BALANCED MATRICES

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Abstract. In this paper we introduce a particular class of matrices. We study the concept of a matrix to be balanced. We study some properties of this concept in the context of matrix operations. We examine the behaviour of various matrix statistics in this setting. The crux will be to understanding the determinants and the eigen-values of balanced matrices. It turns out that there does exist a direct communication among the leading entry, the trace, determinants and, hence, the eigen-values of these matrices of order 2×2. These matrices have an interesting property that enables us to predict their quadratic forms, even without knowing their entries but given their spectrum.

1. Introduction and motivation

Matrix theory is very vast and rich, that there is hardly any concrete introduction even to the uninitiated. Every now and then more and more class of matrices are being introduced. Consider any 2×2 matrix given by

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose we seek the spectrum of A, then we begin by solving the characteristics equation

$$|A - \lambda I| = 0$$

where λ is in the spectrum of A. Similarly, the quadratic form of A is given by

$$F(x, y) := ax^2 + bxy + dy^2$$

for any symmetric matrix A. Thus for a general matrix, we need the entries to be able to write down the quadratic form. In the following sequel, we will study a particular class of matrix whose spectrum can effectively be determined from their entries without neccessarily going through the traditional procedure, and whose quadratic form can easily be determined without even knowing the matrix but given the spectrum. In fact for any such matrix A, it turns out that

$$\sum_{r=1}^{2} a_{i\cdot r} \approx \sum_{s=1}^{2} a_{s\cdot j} \approx \max(M)$$

and

$$|a_{i\cdot 1} - a_{i\cdot 2}| \approx |a_{1\cdot j} - a_{2\cdot j}| \approx \min(M)$$

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where M is the spectrum of A. Consequently the quadratic form associated with any such symmetric matrix is given by

$$F(x,y) :\approx \left(\frac{\lambda_2 - |\lambda_1|}{2}\right)(x+y)^2 + 2|\lambda_1|xy$$

or

$$F(x,y) :\approx \left(\frac{\lambda_2 + |\lambda_1|}{2}\right)(x+y)^2 - 2|\lambda_1|xy.$$

2. **Balanced matrices**

Definition 2.1. Let $\mathbb{M}_{n\times m}(\mathbb{R})$ be the space of $n \times m$ matrices with real entries. Then $A = (a_{ij}) \in \mathbb{M}_{n\times m}(\mathbb{R})$, a non-zero matrix, is said to be horizontally balanced if

$$\sum_{j=1}^{m} a_{r\cdot j}^2 \approx \sum_{j=1}^{m} a_{s\cdot j}^2,$$

for $1 \leq s < r \leq n$. Similarly, It is said to be vertically balanced if

$$\sum_{i=1}^{n} a_{i\cdot r}^2 \approx \sum_{i=1}^{n} a_{i\cdot s}^2,$$

for $1 \leq s < r \leq 1$. Any matrix A is said to be fully balanced if it is both vertically and horizontally balanced.

Example 2.2. Perhaps a good straight-forward example of a fully balanced matrix is the identity matrix, since it abides by the above criterion. Another obvious example of a fully-balanced matrix is given by

$$\lambda \begin{pmatrix} 1 & 1 & \cdots 1 \\ 1 & 1 & \cdots 1 \\ \vdots & & \\ 1 & 1 & \cdots 1 \end{pmatrix}$$

for $\lambda \in \mathbb{R}$. Hence for $A \in \mathbb{M}_3(\mathbb{R})$, the unity matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

the definition 2.1 about fully balanced-matrices holds, for we have

$$1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 1^2 = 3 \quad \text{Horizontally} \tag{2.1}$$

$$1^2 + 1^2 + 1^2 = 1^2 + 1^2 + 1^2 = 3 \quad \text{Vertically.} \tag{2.2}$$

Through out this paper we choose for simplicity to specialize our study to fully-balanced square matrices. Letting the paper to be taken this way brings more questions around.

3. Elementary properties of fully balanced matrices

In this section we examine some properties of fully balanced matrices. We investigate how these properties are preserved under various matrix operations. We prove the theorem for the sums and products of 2×2 matrices. Later, we will prove a result that will enable us to extend these properties to higher order matrices.

Theorem 3.1. *Let $A, B \in \mathbb{M}_n(\mathbb{R})$ be fully-balanced matrices and let $\lambda \in \mathbb{R}$. Then the following remain valid:*

(i) *The transpose A^T is also fully balanced.*
(ii) *The multiple λA is also fully balanced.*
(iii) *The sum of any 2×2 fully-balanced matrix with positive entries is still fully-balanced. In otherword, the notion of balanced balanced matrices is preserved under matrix addition.*
(iii) *The product of any 2×2 fully-balanced matrices with positive entries is still fully balanced.*
(iv) *The inverse of any 2×2 non-singular fully-balanced matrix is still fully balanced. That is, if A is a non-singular fully-balanced 2×2 matrix, then so is A^{-1}.*

Proof. (i) Let $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R})$ be fully-balanced balanced. Then by definition 2.1, it is both vertically and horizontally balanced. Since the transpose of a vertically balanced matrix becomes a horizontally-balanced matrix and vice-versa, it follows that the transpose A^T must be fully balanced.

(ii) The fact that λA is also fully balanced is obvious.

(iii) Consider the 2×2 matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

Then by definition 2.1 the following holds

$$a_1^2 + b_1^2 \approx c_1^2 + d_1^2, \quad a_2^2 + b_2^2 \approx c_2^2 + d_2^2 \tag{3.1}$$

and

$$b_1^2 + d_1^2 \approx a_1^2 + c_1^2, \quad a_2^2 + c_2^2 \approx b_2^2 + d_2^2. \tag{3.2}$$

Using the relation $a_1^2 + b_1^2 \approx c_1^2 + d_1^2$ and $a_1^2 + c_1^2 \approx b_1^2 + d_1^2$, we observe that $c_1^2 \approx b_1^2$. Since the entries are positive, we must have $c_1 \approx b_1$. Using the equation further shows that $a_1 \approx d_1$, $a_2 \approx d_2$ and $b_2 \approx c_2$. Their sum is given by

$$A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.$$

We claim that the matrix $A + B$ is also fully balanced. For we observe that

$$\begin{aligned} (a_1 + a_2)^2 + (b_1 + b_2)^2 &= a_1^2 + a_2^2 + 2|a_1||a_2| + b_1^2 + b_2^2 + 2|b_1||b_2| \\ &= (a_1^2 + b_1^2 + 2|b_1||b_2|) + (a_2^2 + b_2^2 + 2|a_1||a_2|) \\ &\approx (c_1^2 + d_1^2 + 2|b_1||b_2|) + (c_2^2 + d_2^2 + 2|a_1||a_2|) \\ &\approx (c_1^2 + c_2^2 + 2|c_1||c_2|) + (d_1^2 + d_2^2 + 2|d_1||d_2|) \\ &\approx (c_1 + c_2)^2 + (d_1 + d_2)^2 \end{aligned} \tag{3.3}$$

by leveraging the relations in 3.1 and 3.2. Thus the matrix $A+B$ is horizontally balanced. Similarly we observe that

$$\begin{aligned}(b_1+b_2)^2+(d_1+d_2)^2 &= b_1^2+b_2^2+2|b_1||b_2|+d_1^2+d_2^2+2|d_1||d_2| \\ &= (b_1^2+d_1^2+2|d_1||d_2|)+(d_1^2+d_2^2+2|b_1||b_2|) \\ &\approx (a_1^2+a_2^2+2|a_1||a_2|)+(c_1^2+c_2^2+2|c_1||c_2|) \\ &\approx (a_1+a_2)^2+(c_1+c_2)^2\end{aligned}$$

where we have used the relation 3.1 and 3.2. Thus the matrix $A+B$ is also vertically balanced. Therefore it must be fully balanced.

(iv) We now show that their product is also fully balanced. Their product is given by

$$\begin{aligned}AB &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{pmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}(a_1a_2+b_1c_2)^2+(a_1b_2+b_1d_2)^2 &= (a_1^2a_2^2+a_1^2b_2^2)+(b_1^2c_2^2+b_1^2d_2^2)+2a_1a_2b_1c_2+2a_1b_2b_1d_2 \\ &\approx a_1^2(c_2^2+d_2^2)+b_1^2(c_2^2+d_2^2)+2d_1a_2c_1c_2+2b_2c_1d_1d_2 \\ &\approx d_1^2(c_2^2+d_2^2)+c_1^2(a_2^2+b_2^2)+2d_1a_2c_1c_2+2b_2c_1d_1d_2 \\ &\approx (c_1a_2+d_1c_2)^2+(c_1b_2+d_1d_2)^2\end{aligned}$$

and the product is horizontally balanced. A similar argument will show that, the product is also vertically balanced. Therefore, the product is fully balanced.

(iv) The fact that A^{-1} is fully balanced, given that A is fully balanced is obvious. $\square$

Corollary 3.1. The space $\mathcal{N} := \left(\mathcal{B}_2(\mathbb{R}^+)\cup\mathcal{B}_2(\mathbb{R}^-),+,\cdot\right)$ is a vector space over $\mathbb{R}$.

Proof. It is easy to verify from Theorem 3.1 that the set $\mathcal{B}_2(\mathbb{R}^+)\cup\mathcal{B}_2(\mathbb{R}^-)$ is a group under addition. The properties of scalar multiplication also remains valid, and the result follows immediately. $\square$

Remark 3.2. It is important to notice that the set $\mathcal{B}_2(\mathbb{R})$ may not be a group under multiplication.

4. Trace, determinants and eigen-values associated with balanced matrices

In this section we examine various statistics associated with balanced matrices. We study the behaviour of their trace, their determinants, their eigen-values, their eigen-vectors and their corresponding interplay in this setting.

Proposition 4.1. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a fully-balanced square matrix with positive real entries. Then $a < \epsilon$ if and only if $Tr(A) \leq N_\epsilon$ for any $\epsilon > 0$.

Proof. By invoking Theorem 3.1, the result follows immediately. □

Remark 4.1. Theorem 3.1 relates the leading entry of a 2×2 fully-balanced matrix to their trace. Indeed if the leading entry is small enough then the trace must not be too big. Similarly if the leading entry is somewhat large then their trace must be large. This property is archetypical of balanced matrices.

Proposition 4.1 does highlights the importance of balanced matrices. It tells us for the most part that the leading entry or more generally the diagonal entry of any 2×2 fully-balanced matrices has a profound connection with their eigen-values, and hence influences their eigen-vectors. Indeed by using the well-known elementary relation

$$\lambda_1 + \lambda_2 = Tr(A),$$

where λ_1, λ_2 are the eigen-values of the fully-balanced matrix A, then by leveraging Proposition 4.1, we observe that if the leading entry is small enough then each of the eigen-values must not be too big. Similarly if the leading entry is somewhat large then at least one of the eigen-values must be large. A similar description could be carried out to relate the leading entry of balanced matrices to their determinants, using the well-known elementary relation (See [2])

$$det(A) = \lambda_1 \lambda_2,$$

where each λ_i for $1 \leq i \leq 2$ is an eigen-value of A. This is a description characteristic of very rare class of matrices of which balanced matrices is a sub-class.

Definition 4.2. A matrix A is said to be in reduced row echelon form if the following are satisfied

- (i) Any non-zero row has leading term 1 and zero elsewhere.
- (ii) Every row with all 0's must be at the bottom.
- (iii) The leading term of the mth row must be at the right of the leading term of the $(m-1)$ th row.
- (iv) Any column that has a leading entry has zero in all other directions.

Any given matrix can be reduced to a reduced row echelon form, reviewed in definition 4.2 by series of row-echelon reduction. The three most basic types are Type 1, Type 2 and Type 3. Type 1 involves taking the multiple of each entry on a given row. Type 2 involves interchanging or the switching of two rows, where as Type 3 consist of adding the output of Type 1 to another row. The offshoot of any of these operations on the idenitity matrix is an elementary matrix. More precisely applying Type 1 on a given matrix produces an elementary matrix of Type 1, Type 2 produces an elementary matrix of Type 2 and a similar result for Type 3 operation holds. To the uninitiated who seeks details on these kinds of operations see [2, 1].

Balanced matrices are very important theoritically and could have real use application in areas of applied mathematics. The simple and the most basic example of a fully-balanced matrix, as we have seen, is the identity matrix. The determinant of this matrix is always 1. This gives us a clue of the distribution of balanced matrices. Next we state and prove the following key lemma.

Lemma 4.3. *Let $A \in \mathcal{B}_n(\mathbb{R})$ be the space of fully-balanced square matrices and let R_A be the reduced row echelon form of A. Then we have*

$$R_A = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Proof. The proof follows by applying elementary row reductions of various types on A, highlighted in the ensuing discussion. □

Remark 4.4. Henceforth, when we say a balanced matrix it will imply a fully-balanced matrix. Otherwise we will specify the context of balanced matrix. Next we step an inch towards understanding the distribution of the determinants of balanced matrices, by stating a theorem that tells us that any fully-balanced matrix must be invertible.

Theorem 4.5. *Let A be any fully-balanced square matrix with real entries. If the modulus of the entries of A are not all the same, then $|det(A)| > 0$.*

Proof. Let $A \in \mathcal{B}_n(\mathbb{R})$, the space of fully-balanced square matrices. Then by Lemma 4.3

$$R_A = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Next we observe that (See, [1])

$$R_A = E_1 E_2 \cdots E_k A,$$

where E_i is the i $(i = 1, 2 \cdots k)$ elementary matrix described in the forgone discussion. We remark that each E_i is invertible, thus it follows that

$$\begin{aligned} 1 &= det(E_1 E_2 \cdots E_k A) \\ &= det(A) \prod_{i=1}^{k} det(E_i). \end{aligned}$$

Since each of the E_i's are invertible, $det(E_i) \neq 0$ for all $i = 1, 2 \cdots k$ and we find that

$$det(A) = \frac{1}{\prod\limits_{i=1}^{k} det(E_i)}$$

and the proof of the theorem is complete. □

Remark 4.6. Theorem 4.5 is stronger than what was originally stated. It gives us a quantitative picture of the nature of the determinants of square balanced-matrices. More than this, it reduces the study of the nature of the determinants of square balanced matrices to the study of the determinants of the associated elementary matrices.

Eigen values and eigen-vectors are extremely important statistics in the study of matrices. Knowing these two for any matrix can be usefull in practice. The quest to find an eigen-value and, hence, eigen vector features very often in other various applied areas such as physics. The next result helps us to predict upto a smaller error eigen-values and hence eigen-vectors of balanced matrices, without having to undergo the traditional procedure. This result relates the sums and differences of the entries of balanced matrices to the least and worst eigenvalue for 2×2 balanced matrices. It will be great to extend this result to matrices of higher orders. But for the time being we content ourselves with the following:

Theorem 4.7. *Let $A \in \mathbb{B}_2(\mathbb{R}^+)$, the spaces of 2×2 balanced matrices with each $a_{ij} \geq 1$. If $M = \{|\lambda_1|, |\lambda_2|\}$ is the set of eigen-values of A, then*

$$\sum_{r=1}^{2} a_{i \cdot r} \approx \sum_{s=1}^{2} a_{s \cdot j} \approx \max(M)$$

for $1 \leq s, r \leq 2$ and

$$|a_{i \cdot 1} - a_{i \cdot 2}| \approx |a_{1 \cdot j} - a_{2 \cdot j}| \approx \min(M)$$

where $1 \leq i, j \leq 2$.

Proof. Consider the fully-balanced matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then by recalling the well-known elementary relation (See [2])

$$det(B) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = Tr(B)$$

for any matrix B, we can write

$$\begin{aligned} det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ &\approx a^2 - b^2. \end{aligned} \tag{4.1}$$

Since $Tr(A) = \lambda_1 + \lambda_2$, then it follows that $|a - b| \approx |\lambda_1|$ and $a + b \approx |\lambda_2|$ or vice-versa. Similarly we can write

$$\begin{aligned} det(A) &= \lambda_1 \lambda_2 \\ &= ad - bc \\ &\approx d^2 - c^2. \end{aligned} \tag{4.2}$$

Again using the relation $Tr(A) = \lambda_1 + \lambda_2$, then it follows that $|c - d| \approx |\lambda_1|$ and $c + d \approx |\lambda_2|$ or vice-versa. Again it follows that

$$\begin{aligned} det(A) &= \lambda_1\lambda_2 \\ &= ad - bc \\ &\approx a^2 - c^2. \end{aligned} \tag{4.3}$$

Using the relation $Tr(A) = \lambda_1+\lambda_2$, then it follows that $|a-c| \approx |\lambda_1|$ and $a+c \approx |\lambda_2|$ or vice-versa. Also, we have

$$\begin{aligned} det(A) &= \lambda_1\lambda_2 \\ &= ad - bc \\ &\approx d^2 - b^2 \end{aligned} \tag{4.4}$$

and it follows that $|b - d| \approx |\lambda_1|$ and $b + d \approx \lambda_2$ or vice-versa, by using the relation $Tr(A) = \lambda_1 + \lambda_2$. Without loss of generality, we let $\lambda_2 = \max(M)$ and $\min(M) = \lambda_1$. Then it follows that $b + d \approx a + c \approx a + b \approx c + d \approx \lambda_2 = \max(M)$ and $|b-d| \approx |a-c| \approx |c-d| \approx |a-b| \approx \lambda_1 = \min(M)$. For suppose $b+d \approx |b-d|$, then it follows that either $d \approx 0$ or $b \approx 0$, which contradicts the minimality of each of the a_{ij}'s. Similarly, let us suppose that $b + d \approx |a - c|$. Then it follows that

$$\begin{aligned} b + d &\approx a - c \\ &\approx d - c \end{aligned}$$

and we have that $b \approx -c$ if and only if $c \approx 0$, which violates the minimality of a_{ij} for $1 \leq i, j \leq 2$. Also in the case where $b+d = -(a-c)$, then it follows that $d \approx 0$, which is a contradiction. Again if $b + d \approx |c - d|$, then we see that

$$\begin{aligned} b + d &\approx c - d \\ &\approx b - d \end{aligned}$$

and it follows that $d \approx 0$. On the other hand, we will have that $c \approx 0$, both of which contradicts the minimality of a_{ij}. Thus by leveraging the fact that A is fully-balanced, in a similar manner for other cases the result follows immediately. ☐

Corollary 4.1. Let $A_1, A_2 \in \mathcal{B}_2(\mathbb{R}^+)$ with $a_{ij} \geq 1$ and let $E_{\max}(A_1)$ denotes the maximum eigen-value of A_1. Then

$$E_{\max}(A_1 + A_2) \approx E_{\max}(A_1) + E_{\max}(A_2).$$

Proof. Consider the 2×2 fully-balanced matrices given by

$$A_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{and} \quad A_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then by Theorem 3.1, their sum

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

is also fully-balanced. By Theorem 4.7, $E_{\max}(A_1 + A_2) \approx b_1 + b_2 + d_1 + d_2$, and the result follows immediately. ☐

Remark 4.8. Before we state the next result, we review the following terminologies concerning matrices in general.

Definition 4.9. By a block $n \times m$ matrix, we mean any matrix of the form

$$A = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{pmatrix},$$

and where each C_{ij} is a submatrix of A for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 4.10. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ given by

$$A := \begin{pmatrix} a_{1\cdot 1} & a_{1\cdot 2} \cdots & & a_{1\cdot n} \\ a_{2\cdot 1} & a_{2\cdot 2} \cdots & & a_{2\cdot n} \\ \vdots & & & \\ & & & \\ a_{m\cdot 1} & a_{m\cdot 2} & \ddots & a_{m\cdot n} \end{pmatrix}.$$

Then we say a matrix B is an interior of A if it is a submatrix of A.

Conjecture 4.1. Let $A \in \mathcal{B}_n(\mathbb{R})$, the space of square balanced-matrices. Then there exist some interior of A that is also balanced.

Remark 4.11. By thinking of a matrix as a system, Conjecture 4.1 roughly speaking conveys the notion that, if a bigger system is balanced, then there must be a subsystem that is also balanced.

5. **Discrepancies of fully-balanced matrices**

In this section, in the spirit of proving some weaker versions of Conjecture 4.1 we introduce the notion of discrepancy of fully-balanced matrices. It turns out that Conjecture 4.1 is somewhat easier to attack in this setting.

Definition 5.1. Let $A \in \mathbb{M}_{n \times m}(\mathbb{R})$. Then by the discrepancy of the matrix A along rows, we mean the value

$$\sum_{j=1}^{m} a_{ij}.$$

Similarly, by the discrepancy along columns, we mean the value

$$\sum_{i=1}^{n} a_{ij}.$$

Definition 5.2. Let $A \in \mathbb{M}_{m \times n}(\mathbb{R})$ and let

$$M = \frac{1}{m} \sum_{j=1}^{m} a_{ij}.$$

Then we say the discrepancy is fair along rows if $|M - a_{ij}| < \epsilon$ for $1 \leq j \leq m$, where $\epsilon > 0$ is very small.

The discrepancy is unfair along rows if for some a_{ij} $(j = 1, 2 \ldots m)$, there exist some n_0 such that

$$|M - a_{ij}| > N$$

for all $N \geq n_0$.

Remark 5.3. Next we prove some few propositions concerning fully-balanced matrices, in the context of discrepancy.

Theorem 5.4. *Let $A \in \mathcal{B}_{n\times m}(\mathbb{R}^{+})$, the space of fully-balanced $n \times m$ matrices. Then A has a fair discrepancy along rows if and only if it has a fair discrepancy along columns.*

Proof. Let $A \in \mathcal{B}_{n\times m}(\mathbb{R}^{+})$ and suppose A has a fair discrepancy along rows. Then it follows that for each $1 \leq i \leq n$

$$|M_i - a_{ij}| < \epsilon$$

for all $1 \leq j \leq m$. Since A is fully-balanced, it follows that $M_1 \approx M_2 \approx \cdots M_n$ and it follows that $a_{1j} \approx a_{2j} \cdots \approx a_{nj}$ for all $1 \leq j \leq m$. It follows that A must have a fair discrepancy along coloumns. For suppose A has an unfair discrepancy along columns. Then that would mean

$$|M_i - a_{ij}| \geq N$$

for all $1 \leq j \leq m$ under each $1 \leq i \leq n$, with $N \geq n_0$ for some n_0, a contradiction. The converse, on the other hand, follows similar approach. □

Next we prove a weaker version of Theorem 4.1, which states that some interior of a fully-balanced matrix must be balanced. Infact it turns out that we can make this result a little bit stronger by imposing some discrepancy conditions of the fully-balanced matrix in question. The following result in that direction is a consequence of Theorem 5.4.

Corollary 5.1. Every interior of a fully-balanced matrix, with fair discrepancy along rows or columns is still fully-balanced.

Proof. This follows from Theorem 5.4. □

Theorem 5.5. *Let $A \in \mathcal{B}_{n\times m}(\mathbb{R}^{+})$, the space of fully-balanced $n \times m$ matrices. If A has a fair discrepancy along exactly one row, then it must have an unfair discrepancy along columns.*

Proof. Let $A \in \mathcal{B}_{n\times m}(\mathbb{R}^{+})$ and suppose A has a fair discrepancy on exactly one row, then without loss of generality we can write

$$|M_1 - a_{1j}| < \epsilon$$

for all $1 \leq j \leq m$, and where M_1 is the average discrepancy along row 1. Again, it follows that there must be unfair discrepancy along rows, so that there exist an n_0 such that for all $N \geq n_0$

$$|M_{i_0} - a_{i_0 j_0}| \geq N$$

for some fixed $1 \leq i_0 \leq n$ and $1 \leq j_0 \leq m$. Since A is fully-balanced, it follows that

$$|M_1 - a_{i_0 j_0}| \geq N - \delta$$

for some $\delta > 0$ sufficiently small. Thus, we can write

$$\begin{aligned} |a_{1j} - a_{i_0 j_0}| &> N - \delta - |M_1 - a_{1j}| \\ &= N - \delta - \epsilon, \end{aligned}$$

where $\epsilon > 0$ and $\delta > 0$ are sufficiently small. This implies that A has an unfair discrepancy along columns, and the proof of the theorem is complete. □

Remark 5.6. Theorem 5.5 and Theorem 5.4 tells us that we can switch any form of discrepancy between rows and columns for fully-balanced matrices. In other words, columns and rows of fully-balanced matrices talk to each other in terms of discrepancy. This would or may not be possible for general matrices, making these matrices very intriguing and as well interesting.

Proposition 5.1. *Let $A \in \mathcal{B}_{2\times 2}(\mathbb{R}^+)$. If A has a fair discrepancy on exactly one row, then it must have fair discrepancy on rows.*

Proof. Specify $A \in \mathcal{B}_{2\times 2}(\mathbb{R}^+)$ given by

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose A has a fair discrepancy along exactly one row. Without loss of generality, let us assume the fair discrepancy occurs on the first row, then by Theorem 5.4, it must be that $a \approx b$. Since A is fully-balanced, Theorem 3.1 tells us that $a \approx b \approx c \approx d$. This implies that A has a fair discrepancy on rows, and the proof of the proposition is complete. □

Conjecture 5.1. Let $\epsilon > 0$ and let $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ be a fully balanced matrix. The average discrepancy along rows is given by

$$M = \frac{1}{m}\sum_{j=1}^{m} a_{ij}.$$

If

$$|M_i - a_{ij}| < \epsilon$$

for a fixed $1 \leq i \leq n$, then $|M_i - a_{ij}| < \epsilon$ for all $1 \leq i \leq n$.

Remark 5.7. Conjecture 5.1 tells us that if the discrepancy of a fully-balanced matrix along a given row is fair, then it must be fair on all other rows. In other words, a fair discrepancy on a given row is propagated to all other rows.

In general the determinant of matrices is not an approximate homomorphism. That is, the determinant of the sums of matrices may not have the same distribution as the sum of each determinant. These two statistics may be close to each other and could very well be far from each other. Here is where the concept of balanced matrices plays an important role. Given k distinct matrices, we say the determinant is an approximate homomorphism if the relation holds:

$$det\left(\sum_{k=1}^{n} A_k\right) \approx \sum_{k=1}^{n} det(A_k).$$

The next result clarifies and gives a more formall context to the ensuing discussion.

Theorem 5.8. *Let $A, B \in \mathcal{B}_2(\mathbb{R}^+)$, where $\mathcal{B}_2(\mathbb{R}^+)$ is the space of 2×2 balanced-matrices with $a_{ij} \geq 1$ and $b_{ij} \geq 1$. Let $\mathcal{M} = \{|\lambda_1|, |\lambda_2|\}$ be the spectrum of A. If $\min(\mathcal{M}) \approx 0$ and B has a fair discrepancy along rows or columns, then*

$$det(A + B) \approx det(A) + det(B).$$

Proof. Consider the 2×2 fully-balanced matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Then, by Theorem 3.1 we have the fully-balanced matrix

$$A + B := \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix}.$$

It follows that

$$\begin{aligned} det(A+B) &= (a_1+b_1)(a_4+b_4) - (a_2+b_2)(a_3+b_3) \\ &= (a_1a_4 - a_2a_3) + (b_1b_4 - b_2b_3) + (a_1b_4 + b_1a_4 - a_2b_3 - b_2a_3) \\ &\approx det(A) + det(B) + 2(b_1a_1 - a_2b_2) \\ &\approx det(A) + det(B) + 2b_1(a_1 - a_2) \end{aligned}$$

where we have utilised the fact that A and B are fully-balanced matrices, and that B has a fair discrepancy along rows or columns. By using the fact that $\min(\mathcal{M}) \approx 0$, then the result follows from Theorem 4.7. □

Remark 5.9. Theorem 5.8 tells us that the determinant can be made an approximate homomorphism on any two fully-balanced matrices of not-too-small entries, by making the least element in the spectrum of one matrix negligible and avoiding outliers in the entries and rows of the second.

Conjecture 5.2. Let $A, B \in \mathcal{B}_n(\mathbb{R}^+)$, where $\mathcal{B}_n(\mathbb{R}^+)$ is the space of $n \times n$ balanced-matrices with $a_{ij} \geq 1$ and $b_{ij} \geq 1$. Let $\mathcal{M} = \{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$ be the spectrum of A. If $\min(\mathcal{M}) \approx 0$ and B has a fair discrepancy along rows or columns, then

$$det(A+B) \approx det(A) + det(B).$$

6. Quadratic forms associated with balanced matrices

In this section we examine various forms associated with balanced matrices. For the time being we study the quadratic forms associated with fully-balanced 2×2 matrices. We review therefore the following definitions.

Definition 6.1. Let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any symmetric matrix. Then by the quadratic form of A, we mean any expression of the form

$$F(x, y) := ax^2 + 2bxy + dy^2.$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a fully-balanced symmetric matrix. Then the associated quadratic form can be written as

$$\begin{aligned}F(x,y) &:= ax^2+2bxy+dy^2\\ &\approx a(x^2+y^2)+2bxy\\ &\approx a(x+y)^2+2(b-a)xy.\end{aligned}$$

By using Theorem 4.7, we can write

$$\begin{aligned}F(x,y) &:\approx a(x+y)^2+2(b-a)xy.\\ &\approx \left(\frac{\lambda_2-|\lambda_1|}{2}\right)(x+y)^2+2|\lambda_1|xy\end{aligned}$$

if $b>a$. Similarly if $b<a$, then the quadratic form looks a lot like

$$\begin{aligned}F(x,y) &= ax^2+2bxy+dy^2\\ &\approx \left(\frac{\lambda_2+|\lambda_1|}{2}\right)(x+y)^2-2|\lambda_1|xy,\end{aligned}$$

where λ_2 and λ_1 are the worst and the least eigen-values of A respectively. More formally we launch a proposition concerning the quadratic forms associated with 2×2 fully-balanced symmetric matrices.

Proposition 6.1. *Let A be a fully-balanced 2×2 symmetric matrix, such that $a_{ij}\geq 1$ for $1\leq i,j\leq 2$. Let $\mathcal{N}:=\{|\lambda_1|,|\lambda_2|\}$ be the spectrum of A, and where $\max(\mathcal{N})=|\lambda_2|$ and $\min(\mathcal{N})=|\lambda_1|$. Then one of the following is an approximation of the quadratic form of A*

$$F(x,y) :\approx \left(\frac{\lambda_2-|\lambda_1|}{2}\right)(x+y)^2+2|\lambda_1|xy$$

or

$$F(x,y) :\approx \left(\frac{\lambda_2+|\lambda_1|}{2}\right)(x+y)^2-2|\lambda_1|xy.$$

Proof. The result follows from the ensuing discussion concerning quadratic forms of fully-balanced matrices. □

Remark 6.2. Proposition 6.1 tells us that we do not neccessarily need the entries of a 2×2 symmetric fully-balanced matrices to compute the values of their quadratic forms. Given the eigen-values of A, we can with some precision predict the quadratic form of any fully-balanced 2×2 symmetric matrices, without knowing the entries.

7. **End remarks**

In this paper we have introduced the concept of balanced matrices, where we studied various matrix statistics underlying this concept. Much emphasis was placed on 2×2 fully-balanced matrices. This is just the begining of a series of papers regarding this concept. There is much optimistic work in progress to extend these results for lower order square matrices to matrices of higher orders. Another quest, in the not too distant future, will be to find if there really exist some bit of interaction between this class of matrices and matrices in general. This could provide a new window through which to study matrix theory.

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References

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