THE COMPRESSION METHOD AND APPLICATIONS

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ABSTRACT. In this paper we introduce and develop the method of compression of points in space. We introduce the notion of the mass, the rank, the entropy, the cover and the energy of compression. We leverage this method to prove some class of inequalities related to Diophantine equations. In particular, we show that for each $L < n-1$ and for each $K > n-1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that

$$\frac{1}{K^n} \ll \prod_{j=1}^{n} \frac{1}{x_j} \ll \frac{\log(\frac{n}{L})}{nL^{n-1}}$$

and that for each $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and some $s \geq 2$ such that

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s\frac{n}{L^{s-1}}.$$

1. Introduction

The Erdós-Straus conjecture is the assertion that for each $n \in \mathbb{N}$ for $n \geq 3$ there exist some $x_1, x_2, x_3 \in \mathbb{N}$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}.$$

More formally the conjecture states

Conjecture 1.1. For each $n \geq 3$, does there exist some $x_1, x_2, x_3 \in \mathbb{N}$ such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{4}{n}?$$

Despite its apparent simplicity, the problem still remain unresolved. However there has been some noteworthy partial results. For instance it is shown in [1] that the number of solutions to the Erdós-Straus Conjecture is bounded poly-logarithmically on average. The problem is also studied extensively in [2] and [3]. The Erdós-Straus conjecture can also be rephrased as a problem of an inequality. That is to say, the conjecture can be restated as saying that for all $n \geq 3$ the inequality holds

$$c_1\frac{3}{n} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2\frac{3}{n}$$

for $c_1 = c_2 = \frac{4}{3}$ for some $x_1, x_2, x_3 \in \mathbb{N}^3$. Motivated by this version of the problem, we introduce the method of compression. This method comes somewhat close to

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addressing this problem and its variants. Using this method, we managed to show that

Theorem 1.1. *For each $L \in \mathbb{N}$ with $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$c_1 \frac{n}{L} \leq \sum_{j=1}^{n} \frac{1}{x_j} \leq c_2 \frac{n}{L}$$

for some $c_1, c_2 > 1$. In particular, for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_3 \neq x_1$ such that

$$c_1 \frac{3}{L} \leq \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq c_2 \frac{3}{L}$$

for some $c_1, c_2 > 1$.

Perhaps more general is the result

Theorem 1.2. *For each $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and some $s \geq 2$ such that*

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s \frac{n}{L^{s-1}}.$$

Theorem 1.3. *For each $L < n-1$ and for all $s \geq 2$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for $1 \leq i < j \leq n$ such that*

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \ll \log^s \left(\frac{n}{L}\right).$$

2. **Compression**

Definition 2.1. By the compression of scale $m \geq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $m \geq 1$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. $\square$

3. **The mass of compression**

Definition 3.1. By the mass of a compression of scale $m \geq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Remark 3.2. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

Proposition 3.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then the estimates holds*

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\operatorname{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\operatorname{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

The estimates obtained for the mass of compression is quite suggestive. It restricts the entries of any of our choice of tuple to be distinct. After a little heuristics, It can be seen the left estimate for the mass of compression tends to be almost flawed if we allow for tuples with at least two similar entries. Thus in building this Theory, and with all the results we will obtained, we will enforce that the entries of any choice of tuple is distinct.

3.1. **Application of mass of compression.**

In this section we apply the notion of the mass of compression to the Erdós-Straus conjecture.

Theorem 3.3. *There exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for each $n \geq 2$ with $x_j \geq 1$ such that*

$$m \frac{n}{L_1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \frac{n}{L_2}$$

for some $L_1, L_2 \in \mathbb{N}$.

Proof. First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ such that $\sup(x_j) > \mathrm{Inf}(x_j) > n-1$ for $j = 1, \ldots n$. Then from Proposition 3.1, we have the upper bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &\ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)\\ &= m\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{n-1}{\mathrm{Inf}(x_j)}\right)^k\\ &\ll m\frac{n}{\mathrm{Inf}(x_j)}.\end{aligned}$$

The lower bound also follows by noting that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &\gg m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1}\\ &= m\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{n-1}{\sup(x_j)}\right)^k\\ &\gg m\frac{n}{\sup(x_j)}\end{aligned}$$

and the inequality follows by taking $\sup(x_j) = L_1$ and $\mathrm{Inf}(x_j) = L_2$. □

Theorem 3.3 is redolent of the Edòs-Strauss conjecture. Indeed It can be considered as a weaker version of the conjecture. It is quite implicit from Theorem 3.3 that there are infinitely many points in $\mathbb{N}^n$ that satisfy the inequality with finitely many such exceptions. Therefore in the opposite direction we can assert that there are infinitely many $L_1, L_2 \in \mathbb{N}$ that satisfies the inequality. We state a consequence of the result in Theorem 3.3 to shed light on this assertion.

Corollary 3.1. For each $L \in \mathbb{N}$ with $L > n-1$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that

$$\frac{n}{L} \ll \sum_{j=1}^{n}\frac{1}{x_j} \ll \frac{n}{L}$$

In particular, for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_1 \neq x_3$ such that

$$\frac{3}{L} \ll \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \ll \frac{3}{L}.$$

Proof. First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $\sup(x_j) > \mathrm{Inf}(x_j) > n-1$. By taking $K = \sup(x_j)$ and $L = \mathrm{Inf}(x_j)$ for any such points, it follows that

$$\frac{n}{L} \ll \sum_{j=1}^{n}\frac{1}{x_j} \ll \frac{n}{K} \ll \frac{n}{L}.$$

The special case follows by taking $n = 3$. □

It is important to recognize that the condition $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ all $1 \leq i < j \leq n$ in the statement of the result is not only a quantifier but a requirement; otherwise, the estimate for the mass of compression will be flawed completely. To wit, suppose that we take $x_1 = x_2 = \ldots = x_n$, then it will follow that $\operatorname{Inf}(x_j) = \sup(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\operatorname{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\operatorname{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimates to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i < j \leq n$. Thus our Theory will be built on this assumption, that any tuple we use has to have distinct entry. Since all other statistic will eventually depend on the mass of compression, this assumption will be highly upheld.

Remark 3.4. The result can be interpreted as saying that for each $L \geq 3$ there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ such that

$$c_1\frac{3}{L} \leq \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \leq c_2\frac{3}{L}$$

for some constants $c_1, c_2 > 1$. The Erdós-Straus conjecture will follow if we can take $c_1 = c_2 = \frac{4}{3}$. Investigating the scale of these constants is the motivation for this Theory and will be developed in the following sequel.

Theorem 3.5. *For each $K > n-1$ and for each $L < n-1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{n}{K} \ll \sum_{j=1}^{n}\frac{1}{x_j} \ll \log\left(\frac{n}{L}\right).$$

Proof. Let us choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $\operatorname{Inf}(x_j) < n-1$ and $\sup(x_j) > n-1$. Then we set $L = \operatorname{Inf}(x_j)$ and $K = \sup(x_j)$, then the result follows from the estimate in Theorem 3.1. □

Remark 3.6. Next we expose one consequence of Theorem 3.5.

Corollary 3.2. For each $K > 2$, there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_i \neq x_j$ for all $1 \leq i < j \leq 3$ such that

$$c_1\frac{3}{K} \leq \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \leq c_2\log 3$$

for some $c_1, c_2 > 1$.

4. **The rank of compression**

In this section we introduce the notion of the rank of compression. We launch the following language in that regard.

Definition 4.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ then by the rank of compression, denoted $\mathcal{R}$, we mean the expression

$$\mathcal{R}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)] = \left|\left|\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)\right|\right|.$$

Remark 4.2. It is important to notice that the rank of a compression of scale $m \geq 1$ is basically the distance of the image of points under compression from the origin. Next we relate the rank of compression of scale $m \geq 1$ with the mass of a certain compression of scale 1.

Proposition 4.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then we have*

$$\mathcal{R} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = m^2 \mathcal{M} \circ \mathbb{V}_1 \left[\left(x_1^2, x_2^2, \ldots, x_n^2 \right) \right].$$

Proof. The result follows from definition 4.1 and definition 3.1. $\square$

Remark 4.3. Next we prove upper and lower bounding the rank of compression of scale $m \geq 1$ in the following result. We leverage pretty much the estimates for the mass of compression of scale $m \geq 1$.

Theorem 4.4. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then we have*

$$m\sqrt{\log\left(1 - \frac{n-1}{\sup(x_j^2)}\right)^{-1}} \ll \mathcal{R} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] \ll m\sqrt{\log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j^2)}\right)}$$

Proof. The result follows by leveraging Proposition 4.1 and Proposition 3.1. $\square$

4.1. **Application of rank of compression.** In this section we expose one consequence of the rank of compression. We apply this to estimate the second moment unit sum of the Erdós Type problem. We state this more formally in the following result.

Theorem 4.5. *For each $L > \sqrt{n-1}$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{n}{L^2} \ll \sum_{j=1}^{n} \frac{1}{x_j^2} \ll \frac{n}{L^2}.$$

In particular for each $L \geq 2$, there exist some $(x_1, x_2, x_3) \in \mathbb{N}^3$ with $x_1 \neq x_2$, $x_2 \neq x_3$ and $x_1 \neq x_3$ and some constant $c_1, c_2 > 1$ such that

$$c_1 \frac{3}{L^2} \leq \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \leq c_2 \frac{3}{L^2}.$$

Proof. Let us choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ in Theorem 4.4 such that $L = \mathrm{Inf}(x_j)$ with $L^2 > n-1$. Then the inequality follows immediately. The special case follows by taking $n = 3$. $\square$

Remark 4.6. Next we present a second moment variant inequality of the unit sum of positive integers in the following statement.

Corollary 4.1. For each $L \geq 3$, there exist some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5$ with $x_i \neq x_j$ for all $1 \leq i < j \leq 5$ and some constant $c_1, c_2 > 1$ such that

$$c_1 \frac{5}{L^2} \leq \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2} + \frac{1}{x_5^2} \leq c_2 \frac{5}{L^2}.$$

5. **The entropy of compression**

In this section we launch the notion of the entropy of compression. Intuitively, one could think of this concept as a criteria assigning a weight to the image of points under compression. We provide some quite modest estimates of this statistic and exploit some applications, in the context of some Diophantine problems.

Definition 5.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. By the entropy of a compression of scale $m \geq 1$ we mean the map $\mathcal{E} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{E}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \prod_{i=1}^{n} \frac{m}{x_i}.$$

Remark 5.2. Next we relate the mass of a compression to the entropy of compression and deduce reasonable good bounds for our further studies. We could in fact be economical with the bounds but they are okay for our needs.

Proposition 5.1. *For all $n \geq 2$, we have*

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = m\mathcal{M}\Bigg(\mathbb{V}_1\Bigg[\Bigg(\prod_{i\neq 1}\frac{1}{x_i}, \prod_{i\neq 2}\frac{1}{x_i}, \ldots, \prod_{i\neq n}\frac{1}{x_i}\Bigg)\Bigg]\Bigg) \times \mathcal{E}(\mathbb{V}_1[(x_1, x_2, \ldots, x_n)]).$$

Proof. By Definition 3.1, we have

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= \sum_{i=1}^{n} \frac{m}{x_i}\\ &= m\frac{\displaystyle\sum_{\sigma:[1,n]\longrightarrow[1,n]} \prod_{\substack{n-1\\ \sigma(i)\neq\sigma(j)\\ i\neq j\\ i\in[1,n]}} x_{\sigma(i)}}{\displaystyle\prod_{i=1}^{n} x_i}\end{aligned}$$

The result follows immediately from this relation. $\square$

Proposition 5.2. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then we have*

$$\frac{\log(1 - \frac{n-1}{\sup(x_j)})^{-1}}{n\sup(x_j)^{n-1}} \ll \mathcal{E}(\mathbb{V}_1[(x_1, x_2, \ldots, x_n)]) \ll \frac{\log(1 + \frac{n-1}{\operatorname{Inf}(x_j)})}{n\operatorname{Inf}(x_j)^{n-1}}.$$

Proof. The result follows by using the relation in Proposition 5.1 and leveraging the bounds in Proposition 3.1, and noting that

$$\mathcal{M}\Bigg(\mathbb{V}_1\Bigg[\Bigg(\prod_{i\neq 1}\frac{1}{x_i}, \prod_{i\neq 2}\frac{1}{x_i}, \ldots, \prod_{i\neq n}\frac{1}{x_i}\Bigg)\Bigg]\Bigg) \leq n\sup(x_j)^{n-1}$$

and

$$\mathcal{M}\Bigg(\mathbb{V}_1\Bigg[\Bigg(\prod_{i\neq 1}\frac{1}{x_i}, \prod_{i\neq 2}\frac{1}{x_i}, \ldots, \prod_{i\neq n}\frac{1}{x_i}\Bigg)\Bigg]\Bigg) \geq n\operatorname{Inf}(x_j)^{n-1}.$$

$\square$

5.1. **Applications of the entropy of compression.** In this section we lay down one striking and a stunning consequence of the entropy of compression. One could think of these applications as analogues of the Erdós type result for the unit sums of triples of the form (x_1, x_2, x_3). We state two consequences of these estimates in the following sequel.

Theorem 5.3. *For each $L > n-1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{1}{L^n} \ll \prod_{i=1}^{n} \frac{1}{x_i} \ll \frac{1}{L^n}.$$

Proof. Let us choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $L > n-1$ with $\mathrm{Inf}(x_j) = L$, then the result follows immediately in Proposition 5.2. □

Theorem 5.3 tells us that for some tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ there must exist some constant $c_1, c_2 > 1$ such that we have the inequality

$$\frac{c_1}{L^n} \leq \prod_{j=1}^{n} \frac{1}{x_j} \leq \frac{c_2}{L^n}.$$

Next we present a second application of the estimates of the entropy of compression in the following sequel.

Theorem 5.4. *For each $L < n-1$ and for each $K > n-1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{1}{K^n} \ll \prod_{j=1}^{n} \frac{1}{x_j} \ll \frac{\log(\frac{n}{L})}{nL^{n-1}}.$$

Proof. Let us choose a tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $\sup(x_j) = K > n-1$ and $L = \mathrm{Inf}(x_j) < n-1$, then the result follows immediately. □

Corollary 5.1. For each $L < 4$ and for each $K > 4$, there exist some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5$ with $x_i \neq x_j$ for all $1 \leq i < j \leq 5$ and some constant $c_1, c_2 > 1$ such that

$$\frac{c_1}{K^5} \leq \frac{1}{x_1} \times \frac{1}{x_2} \times \frac{1}{x_3} \times \frac{1}{x_4} \times \frac{1}{x_5} \leq c_2 \frac{\log 5}{5L^4}.$$

Proof. The result follows by taking $n = 5$ in Theorem 5.3. □

6. Compression gap

In this section we introduce the notion of the gap of compression. We investigate this concept in-depth and in relation to the already introduced concepts.

Definition 6.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale m $\mathbb{V}_m$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left| \left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right) \right|\right|$$

The gap of compression is a definitive measure of the chasm between points and their image points under compression. We can estimate this chasm by relating the compression gap to the mass of an expansion in the following ways.

Proposition 6.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2 \mathcal{M} \circ \mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

Proof. The result follows by using using Definition 6.1 and Definition 3.1. □

Remark 6.2. We are now ready to provide an estimate for the gap of compression.

Theorem 6.3. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\sup(x_j^2) + m^2 \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \gg n\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{n-1}{\sup(x_j^2)}\right)^{-1} - 2mn.$$

Proof. The result follows by exploiting Proposition 3.1 in Proposition 6.1 and noting that

$$n\mathrm{Inf}(x_j^2) \leq \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] \leq n\sup(x_j^2).$$

□

6.1. **Application of the compression gap.** In this section we give one striking application of the notion of the gap of compression. It applies to all points $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$.

Theorem 6.4. *Let $n \leq m$, then for each $L > \sqrt{n-1}$ there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{m^2 n}{L} \ll \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right| \ll \frac{m^{\frac{3}{2}}}{L}.$$

Proof. For $m \geq n$, choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $\mathrm{Inf}(x_j) > \sqrt{n-1}$ and set $\sup(x_j) = K$ and $L = \mathrm{Inf}(x_j)$. Then the result follows from the estimate in Theorem 6.3. □

Theorem 6.5. *Let $m \geq n$. For each $L < \sqrt{n-1}$ and each $K > \sqrt{n-1}$ there exist some $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{m\sqrt{n}}{K} \ll \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right| \ll m\sqrt{\log(\frac{n}{L})}.$$

Proof. Let $m \geq n$. Then In Theorem 6.3 choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_1 \neq x_j$ for all $1 \leq i < j \leq n$ such that $\mathrm{Inf}(x_j) < \sqrt{n-1}$ and $\sup(x_j) > \sqrt{n-1}$ and set $\mathrm{Inf}(x_j) = L$ and $\sup(x_j) = K$. Then the result follows immediately. □

7. **The energy of compression**

In this section we introduce the notion of the energy of compression. We launch more formally the following language.

Definition 7.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$ for $n \geq 2$, then by the energy dissipated under compression on $(x_1, x_2, \ldots, x_n)$, denoted $\mathbb{E}$, we mean the expression

$$\mathbb{E} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] \times \mathcal{E}(\mathbb{V}_m\left[\left(x_1, x_2, \ldots, x_n\right)\right]).$$

Remark 7.2. Given that we have obtained upper and lower bounds for the compression gap and the entropy of any points under compression, we can certainly get control on the energy dissipated under compression in the following proposition.

Proposition 7.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then we have*

$$\mathbb{E} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \ll \frac{1}{(\mathrm{Inf}(x_j))^{n-1}\sqrt{n}} \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

and

$$\mathbb{E} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \gg \frac{1}{\sqrt{n}(\sup(x_j))^{n-1}} \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1}.$$

Proof. The result follows by plugging the estimate in 6.3 and 5.2 into definition 7.1. ☐

7.1. **Applications of the energy of compression.**

In this section we give some consequences of the notion of the energy of compression.

Theorem 7.3. *For each $K > n - 1$ and for each $L < n - 1$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that*

$$\frac{\sqrt{n}}{K^n} \ll \frac{\left|\left|\left(x_1 - \frac{1}{x_1}, x_2 - \frac{1}{x_2}, \ldots, x_n - \frac{1}{x_n}\right)\right|\right|}{x_1 x_2 \cdots x_n} \ll \frac{\log\left(\frac{n}{L}\right)}{L^{n-1}\sqrt{n}}.$$

Proof. First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ such that $\mathrm{Inf}(x_j) < n - 1$ and $\sup(x_j) > n - 1$. Now set $K = \sup(x_j)$ and $\mathrm{Inf}(x_j) = L$, then the result follows by exploiting the estimates in Proposition 7.1. ☐

Corollary 7.1. For each $K \geq 5$ and for each $L < 4$, there exist some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5$ with $x_i \neq x_j$ for all $1 \leq i < j \leq 5$ such that

$$\frac{\sqrt{5}}{K^5} \ll \frac{\left|\left|\left(x_1 - \frac{1}{x_1}, x_2 - \frac{1}{x_2}, x_3 - \frac{1}{x_3}, x_4 - \frac{1}{x_4}, x_5 - \frac{1}{x_5}\right)\right|\right|}{x_1 x_2 \cdots x_5} \ll \frac{\log\left(\frac{5}{L}\right)}{L^4\sqrt{5}}.$$

Proof. The result follows by taking $n = 5$ in Theorem 7.3. ☐

8. **The cover of compression**

In this section we introduce the notion of the cover of compression. The cover of compression is basically the s-fold direct product of compression on points in space. We launch the following language in that regard.

Definition 8.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the s-fold cover of compression on the point, we mean the direct product

$$\otimes_{m=1}^{s} \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$$

Remark 8.2. Next we show that we can get control on the mass of the s-fold cover of any compression by the s powers of the mass of the s th compression.

Proposition 8.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ for $n \geq 2$, then we have the estimate*

$$\mathcal{M} \circ \otimes_{m=1}^{s} \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] \gg s! \left[\log^s \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} - s\frac{n}{\mathrm{Inf}(x_j)^{s-1}}\right]$$

and

$$\mathcal{M} \circ \otimes_{m=1}^{s} \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] \ll s! \left[\log^s \left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right) - s\frac{n}{\sup(x_j)^{s-1}}\right].$$

Proof. First we notice that by an application of Stirling formula we have

$$\begin{aligned}\mathcal{M} \circ \otimes_{m=1}^{s} \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] &= s! \sum_{j=1}^{n} \frac{1}{x_j^s} \\ &= s! \left[\left(\sum_{j=1}^{n} \frac{1}{x_j}\right)^s - s\sum_{j=1}^{n} \prod_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{1}{x_i}\right]\end{aligned}$$

The estimate follows by plugging the upper bound in Proposition 3.1 into this estimate and noting that

$$\frac{n}{\sup(x_j)^{s-1}} \leq \sum_{j=1}^{n} \prod_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{1}{x_i} \leq \frac{n}{\mathrm{Inf}(x_j)^{s-1}}.$$

∎

8.1. **Application of the cover of compression.** In this section we present some consequences of the cover of compression. We provide two applications in the context of a Diophantine problem. We generalize the result in Theorem 3.3 at the compromise of some slightly worst implicit constants.

Theorem 8.3. *For each $L > n - 1$ there exist some $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and some $s \geq 2$ such that*

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \gg s\frac{n}{L^{s-1}}.$$

Proof. First choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ such that $\mathrm{Inf}(x_j) = L > n - 1$ and $K = \sup(x_j)$. By choosing $s \geq 2$ such that $n^{s-1} < L$ then the inequality follows from the estimates in Proposition 8.1. ∎

Theorem 8.3 can be thought of as a one-sided generalization of Theorem 3.3.

Theorem 8.4. *For each $L < n-1$ and for all $s \geq 2$, there exist some $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for $1 \leq i < j \leq n$ such that*

$$\sum_{j=1}^{n} \frac{1}{x_j^s} \ll \log^s \left(\frac{n}{L}\right).$$

Proof. Let us choose $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for $1 \leq i < j \leq n$ such that $\mathrm{Inf}(x_j) < n-1$ and $\sup(x_j) > n-1$. Again set $\mathrm{Inf}(x_j) = L$ and $\sup(x_j) = K$, then the result follows from the estimates in Proposition 8.1. $\square$

Corollary 8.1. For each $L < 4$ and for all $s \geq 2$, there exist some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5$ such that

$$\frac{1}{x_1^s} + \frac{1}{x_2^s} + \frac{1}{x_3^s} + \frac{1}{x_4^s} + \frac{1}{x_5^s} \ll \log^s(5/L).$$

9. **The measure and cost of compression**

In this section we introduce the notion of the measure and the cost of compression. We launch the following languages.

Definition 9.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$ for $n \geq 2$. Then by the measure of compression on $(x_1, x_2, \ldots, x_n)$, denoted $\mathcal{N}$, we mean the expression

$$\mathcal{N} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\mathcal{E}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) - \mathcal{E}(\mathbb{V}_m\left[\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)\right])\right|.$$

The corresponding cost of compression, denoted $\mathcal{C}$ is given

$$\mathcal{C} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathcal{N} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] \times \mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

Next we estimate from below and above the measure and the cost of compression in the following sequel. We leverage the estimates established thus far to provide these estimates.

Proposition 9.1. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then the following estimates remain valid*

$$\mathcal{N} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \ll \sup(x_j)^n$$

and

$$\mathcal{N} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \gg \mathrm{Inf}(x_j)^n.$$

Proof. The result follows by exploiting the estimates in Theorem 5.2 in definition 9.1. $\square$

Proposition 9.2. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then we have*

$$\mathcal{C} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \ll \sup(x_j)^{n+1}\sqrt{n}$$

and

$$\mathcal{C} \circ \mathbb{V}_1[(x_1, x_2, \ldots, x_n)] \gg \mathrm{Inf}(x_j)^{n+1}\sqrt{n}.$$

Proof. The result follows by leveraging various estimates developed. $\square$

10. **Final remarks**

The method of compression could be a potentially useful and as well powerful tool for resolving the Erdós-Straus conjecture. It can also find its place as a toolbox for quite a good number of Diophantine problem. The theory as it stands is still open to further development, which we do not pursue in this current version. One area that could be tapped is to investigate the geometry of **compression**. That is, to analyze the topology and the geometry of this concept.
[1].

References

1. Elsholtz, C and Tao, T *Counting the number of solutions to the Erdős–Straus equation on unit fractions*, Journal of the Australian Mathematical Society, vol. 94:1, 2013, Cambridge University Press, pp 50–105.
2. Yamamoto, K *ON THE DIOPHANTINE EQUATION 4/n= 1/x+ 1/y+ 1/z*, Memoirs of the Faculty of Science, Kyushu University. Series A, Mathematics, vol. 19:1, Department of Mathematics, Faculty of Science, Kyushu University, 1965, pp 37–47.
3. Yang, Xun Qian *A Note on 4n=1x+1y+1z*, Proceedings of the American Mathematical society, JSTOR, 1982, pp 496–498.

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