# THE THEORY OF THE COLLATZ PROCESS

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Abstract. In this paper we introduce and develop the theory of the Collatz process. We leverage this theory to study the Collatz conjecture. This theory also has a subtle connection with the infamous problem of the distribution of Sophie germain primes. We also provide several formulation of the Collatz conjecture in this language.

## 1. **Introduction and motivation**

Recall the Collatz function, the arithmetic function of the form

**Definition 1.1.** Let $a \in \mathbb{N}$, then the Collatz function is the piece-wise function

$$\mathcal{C}(a) = \begin{cases} \frac{a}{2} \quad \text{if} \quad a \equiv 0 \pmod 2 \\ 3a+1 \quad \text{if} \quad a \equiv 1 \pmod 2. \end{cases}$$

Then Collatz conjecture, which is one of the acclaimed hardest but easy to state problems is the assertion that

**Conjecture 1.1.** Let $\mathcal{C}$ be the Collatz function, then $\min\{\mathcal{C}^s(b)\}_{s=0}^{\infty} = 1$ for any $b \in \mathbb{N}$.

The conjecture has long been studied and hence the vast literature and surveys concerning the study. For instance the problem has been given a fair treatment in the following surveys [1], [2], [3]. Motivated by this problem we introduce the subject of the Collatz process. We develop this theory and it turns out incidentally that it is connected to other open problems such as the problem concerning the distribution of the Sophie germain primes.

## 2. **Modified Collatz function and the Collatz process**

In this section we introduce s slight variant of the Collatz function and introduce the notion of the Collatz process. We introduce the notion of the backward Collatz process and the generator of the Collatz process.

**Definition 2.1.** Let $a \geq 1$, then the Collatz function is the piece-wise function

$$f(a) = \begin{cases} \frac{a}{2} \quad \text{if} \quad a \equiv 0 \pmod 2, \ a > 1 \\ 3a+1 \quad \text{if} \quad a \equiv 1 \pmod 2, \ a > 1 \\ 1 \quad \text{If} \quad a = 1. \end{cases}$$

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**Definition 2.2.** Let $f$ be the Collatz function and let $a \geq 1$. Then by the Collatz process on $a$, we mean the sequence $\{f^s(a)\}_{s=1}^{\infty}$ for $s \in \mathbb{N}$. The sequence $\{\mathrm{Inf}\{f^{-s}(a)\}\}_{s=1}^{\infty}$ is the backward Collatz process and $a$ is said to be the generator of the Collatz process if each $a_n \in \{\mathrm{Inf}\{f^{-s}(a)\}\}_{s=1}^{\infty}$ is of the same parity.

**Proposition 2.1.** *Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If $b$ is the generator, then each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$ must satisfy*

$$a_n \equiv 0 \pmod 2.$$

*Proof.* Let $f$ be the Collatz function with generator $b \in \mathbb{N}$. First we see that $f^{-1}(b) \not\equiv 1 \pmod 2$; Otherwise it would mean $f^{-2}(b) \equiv 0 \pmod 2$ contradicting the assumption that $b$ is the generator of the process. It suffices to consider the case $m \geq 2$. Suppose for the sake of contradiction that there exist some $a_m \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$; in particular, let $a_m = f^{-m}(b)$ such that $a_m \equiv 1 \pmod 2$ for $m \geq 2$. Then under the Collatz process, we must have

$$3f^{-m}(b) = f^{-(m-1)}(b) - 1.$$

It follows that $f^{-(m-1)}(b) \equiv 0 \pmod 2$, thereby contradicting the underlying assumption that $b$ is the generator of the process. □

**Proposition 2.2.** *Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If $b \in \mathbb{N} \setminus \{1\}$ is the generator of the process, then $b \notin \{f^s(b)\}_{s=1}^{\infty}$.*

*Proof.* Let $f$ be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ with generator $b \in \mathbb{N} \setminus \{1\}$. Suppose on the contrary that $b \in \{f^s(b)\}_{s=1}^{\infty}$. Then it follows that there exist some $s \geq 1$ such that $f^s(b) = b$. Thus we obtain the following chains of equality

$$b = f^s(b) = f^{2s}(b) = f^{3s}(b) = \ldots$$

for some $s \geq 1$. It follows from Proposition 2.1 that each $a_n \in \{f^s(b)\}_{s=m;m\geq 1}^{\infty}$ must satisfy the parity condition $a_n \equiv 0 \pmod 2$, since $b$ is the generator of the process. This is not true since $f$ is the Collatz function. □

*Remark* 2.3. Next we prove the unicity of generators of the Collatz process.

**Proposition 2.3.** *The generator of any Collatz process is unique.*

*Proof.* Let $f$ be the Collatz function with the corresponding processes $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$, where $a, b \in \mathbb{N}$ are the two generators such that $a \neq b$ with $a, b > 1$. Then it follows that for any $m \geq 1$, there exist some $r > 1$ such that we have $f^m(b) = f^r(a)$. Without loss of generality, we let $m > r$ so that we have $f^{m-r}(b) = a$. It follows that $a \in \{f^s(b)\}_{s=1}^{\infty}$. Appealing to Proposition 2.2, then $a \notin \{f^s(a)\}_{s=1}^{\infty}$ and it follows that $\{f^s(a)\}_{s=1}^{\infty} \neq \{f^s(b)\}_{s=1}^{\infty}$, thereby contradicting the assumption that $a, b$ are two distinct generators of the process. □

It is very important to point out that a Collatz process may or may not have a generator. If a Collatz process has a generator, then we say the generator is finite; On the other hand, If it has no generator then we say the generator is infinite. Next we expose the parity of a generator of a Collatz process.

**Proposition 2.4.** *Let $f$ be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$. If $b \neq 1$ is the generator of the process, then $b \equiv 1 \pmod 2$.*

*Proof.* Let $f$ be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ and suppose $b \in \mathbb{N}$ is the generator of the process. Then by definition each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$ has the same parity and must satisfy $a_n \equiv 0 \pmod 2$. Suppose on the contrary that $b \equiv 0 \pmod 2$, then we choose $m \geq 1$ for $m = \mathrm{Inf}(s)_{s=1}^{\infty}$ such that $f^m(b) \equiv 1 \pmod 2$. Then it follows that each

$$a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty} \cup \{b, f(b), f^2(b), \ldots, f^{m-1}(b)\}$$

has the same parity. It follows that $f^m(b)$ is the generator of the process $\{f^s(b)\}_{s=m+1;m\geq 1}^{\infty}$. Since $\{f^s(b)\}_{s=m+1;m\geq 1}^{\infty} \subset \{f^s(b)\}_{s=1}^{\infty}$, It follows that for some $f^r(b) \in \{f^s(b)\}_{s=m+1;m\geq 1}^{\infty}$, there exist some $f^t(b) \in \{f^s(b)\}_{s=1}^{\infty}$ such that $f^r(b) = f^t(b)$. It then follows that $f^k(b) = b$ for $k \geq 1$. This relation is absurd under the Collatz function. □

## 3. The order and index under the Collatz process

In this section we introduce the notion of the order and the index of positive integers under the Collatz process. We study the convergence and the divergence of the Collatz process. We launch the following terminology to aid our inquiry.

**Definition 3.1.** Let $f$ be the Collatz function and $a > 1$, then the order of $a$ under the Collatz process is the least value of $m$ such that $f^m(a) = 2^k$. The value of $k$ is the index of $a$ under the Collatz process. The number $a$ is said to have finite order and a finite index if and only if it converges under the Collatz process. Otherwise we say it diverges under the Collatz process. We denote by $\tau_f(a)$ and $\mathrm{Ind}_f(a)$ the period and the index of $a$ under the Collatz process. In the case $a$ diverges under the process, then $\tau_f(a) = \infty$ and $\mathrm{Ind}_f(a) = \infty$.

In light of definition 3.1, the Collatz conjecture can be restated in the following manner:

**Conjecture 3.1** (Collatz)**.** Let $f$ be the Collatz function and $\{f^s(a)\}_{s=1}^{\infty}$ for $a \in \mathbb{N}$ be a Collatz process, then $\tau_f(a) < \infty$.

The above conjecture can also be expressed in a more quantitative form. In other words, It suffice to resolve the Collatz conjecture by showing that

**Conjecture 3.2** (Collatz)**.** Let $f$ be the Collatz function and $\{f^s(b)\}_{s=1}^{\infty}$ an arbitrary Collatz process. Then

$$\sum_{s=1}^{\infty} \log(f^s(b)) < \infty.$$

**Proposition 3.1.** *Let $f$ be the Collatz function and let $a > 1$ with $\Omega(a) = 2$ such that $a \equiv 0 \pmod 2$, then*

$$f(r) - 1 = 3f(a)$$

*where $a = 2r$ with $r \equiv 1 \pmod 2$.*

*Proof.* Since $a$ is even, it follows from definition 2.1 that the right hand side must be $3r$. Under the condition that $\Omega(a) = 2$ with $r \equiv 1 \pmod 2$, the result follows by definition 2.1. $\square$

*Remark* 3.2. Next we show that primes in a certain congruence class should, by necessity, have large order in as much as their index under the Collatz process is large.

**Theorem 3.3.** *Let $f$ be the Collatz function and $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. If $\mathrm{Ind}_f(p) > 1$, then $\tau_f(p) > 1$.*

*Proof.* Let $p > 3$ be a prime, then under the Collatz process 3.1, it follows that $f^{\tau_f(p)}(p) = 2^{\mathrm{Ind}_f(p)}$. Suppose on the contrary $\tau_f(p) = 1$, then under the assumption $\mathrm{Ind}_f(p) > 1$ it follows that $2^{\mathrm{Ind}_f(p)} + 1 \equiv 1 \pmod 4$. Then it must certainly be that $2^{\mathrm{Ind}_f(p)} - 1 \equiv 3 \pmod 4$, so that under the Collatz process we have

$$\begin{aligned} 3p &\equiv 3 \pmod 4 \\ \Longleftrightarrow p &\equiv 1 \pmod 4 \end{aligned}$$

thereby contradicting the residue class of the prime $p > 3$. $\square$

*Remark* 3.4. Next we establish a converse of Theorem 3.3 in the following proposition.

**Proposition 3.2.** *Let $f$ be the Collatz function with the corresponding convergent Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$. If $\tau_f(b) \geq 2$, then $\mathrm{Ind}_f(b) > 1$.*

*Proof.* Let $f$ be the Collatz function with a corresponding convergent Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Let $\tau_f(b) \geq 2$ and suppose on the contrary that $\mathrm{Ind}_f(b) = 1$, then we can write $f^{\tau_f(b)}(b) = 2$. Since $\tau_f(b) \geq 2$, we can write $f^{\tau_f(b)-1}(b) = f^{-1}(2) = 4$. This contradicts the minimality of $\tau_f(b)$, since $f^{\tau_f(b)-1}(b) \in \{f^s(b)\}_{s=1}^{\infty}$. $\square$

## 4. Relative speed of the Collatz process

In this section we introduce the notion of the relative speed of a Collatz process.

**Definition 4.1.** Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(a)\}_{s=1}^{\infty}$. Then by the speed of the $j$ th Collatz process relative to the $k$ th Collatz process, we mean the expression

$$\nu(f^j(a), f^k(a)) = \frac{|f^k(a) - f^j(a)|}{|k - j|}.$$

The Collatz conjecture can also be framed in the language of the relative speed of the Collatz process as

**Conjecture 4.1.** Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$, then there exist some $1 \leq j < k$ such that $\nu(f^j(b), f^k(b)) = 2^r$ for some $r \in \mathbb{N}$.

It follows from definition 4.1 that the relative speed of the Collatz process must satisfy the inequality

$$|f^k(b) - f^j(b)| = |k-j|\nu(f^j(b), f^k(b)) \leq f^j(b) + f^k(b).$$

Thus the Collatz conjecture is equivalent to establishing the inequality

**Conjecture 4.2.** Let $f$ be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$, then there exist some $k \geq 1$ such that the inequality is valid

$$2^r \leq \nu(f^{k+1}(b), f^k(b)) \leq 2^m$$

for some $m, r \in \mathbb{N}$.

## 5. **The sub-Collatz process**

In this section we introduce the notion of the sub-Collatz process. We establish a relationship between the order and the index of a number under the assumption that the Collatz process converges. We launch the following language.

**Definition 5.1.** Let $f$ be the Collatz function. Then the Collatz process $\{f^t(a)\}_{t=1}^{\infty}$ is said to be a sub-Collatz process of the Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $s, t \in \mathbb{N}$ if $\{f^t(a)\}_{t=1}^{\infty} \subseteq \{f^s(b)\}_{s=1}^{\infty}$. It is said to be proper if $\{f^t(a)\}_{t=1}^{\infty} \subset \{f^s(b)\}_{s=1}^{\infty}$. The Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ is said to be full if $b$ is the generator of the process.

*Remark* 5.2. Next we state a result that indicates that Collatz processes are indistinguishable once they overlap.

**Proposition 5.1.** *Let $\{f^s(a)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty}$ be full Collatz processes. If $\{f^s(a)\}_{s=1}^{\infty} \cap \{f^s(b)\}_{s=1}^{\infty} \neq \emptyset$, then $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$ and $a = b$.*

*Proof.* Let $\{f^s(a)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty}$ be full Collatz processes and suppose $\{f^s(a)\}_{s=1}^{\infty} \cap \{f^s(b)\}_{s=1}^{\infty} \neq \emptyset$. It follows that there exist some $t, m \geq 1$ such that $f^t(a) = f^m(b)$. Thus we obtain the following chains of equality

$$f^{(t+1)}(a) = f^{(m+1)}(b), \quad f^{(t+2)}(a) = f^{(m+2)}(b), \ldots f^{t+j}(a) = f^{(m+j)}(b)$$

for all $t \geq 1$. It follows that $\{f^s(a)\}_{s=t}^{\infty} = \{f^s(b)\}_{s=m}^{\infty}$. It follows that $\{f^s(a)\}_{s=1}^{\infty} \subseteq \{f^s(b)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty} \subseteq \{f^s(a)\}_{s=1}^{\infty}$. This implies that $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$. Since the processes $\{f^s(a)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty}$ are full, It follows by definition 5.1 that $a$ and $b$ are two generators of the process and by appealing to Proposition 2.3, It follows that $a = b$. □

**Proposition 5.2.** *Let $f$ be the Collatz function with the corresponding full process $\{f^s(b)\}_{s=1}^{\infty}$. If the generator is trivial, then each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}$ must be of the form $c_n = 2^n - 1$.*

*Proof.* Let $f$ be the Collatz function and suppose $\{f^s(b)\}_{s=1}^{\infty}$ is a full process. Then it follows that $b$ is the generator of the process and $\{\mathrm{Inf}\{f^s(b)\}\}_{s=1}^{\infty}$ is the backward Collatz process, so that for each $a_n \in \{\mathrm{Inf}\{f^s(b)\}\}_{s=1}^{\infty}$ satisfies the parity condition $a_n \equiv 0 \pmod 2$. Since $\frac{f^{m+1}(b)}{2} = f^{-m}(b)$ for $m \geq 1$, It follows that $a_n = 2^{n-1}f^{-1}(b)$. Since the process is trivial, It follows that $b = 1$ and $f^{-1}(1) = 2$ and the result follows immediately. □

There does appear an important relation between the index and the order of primes generators. In light of this we state the following conjecture

**Conjecture 5.1.** Let $f$ be the Collatz function and $\{f^s(b)\}_{s=1}^{\infty}$ be a full Collatz process. If $\{f^s(b)\}_{s=1}^{\infty}$ converges, then $b$ is prime if and only if

$$\mathrm{Ind}_f(b) = \tau_f(b) + 1.$$

## 6. Distribution of the Collatz process

In this section we study the local and the global distribution of any Collatz process. We focus our study to the existence of primes in any Collatz process under a given generator.

**Proposition 6.1.** *Let $f$ be the Collatz function with the corresponding full process $\{f^s(b)\}_{s=1}^{\infty}$ for $b > 1$. If $f^2(b)$ is prime, then $2f^2(b) - 1$ cannot be prime.*

*Proof.* Let $f$ be the Collatz function with the corresponding full Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Then $b$ is the generator and it follows from Proposition 2.4 that $b \equiv 1 \pmod 2$, so that $f(b) \equiv 0 \pmod 2$. Then it is certainly the case that $f^2(b) = \frac{f(b)}{2} = \frac{3b+1}{2}$. The claim follows from this relation. □

*Remark* 6.1. Next we expose the Theory to the infamous problem concerning the distribution of the Sophie germain primes. It reduces the problem entirely to knowing the existence and the distribution of consecutive primes in the unit left translates of the backward Collatz process.

**Theorem 6.2.** *Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Let $\{f^s(b)\}_{s=1}^{\infty}$ be a full process and $f^{-k}(b), f^{-(k+1)}(b) \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$, the backward Collatz process. If $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1$ are both prime, then $f^{-k}(b) - 1$ must be a Sophie germain prime. Moreover there are infinitely many Sophie germain primes if there are infinitely many consecutive primes in $\{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$.*

*Proof.* Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Since the process $\{f^s(b)\}_{s=1}^{\infty}$ is full, It follows that $b$ must be the generator of the process. It follows from Proposition 2.1 each $f^{-m}(b) \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ must satisfy the parity condition $f^{-m}(b) \equiv 0 \pmod 2$. Then under the Collatz process, we obtain the following increasing sequence

$$\ldots > f^{-(m+2)}(b) > f^{-(m+1)}(b) > f^{-m}(b) > \ldots > f^{-1}(b)$$

with each term satisfying the equality $f^{-(j+1)}(b) = 2f^{-j}(b) \Longleftrightarrow f^{-(j+1)}(b) - 1 = 2(f^{-j}(b) - 1) + 1$. Under the assumption that $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1$ are both prime, then the result follows immediately. If there are infinitely many consecutive primes $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1 \in \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$, then it will follow that there are infinitely many Sophie germain primes. □

Theorem 6.2 relates the Collatz process to the problem of the distribution of Sophie germain primes. Indeed it sets out the idea that studying the problem on the set of integers is in some way silly. Rather it will much more technique convenient to study the unit left translate of the sequence arising from the backward Collatz process.

*Remark* 6.3. It turns out that certain Collatz process which when undergo a unit left translate has most of its elements being prime. This notion is exemplified in the following result.

**Theorem 6.4.** *Let $f$ be a Collatz function, with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If the process is full, then $\mathcal{M} = \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$ contains a prime.*

*Proof.* Let $f$ be the Collatz function with the corresponding full process $\{f^{-s}(b)\}_{s=1}^{\infty}$. Then it follows that $b$ is the generator of the process. It follows that the sequence $\{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ is the backward Collatz process. By Proposition 2.4, each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ must satisfy the parity condition $a_n \equiv 0 \pmod 2$ and additionally that

$$a_n = 2^{n-1} f^{-1}(b) - 1$$

for $n \geq 1$ with $f^{-1}(b) \in \{\mathrm{Inf}\{f^{-s}\}_{s=1}^{\infty}$. It follows that $a_n$ must be prime for some $n \geq 1$. □

**Conjecture 6.1.** Let $f$ be a Collatz function, with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If the process is full, then $\mathcal{M} = \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$ contains infinitely consecutive primes and

$$\lim_{s \longrightarrow \infty} \frac{\#(\mathcal{M} \cap \rho)}{\#\mathcal{M}} = 1$$

where $\rho$ is the set of all primes.

Conjecture 6.1 is equivalent to the problem of deciding on the distribution of the Sophie germain prime, which is still an unsolved problem in the subject and we do not pursue in this paper. The Collatz process with a given generator exhibits other stunning and subtle properties in terms of the primality of the terms. In light of this we make the following conjectures

**Conjecture 6.2.** Let $p > 2$ be prime with the corresponding Collatz process such that $p \equiv 3 \pmod 4$, then there exist $n \in \{f^s(p)\}_{s=1}^{\infty}$ such that $\mu(n) \neq 0$.

**Conjecture 6.3.** If $\{f^s(b)\}_{s=1}^{\infty}$ is a full Collatz process, then there exist an odd prime $p \in \{f^s(b)\}_{s=1}^{\infty}$.

**Conjecture 6.4.** Let $f$ be the Collatz function with the corresponding Collatz process $\{f^s(a)\}_{s=1}^{\infty}$. If the process $\{f^s(a)\}_{s=1}^{\infty}$ is full, then $\Omega(a) \leq 2$.

[1].

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