclosed region an infinite number of boundary conditions might be possible for several distributions outside the enclosure[maintaining conditions like fields going to zero at an infinite distance]. But for any global source distribution we do not expect an infinitude of boundary conditions.

Introduction

For a specified source distribution in a finite [or a semi-infinite]enclosed region an infinite number of boundary conditions[1] might be possible for several distributions outside the enclosure maintain conditions like fields going to zero at infinite distances or other mathematically imposedconditions. But for any global source distribution we do not expect an infinitude of boundary conditions. Nevertheless in our analysis we derive the abnormal possibility of having an infinite number of boundary conditions for any global distribution of charges and currents.

Maxwell's Equations and some Mathematical Consequences

We first write the traditional Maxwell's equations[2] in the SI system using conventional notations:

$$\nabla\vec{E}=\frac{\rho}{\varepsilon_0} \qquad (1.1)$$

$$\nabla\vec{B}=0 \qquad (1.2)$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (1.3)$$

$$\nabla\times\vec{B}=\mu_0\vec{J}+\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \qquad (1.4)$$

We may replace $\vec{E}$ by $\vec{E}$+∇λand $\vec{B}$ by $\vec{B}+\nabla\chi$ with the condition that λ and χ *are time independent* scalars and that they satisfy Laplace's equation:

1) $\nabla^2\lambda = 0$ (2.1)
2) $\nabla^2\chi = 0$ (2.2)

[Prime below does not denote differentiation but transformation]
Using (1.1) and (2.1) we have:
$\nabla\overrightarrow{\mathrm{E}'} = \nabla(\vec{\mathrm{E}} + \nabla\lambda) = \nabla\,\vec{\mathrm{E}} + \nabla^2\lambda = \nabla\vec{E} = \frac{\rho}{\varepsilon_0}$since$\nabla^2\lambda = 0$
Using (1.2) and (2.2) we have
$\nabla\overrightarrow{\mathrm{B}'} = \nabla(\vec{\mathrm{B}} + \nabla\chi) = \nabla\vec{B} + \nabla^2\chi = 0$since$\nabla^2\chi = 0$
Using $\nabla \times \nabla\lambda = 0$ and $\nabla\chi$ independent of time we have
$\nabla \times \overrightarrow{\mathrm{E}'} = \nabla \times (\vec{E} + \nabla\lambda) = -\frac{\partial(\vec{B}+\nabla\chi)}{\partial t} = -\frac{\partial\vec{B}'}{\partial t}$since$\lambda$ is independent of time

$$\nabla \times \vec{\mathrm{B}}' = \nabla \times (\vec{\mathrm{B}} + \nabla\chi) = \nabla \times \vec{B} + \nabla^2\chi = \nabla \times \vec{B}$$

[Since $\nabla^2\chi = 0$]
Or,
Using $\nabla \times \nabla\chi = 0$ and $\nabla\lambda$ independent of time we have

$$\nabla \times \vec{\mathrm{B}}' = \nabla \times (\vec{B} + \nabla\chi) = \mu_0\vec{j} + \varepsilon_0\mu_0\frac{\partial(\vec{E} + \nabla\lambda)}{\partial t} = \mu_0\vec{j} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

[since$\lambda$is independent of time]
We have :
If $\vec{E}$,$\vec{B}$ are solutions to Maxwell's equations for configuration inside a region,
$\vec{E}' = \vec{E} + \nabla\lambda$and$\vec{B}' = \vec{B} + \nabla\chi$ will also be solutions for the same source configuration

Provided

1) $\nabla^2\lambda = 0$
2) $\nabla^2\chi = 0$
3) And λ and χ are time independent quantities.

It is an inherent fact in our transformations that charge density and current densities remain unaltered by these transformations that is by $\vec{E}' = \vec{E} + \nabla\lambda$and$\vec{B}' = \vec{B} + \nabla\chi$, λ and χ being time independent.

Indeed

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0\nabla\vec{E} \quad (3)$$

On transformation

$$\rho' = \varepsilon_0 \nabla \vec{E}' = \varepsilon_0 \vec{\nabla}\left(\vec{\mathrm{E}} + \vec{\nabla}\lambda\right) = \varepsilon_0 \vec{\nabla}\vec{\mathrm{E}} + \nabla^2 \lambda$$

Since $\nabla^2 \lambda = 0$ by our choice

$$\rho' = \varepsilon_0 \vec{\nabla}\vec{\mathrm{E}} = \rho \Rightarrow \rho' = \rho \quad (4)$$

Again

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad (5)$$

$$\vec{j} = \frac{1}{\mu_0}\left[\nabla \times \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right] \quad (6)$$

On transformation we have

$$\vec{j}' = \frac{1}{\mu_0}\left[\nabla \times \left(\vec{B} + \vec{\nabla}\chi\right) + \varepsilon_0 \mu_0 \frac{\partial\left(\vec{E} + \vec{\nabla}\lambda\right)}{\partial t}\right]$$

$$\vec{j}' = \frac{1}{\mu_0}\left[\nabla \times \vec{B} + \vec{\nabla} \times \vec{\nabla}\chi\left(+\vec{\nabla}\chi\right) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \mu_0 \frac{\partial\left(\vec{\nabla}\lambda\right)}{\partial t}\right]$$

Now $\vec{\nabla} \times \vec{\nabla}\chi = 0$ and $\frac{\partial(\vec{\nabla}\lambda)}{\partial t} = 0$ since λ is time independent

$$\vec{j}' = \frac{1}{\mu_0}\left[\nabla \times \vec{B} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right] \quad (7)$$

From (6) and (7) we have,

$$\vec{j}' = \vec{j} \quad (21)$$

Our transformations do not change the distribution of the sources asides maintaining Maxwell's equations[preserving their form].The functions representing charge and current densities do not change. We must keep in mind that our transformations are not the Lorentz transformations [the Lorentz transformations , incidentally, treat $(\rho, \vec{j})$ as a four vector]

With our substitutions[transformations]the macroscopic charge values and the currents remain unaltered. We may come to this conclusion in an equivalent manner[for the macroscopic case] by considering the Integral form of the laws: Gauss law and Ampere's circuital law.

$$\oiint \vec{E} \,.\, d\vec{S} = \frac{q}{\epsilon_0} \quad (8)$$

Since Maxwell's equations are preserved for our transformations, we have,

$$\oiint \vec{E}^{'} . d\vec{S} = \frac{q'}{\epsilon_0} \ (9)$$

$$\frac{q'}{\epsilon_0} = \oiint \vec{E}' . d\vec{S} = \oiint (\vec{E} + \vec{\nabla}\lambda) . d\vec{S} = \oiiint \vec{\nabla}(\vec{E} + \vec{\nabla}\lambda). d\vec{S} = \oiiint (\overrightarrow{\nabla . E} + \nabla^2\lambda) . d\vec{S}$$

$$\oiint \vec{E}' . d\vec{S} = \oiiint \overrightarrow{\nabla . E} . d\vec{S} = \oiint \vec{E} . d\vec{S} = \frac{q}{\epsilon_0} \ (10)$$

For our transformations, for any arbitrary volume,

$$q = q' \ (11)$$

We come to Ampere's circuital law

$$\oint \vec{B} . d\vec{l} = \mu_0 i \quad (12)$$

Since Maxwell's equations are preserved on transformations we have

$$\oint \vec{B}' . d\vec{l} = \mu_0 i' (13)$$

$$\mu_0 i' = \oint \overrightarrow{B'} . d\vec{l} = \oint (\vec{B} + \vec{\nabla}\lambda). d\vec{l} = \oint \vec{B} . d\vec{l} + \oint \vec{\nabla}\lambda . d\vec{l} = \iint \vec{\nabla} \times \vec{B} . d\vec{S} + \iint \vec{\nabla}\vec{\nabla}\lambda \times . d\vec{S}$$
$$= \iint \vec{\nabla} \times \vec{B} . d\vec{S}$$

$$\mu_0 i' = \oint \overrightarrow{B'} . d\vec{l} = \iint \vec{\nabla} \times \vec{B} . d\vec{S} = \oint \vec{B} . d\vec{l} = \mu_0 i$$

For any arbitrary surface if we consider currents passing through it, we have,

$$i' = i \quad (14)$$

Our transformations change the values of $\vec{E}$ and $\vec{B}$ without disturbing the sources.

For a given source distribution we have an infinitude of $(\vec{E} + \vec{\nabla}\lambda, \vec{B} + \vec{\nabla}\chi)$ where $\vec{\nabla}\lambda$ and $\vec{\nabla}\chi$are time independent and also $\nabla^2\lambda = 0$ and $\nabla^2\chi = 0$. The bewildering aspect is that an infinitude of boundary conditions are possible for a given source distribution [distribution of charges and currents.

Conclusion

As claimed an infinitude of boundary conditions are possible for any global distribution of currents and charges.

References

1. Wikipedia, Boundary Value Problem, https://en.wikipedia.org/wiki/Boundary_value_problem

2. Griffiths D J, Introduction to Electrodynamics, Pearson India Education Services Pvt Ltd Copyright © 2015,Appendix: Basic Equations of Electrodynamics

[Will be continued]