SURGICAL ANALYSIS OF FUNCTION

T. AGAMA

Abstract. In this paper we introduce the concept of surgery. This concept ensures that almost all discontinuous functions can be made to be continuous without redefining their support. Inspite of this, it preserves the properties of the original function. Consequently we are able to get a handle on the number of points of discontinuities on a finite interval by having an information on the norm of the repaired function and vice-versa.

1. Introduction and motivation

Continuous functions are considered very tractable functions to work with, but once a function fails to be continuous we need to be extra careful in handling such function for effective analysis. We can still engage in some repair process if the discontinuous function has removable discontinuities in the support. Consider the function $G : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$G(x) = \frac{x-1}{x^2-1},$$

a function that has discontinuities at $x = 1, -1$. One of this is a removal discontinuity and the other is an essential discontinuity. For various classifications of the type of discontinuities, see [1]. In this paper, we introduce the concept of surgery, a method of making a discontinuous function continous on it's support by breaking the function into pieces and replacing the discontinous body with a continous body almost identical to the original. It turns out that this analysis preserves most of the properties of the original function. In that direction we have the following result:

Theorem 1.1. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. Let $\mathcal{N} = \{y_1, y_2, \ldots, y_n\}$ be the points of discontinuities of f. If f accepts surgery, then $||\widehat{f}|| < \delta$ if and only if there exist some $k_0 > 0$ such that $|\mathcal{N}| \geq K$ for all $K \geq k_0$ for arbitrary $\delta > 0$.*

Remark 1.2. It is therefore natural to expect that a discontinuous function could be undefined on its support. But in this paper, the use of the terminology discontinuity would mean the function is defined on its support but fails to be continous. That is, we are entirely ruling out functions that experience blowups at some points in their support.

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2. **Surgery on funtions**

Definition 2.1. Let $f : [a,b] \longrightarrow \mathbb{R}$ and let $x_1 < x_2 < \cdots < x_n$ be the points of discontinuities of f, where $[a,b] \subset \mathbb{R}$. Then we say f accepts surgery if there exist some non-constant continous functions $g_i : [x_i - \epsilon, x_i + \epsilon] \longrightarrow \mathbb{R}$ for $1 \leq i \leq n$ such that $f \asymp g_i$ in the support of g_i, where ϵ is arbitrarily small. Then the surgical representation of f, denoted $\widehat{f}$, is given by

$$\widehat{f} = \tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [x_i - \epsilon, x_i + \epsilon]} \prod_{i=1}^{n} g_{[x_i - \epsilon, x_i + \epsilon]},$$

where

$$\tilde{f} : [a,b] \setminus \cup_{i=1}^{n} [x_i - \epsilon, x_i + \epsilon] \longrightarrow \mathbb{R}.$$

$\tilde{f}$ is said to be the structured part of the representation and each g_i forms a component of the delicate part, and where

$$G_{[c,d]}(x) = \begin{cases} G(x) & \textbf{if} \quad x \in [c,d] \\ 1 & \textbf{otherwise}. \end{cases}$$

The conductor of the surgical representation $\widehat{f}$ of f on any given point on their support will depend greatly on their location in $[a,b]$. That is, if $x \in [x_i - \epsilon, x_i + \epsilon]$ for some $1 \leq i \leq n$, then the value $\widehat{f}(x_i) = g_i(x_i)$. However if

$$x \in \bigcup_{i=1}^{n} [a,b] \setminus [x_i - \epsilon, x_i + \epsilon],$$

then $\widehat{f}(x) = \tilde{f}(x)$. In practice the surgical representation of functions takes on values in the following manner

$$\begin{aligned} \widehat{f}(a) &= \left(\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [x_i - \epsilon, x_i + \epsilon]} \prod_{i=1}^{n} g_{[x_i - \epsilon, x_i + \epsilon]} \right)(a) \\ &= \tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [x_i - \epsilon, x_i + \epsilon]}(a) \prod_{i=1}^{n} g_{[x_i - \epsilon, x_i + \epsilon]}(a) \end{aligned}$$

It follows from this analysis that any discontinous function (resp. continous) can be controlled from below and above by its surgical representation as

$$\left(\frac{\widehat{f}(x)}{D} \right)^{\frac{1}{n+1}} \leq f(x) \leq \left(\frac{\widehat{f}(x)}{C} \right)^{\frac{1}{n+1}}$$

for $D := D(n) > 0$ and $C := C(n) > 0$ for $n > 0$ and $C(n) = 1$ if $n = 0$ with $D(n) = 1$ if $n = 0$, where n is the number of points of discontinuities of f.

Remark 2.2. The notion of surgery of a function f can be thought of in practical terms as identifying sufficiently small neighbourhoods of points of discontinuities of f, removing the portion of the graph of f whose support corresponds to this neighbourhood and replacing it with an equivalently nice function on the same support by glueing both ends to the remnants of the closest body of the original function.

Theorem 2.3. *Let $f : [c,d] \longrightarrow \mathbb{R}$ have finite number of points of discontinuities. If f accepts surgery, then the surgical representation $\widehat{f}$ of f is unique up to a constant dilates of $\widehat{f}$.*

Proof. Suppose $f : [c,d] \longrightarrow \mathbb{R}$ accepts surgery, and let

$$\begin{aligned}\widehat{f} &= \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}\prod_{i=1}^{n} g_{[x_i-\epsilon,x_i+\epsilon]}\\ &= \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}\prod_{i=1}^{n} h_{[x_i-\epsilon,x_i+\epsilon]}.\end{aligned}$$

Then it follows that

$$\frac{\prod\limits_{i=1}^{n} g_{[x_i-\epsilon,x_i+\epsilon]}}{\prod\limits_{i=1}^{n} h_{[x_i-\epsilon,x_i+\epsilon]}} = 1.$$

It follows that $g_{[x_i-\epsilon,x_i+\epsilon]} = h_{[x_i-\epsilon,x_i+\epsilon]}$ for each $1 \leq i \leq n$. Suppose $g_{[x_i-\epsilon,x_i+\epsilon]} \neq h_{[x_i-\epsilon,x_i+\epsilon]}$, then it follows that for some $a \in [x_i-\epsilon, x_i+\epsilon]$, we will certainly have

$$\frac{\prod\limits_{i=1}^{n} g_{[x_i-\epsilon,x_i+\epsilon]}}{\prod\limits_{i=1}^{n} h_{[x_i-\epsilon,x_i+\epsilon]}}(a) \neq 1$$

a contradiction, thereby ending the proof. □

Theorem 2.4. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ each having finite points of discontinuities in the support $[a,b] \subset \mathbb{R}$. If f and g accepts surgery, then we have*

$$\widehat{f+g} = \widehat{f} + \widehat{g}.$$

Proof. Let $f, g : [a,b] \longrightarrow \mathbb{R}$ and suppose f and g accepts surgery, then we can write

$$\widehat{f+g} = (f \tilde{+} g)h_1h_2\cdots h_n.$$

By letting $x_1 < x_2 \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ be the points of discontinuities of f and g respectively, we observe that $(f \tilde{+} g)$ is supported on

(2.1)

$$\begin{aligned}&[a,b]\setminus\bigcup_{i,j=1}^{n}\Big([x_i-\epsilon,x_i+\epsilon]\cap\\ &\qquad [y_j-\epsilon,y_j+\epsilon]\Big) = \bigcap_{i,j=1}^{n}[a,b]\setminus\Big([x_i-\epsilon,x_i+\epsilon]\cap[y_j-\epsilon,y_j+\epsilon]\Big)\\ &\qquad\qquad = \bigcap_{i,j=1}^{n}[a,b]\setminus[x_i-\epsilon,x_i+\epsilon]\cup\bigcap_{i,j=1}^{n}[a,b]\setminus[y_j-\epsilon,y_j+\epsilon]\\ &\qquad\qquad = \bigcup_{i,j=1}^{n}\Big([a,b]\setminus[x_i-\epsilon,x_i+\epsilon]\cap[a,b]\setminus[y_i-\epsilon,y_i+\epsilon]\Big).\end{aligned}$$

It follows from these relation that $f \tilde{+} g = \tilde{f} + \tilde{g}$. Thus

$$\begin{aligned}\widehat{f+g} &= (f \tilde{+} g)h_1h_2\cdots h_n\\ &= (\tilde{f}+\tilde{g})h_1h_2\cdots h_n\\ &= \tilde{f}h_1h_2\cdots h_n + \tilde{g}h_1h_2\cdots h_n\\ &= \widehat{f}+\widehat{g}\end{aligned}$$

and the relation follows immediately. □

Remark 2.5. It is very important to notice that the structured part $\tilde{f}$ of the surgical representation of any function with finite points of discontinuities is still the function f but with the support restricted in some sense.

Corollary 2.1. Let $f_i : [a,b] \longrightarrow \mathbb{R}$ for $i = 1, 2 \ldots n$ each having points of discontinuities on the support, where $[a,b] \subset \mathbb{R}$. If each f_i accepts surgery, then

$$\widehat{\sum_{i=1}^{n} f_i} = \sum_{i=1}^{n} \widehat{f_i}.$$

Proof. The result follows by applying Theorem 2.4. □

Remark 2.6. Corollary 2.1 reinforces the notion that we can get controll on the surgical representation of the sum of function by the surgical representation of each individual function.

Proposition 2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ having points of discontinuities on $[a,b] \subset \mathbb{R}$ and let $\lambda \in \mathbb{R}$. If f accepts surgery, then*

$$\widehat{\lambda f} = \lambda \widehat{f}.$$

Proof. Let $x_1 < x_n \ldots < x_n$ be the points of discontinuities of f and suppose f accepts surgery, then it follows by Theorem 2.3

$$\widehat{\lambda f} = (\tilde{\lambda f})h_1h_2\cdots h_n$$

where $(\tilde{\lambda f})$ by Remark 2.5 is the function λf supported on

$$[a,b] \setminus [x_i - \epsilon, x_i + \epsilon].$$

The conductors on this support are dilates of the conductors of f on the support. Thus we can write $(\tilde{\lambda f}) = \lambda \tilde{f}$ and it follows that

$$\begin{aligned}\widehat{\lambda f} &= (\tilde{\lambda f})h_1h_2\cdots h_n\\ &= \lambda\tilde{f}h_1h_2\cdots h_n\\ &= \lambda\widehat{f},\end{aligned}$$

thereby establishing the relation. □

Corollary 2.2. The surgical representation of functions on a given support is linear. In particular, let $\lambda \in \mathbb{R}$ and suppose $f, g : [a,b] \longrightarrow \mathbb{R}$ both accept surgery, where $[a,b] \subset \mathbb{R}$, then

(i) $\widehat{f+g} = \widehat{f} + \widehat{g}$.
(ii) $\widehat{\lambda f} = \lambda \widehat{f}$.

Proof. This is an assemblage of the result in Theorem 2.4 and Proposition 2.1. □

3. **Ordering of surgical representations**

In this section we assign some bit of ordering to the surgical representations of functions. We launch the following terminologies:

Definition 3.1. Let $f : [a, b] \longrightarrow \mathbb{R}$ and $g : [a, b] \longrightarrow \mathbb{R}$ be functions that accepts surgery, where $[a, b] \subset \mathbb{R}$. Then by $\widehat{f} \ll \widehat{g}$, we mean there exist some constant $c_1 > 0$ such that

$$|\widehat{f}(x)| \leq c_1 \widehat{g}(x)$$

for all common points x in their support. Similarly $\widehat{f} \gg \widehat{g}$ if there exist some constant $c_2 > 0$ such that

$$|\widehat{f}(x)| \geq c_2 \widehat{g}(x)$$

for all points x in the common support. In both relation holds for any function $f : [a, b] \longrightarrow \mathbb{R}$ and $g : [a, b] \longrightarrow \mathbb{R}$ that accepts surgery, then we write

$$\widehat{f} \asymp \widehat{g}.$$

Remark 3.2. Next we state and prove a result that relates the interaction between the structured part of two functions that accepts surgery to their surgical representations. It is essentially saying that once the structured part of their surgical representations are identical then the components of the delicate parts must also be identical and, hence the two functions certainly should be identical.

Theorem 3.3. *Let $f : [a, b] \longrightarrow \mathbb{R} \setminus \{0\}$ and $g : [a, b] \longrightarrow \mathbb{R} \setminus \{0\}$ accepts surgery, where $[a, b] \subset \mathbb{R}$. If $\tilde{f} \asymp \tilde{g}$, then $\widehat{f} \asymp \widehat{g}$ and hence $f \asymp g$.*

Proof. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ and $g : [a, b] \longrightarrow \mathbb{R}$ accepts surgery. Let $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ be their points of discontinuities respectively. Then we observe that $(\tilde{fg})$ is supported on

$$[a, b] \setminus \bigcup_{i,j=1}^{n} \bigg([x_i - \epsilon, x_i + \epsilon] \cap [y_j - \epsilon, y_j + \epsilon]\bigg) = \bigcup_{i,j=1}^{n} \bigg([a, b] \setminus [x_i - \epsilon, x_i + \epsilon] \cap [a, b] \setminus$$
$$[y_j - \epsilon, y_j + \epsilon]\bigg).$$

It follows that $(\tilde{fg}) = \tilde{f}\tilde{g}$. By leveraging this relation, we can write

$$\begin{aligned}
\widehat{fg} &= (\tilde{fg}) h_1 h_2 \cdots h_n \\
&= \tilde{f}\tilde{g} h_1 h_2 \cdots h_n \\
&= \tilde{f}(\tilde{g} h_1 h_2 \cdots h_n) \\
&= \tilde{f}\widehat{g} \\
&= \tilde{g}(\tilde{f} h_1 h_2 \cdots h_n) \\
&= \tilde{g}\widehat{f}.
\end{aligned}$$

It follows from this relation $\tilde{f}\widehat{g} = \tilde{g}\widehat{f}$ if and only if

$$\frac{\tilde{f}}{\tilde{g}} = \frac{\widehat{f}}{\widehat{g}}.$$

Under the condition that $\tilde{f} \asymp \tilde{g}$, it follows that $\widehat{f} \asymp \widehat{g}$, and hence $f \asymp g$, and the proof of the theorem is complete. $\square$

It turns out that the converse of this result also holds. To avoid being clumpsy and ensure orderly presentation, we present the converse as a seperate piece.

Theorem 3.4. *Let $f, g : [a, b] \longrightarrow \mathbb{R} \setminus \{0\}$ accepts surgery, where $[a, b] \subset \mathbb{R}$. If $f \asymp g$, then $\widehat{f} \asymp \widehat{g}$.*

Proof. Suppose $f, g : [a, b] \longrightarrow \mathbb{R} \setminus \{0\}$ accepts surgery. Then the relation holds

$$\frac{\tilde{f}}{\tilde{g}} = \frac{\widehat{f}}{\widehat{g}}.$$

If $f \asymp g$, then $\tilde{f} \asymp \tilde{g}$, since $\tilde{f}$ and $\tilde{g}$ are still the body of f and g with their support restricted in some sense but still covered by the support of f and g. It follows therefore from the above relation that $\widehat{f} \asymp \widehat{g}$, and the proof is complete. $\square$

These two results put together offers us some tranference principle that enables us to examine the order of functions. This principle is phenomenally important and it will be explored in the following sequel. We remark at this point that if a function f is continous on $[a, b] \subset \mathbb{R}$, then the structured part of the surgical representation coincides with f. That is

$$\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [x_i - \epsilon, x_i + \epsilon]} = f.$$

Next we prove that the monotonicity property of a function can be transfered to the surgical representation of functions.

Theorem 3.5. *Let $f : [a, b] \longrightarrow \mathbb{R}$, where $[a, b] \subset \mathbb{R}$, and differ on at least two points in $[a, b]$. If f is non-increasing $(resp.\ non-decreasing)$, then the surgical representation is also non-increasing $(resp.\ non-decreasing)$.*

Proof. Let $f : [a, b] \longrightarrow \mathbb{R}$. In the case f is continuous on $[a, b]$, then it follows from the forgone discussion that $\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]} = f$. Since f is non-increasing, it follows that

$$\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}(x_1) \geq \tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}(x_2)$$

with $x_1 < x_2$ for all $x_1, x_2 \in [a, b]$ and it follows that $\widehat{f}(x_1) \geq \widehat{f}(x_2)$ for any $x_1 < x_2$. In the case f fails to be continuous on $[a, b]$, then let $y_1 < y_2 < \cdots < y_n$ be the points of discontinuities of f and suppose f accepts surgery. Then the structured part of the surgical representation $\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}$ is supported on

$$[a, b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon].$$

Since $\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}$ is still the body of f, it follows that

$$\tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}(x_1) \geq \tilde{f}_{[a,b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]}(x_2)$$

for $x_1 < x_2$ for all $x_1, x_2 \in [a, b] \setminus \cup_{i=1}^{n} [y_i - \epsilon, y_i + \epsilon]$. It follows that the structured part is non-increasing. We complete the proof by showing that each component of the delicate part is also non-increasing. Suppose the contrary that the remaining part of the surgical representation is increasing, that is

$$\prod_{i=1}^{n} h_{[y_i-\epsilon,y_i+\epsilon]}(x_1) < \prod_{i=1}^{n} h_{[y_i-\epsilon,y_i+\epsilon]}(x_2)$$

with $x_1 < x_2$ for all $x_1, x_2 \in [y_i - \epsilon, y_i + \epsilon]$ for some $i = 1, 2, \dots n$. Then it follows that each component of the delicate part is increasing; that is, $h_{[y_i-\epsilon,y_i+\epsilon]}(x_1) < h_{[y_i-\epsilon,y_i+\epsilon]}(x_2)$ for $i = 1, 2 \dots n$. Since each $h_{[y_i-\epsilon,y_i+\epsilon]} \asymp f$ in its support, it follows that

$$h_{[y_i-\epsilon,y_i+\epsilon]}(x) \leq Kf(x)$$

for some contant $K > 0$ in the support $[y_i - \epsilon, y_i + \epsilon]$. It follows that for $x_1 < x_2 < \cdots < x_n$

$$h_{[y_i-\epsilon,y_i+\epsilon]}(x_1) \leq Kf(x_1) \leq h_{[y_i-\epsilon,y_i+\epsilon]}(x_2) \leq Kf(x_2) \leq \cdots \leq h_{[y_i-\epsilon,y_i+\epsilon]}(x_n) \leq Kf(x_n).$$

It follows from this relation

$$f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$$

for $x_1 < x_2 < \cdots < x_n$. Since f is non-increasing, we must have

$$f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n)$$

and it follows that $f(x_1) = f(x_2) = \cdots = f(x_n)$. This is a contradiction since f differs on at least two points. It follows therefore that each component of the delicate part of the surgical representation of f must also be non-decreasing, thereby proving that

$$\widehat{f}(x_1) \geq \widehat{f}(x_2)$$

for $x_1 < x_2$. □

4. **The norm of a surgical representation (weak norm of f)**

In this section we introduce the notion of the norm of the surgical representation of a function. This norm is not realistically the actuall norm of the function. Thus we have coined such a norm a weak norm of f.

Definition 4.1. Let $f : [a, b] \longrightarrow \mathbb{R}$ having points of discontinuities and let f accepts surgery. Then we set

$$||\widehat{f}|| = \sup\{|\tilde{f}(x)| : x \in [a, b] \setminus [x_i - \epsilon, x_i + \epsilon]\}.$$

Since the surgical representation $\widehat{f}$ of a function is a continuous body of f, it follows that the structured part and the components of the delicate part are now continous on $[a, b]$. It follows that they must be bounded. It is also very important to notice that $|| \cdot ||$ on surgical representations of functions is a norm. The following result makes this statement a bit formal.

Proposition 4.1. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ and $g : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ accepts surgery. Let $\lambda \in \mathbb{R}^+$, then the following holds:*

- (i) $||\widehat{f}|| \geq 0$. (*Positivity*)
- (ii) $||\lambda\widehat{f}|| = |\lambda|||\widehat{f}||$. (*Homogeneity*)
- (iii) $||\widehat{f} + \widehat{g}|| \leq ||\widehat{f}|| + ||\widehat{g}||$. (*Triangle inequality*)

Proof.
(i) The fact that $||\widehat{f}|| \geq 0$ follows by definition 4.1.

(ii) Let $\lambda \in \mathbb{R}^{+}$. Then by Corollary 2.2, we can write

$$\begin{aligned}||\lambda\widehat{f}|| &= ||\widehat{\lambda f}||\\ &= \sup\{|(\tilde{\lambda f})(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= \sup\{|\lambda\tilde{f}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= |\lambda|\sup\{\tilde{f}(x) : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= \lambda||\widehat{f}||.\end{aligned}$$

(iii) By Corollary 2.2, we can write

$$\begin{aligned}||\widehat{f} + \widehat{g}|| &= ||\widehat{f+g}||\\ &= \sup\{|(f \tilde{+} g)(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= \sup\{|(\tilde{f} + \tilde{g})(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= \sup\{|\tilde{f}(x) + \tilde{g}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &\leq \sup\{|\tilde{f}(x)| + |\tilde{g}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &\leq \sup\{|\tilde{f}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &+ \sup\{|\tilde{g}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon]\}\\ &= ||\widehat{f}|| + ||\widehat{g}||,\end{aligned}$$

and the triangle inequality follows immediately. □

The Cauchy-swartz inequality is an extremely usefull inequality in the whole of mathematics. In the following sequel, we extend this inequality to the surgical representation of functions. But before then, we examine the following preparatory lemmas.

Lemma 4.2. *Let $\{A_n\}$ and $\{B_n\}$ be any sequence of real numbers. Then the following holds*

(i) $\sup(|A_n||B_n|) = \sup(|A_n|)\sup(|B_n|)$.
(ii) $\sup(|A_n| + |B_n|) = \sup(|A_n|) + \sup(|B_n|)$.
(iii) $\sup(\sqrt{|A_n| + |B_n|}) = \sqrt{\sup(|A_n|) + \sup(|B_n|)}$.

Proof. For a proof see for instance [1], [2]. □

Proposition 4.2. *Let $\{f_n\}$ and $\{g_n\}$ be sequence of functions that accepts surgery, then*

$$||\sum_{n=1}^{s}\widehat{f_n}\widehat{g_n}|| \leq \left(\sum_{n=1}^{s}||\widehat{f_n}||^2\right)^{1/2}\left(\sum_{n=1}^{s}||\widehat{g_n}||^2\right)^{1/2}.$$

Proof. Suppose $\{f_n\}$ and $\{g_n\}$ be sequence of functions that accepts surgery. Then applying Lemma 4.2 and using Proposition 4.1, it follows that

$$\begin{aligned}
||\sum_{n=1}^{s}\widehat{f_n g_n}|| &\leq \sum_{n=1}^{s}||\widehat{f_n g_n}|| \\
&= \sum_{n=1}^{s}||\widehat{f_n}||||\widehat{g_n}|| \\
&= \sum_{n=1}^{s}\sup(|\tilde{f}_n(x)|)\sup(|\tilde{g_n}(x)|) \\
&= \sum_{n=1}^{s}\sup(|\tilde{f}_n(x)||\tilde{g_n}(x)|) \\
&= \sup\bigg(\sum_{n=1}^{s}|\tilde{f}_n(x)||\tilde{g_n}(x)|\bigg) \\
&\leq \sup\bigg(\bigg(\sum_{n=1}^{s}|\tilde{f}_n|^2\bigg)^{1/2}\bigg)\bigg(\sum_{n=1}^{s}|\tilde{g_n}(x)|^2\bigg)^{1/2}\bigg). \\
&= \sup\bigg(\sum_{n=1}^{s}|\tilde{f}_n(x)|^2\bigg)^{1/2}\sup\bigg(\sum_{n=1}^{s}|\tilde{g_n}(x)|^2\bigg)^{1/2} \\
&= \bigg(\sum_{n=1}^{s}\sup(|\tilde{f}_n(x)|^2)\bigg)^{1/2}\bigg(\sum_{n=1}^{s}\sup(|\tilde{g_n}(x)|^2)\bigg)^{1/2} \\
&= \bigg(\sum_{n=1}^{s}||\widehat{f_n}||^2\bigg)^{1/2}\bigg(\sum_{n=1}^{s}||\widehat{g_n}||^2\bigg)^{1/2}
\end{aligned}$$

thereby establishing the relation. □

It is very important to notice that the norm of the surgical representation of any function that accepts surgery depends greatly on the number of points of discontinuities in the support. This is because as we increase the number of points for which f is not well-behaved, essentially points of discontinuities of f, the more we shrink the support of the structured part $\tilde{f}$ and the likelyhood of removing the point for which f majorizes all other conductors of f and so the norm might be very small. The opposite also happens. If a function have very few points of discontinuities, then the support of the structured part may include the point for which f is maximum and hence we might get a somewhat large norm for the surgical representation of f. On the basis of this discussion, we make the following formalism:

Theorem 4.3. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}^+$. Let $\mathcal{N} = \{y_1, y_2, \ldots, y_n\}$ be the points of discontinuities of f. If f accepts surgery, then $||\widehat{f}|| < \delta$ if and only if there exist some $k_0 > 0$ such that $|\mathcal{N}| \geq K$ for all $K \geq k_0$ for arbitrary $\delta > 0$.*

Proof. Let $f:[a,b]\subset \mathbb{R}\longrightarrow \mathbb{R}$ and suppose f accepts surgery. Let us consider the inequality

$$\bigg(\frac{\widehat{f}(x)}{D}\bigg)^{\frac{1}{n+1}}\leq f(x)\leq \bigg(\frac{\widehat{f}(x)}{C}\bigg)^{\frac{1}{n+1}}$$

for $D:=D(n)>0$ and $C:=C(n)>0$ for $n>0$ and $C(n)=1$ if $n=0$ with $D(n)=1$ if $n=0$, where n is the number of points of discontinuities of f. Then it follows that

$$\sup\{|\tilde{f}_{[a,b]\setminus [x_i-\epsilon,x_i+\epsilon]}(x)|\}\leq |f(x)|\leq \bigg(\frac{|\widehat{f}(x)|}{C}\bigg)^{\frac{1}{n+1}}.$$

It follows that for sufficiently large values of n, then $||\widehat{f}||<\delta$. Conversely suppose $||\widehat{f}||<\delta$, then it follows that

$$\bigg(\frac{\widehat{f}(x)}{D}\bigg)^{\frac{1}{n+1}}\leq f(x)\ll \tilde{f}_{[a,b]\setminus [x_i-\epsilon,x_i+\epsilon]}(x).$$

Thus we have

$$\frac{\log\bigg(\frac{\min\{\widehat{f}(x)\}}{D}\bigg)}{\log \delta}\ll n.$$

The result follows by taking

$$k_0=\frac{\log\bigg(\frac{\min\{\widehat{f}(x)\}}{D}\bigg)}{\log \delta}\ll n.$$

□

Remark 4.4. Theorem 4.3 could be viewed as an inverse theorem. It tells us that the norm of surgical representation of functions can be made arbitrarily small by increasing the points of discontinuities in their support. In a similar vein, the very notion of small norms of the surgical representation of a function suggests that the function has finitely many points of discontinuities in their support.

5. **The zeros of surgical representations**

In this section we examine the notion of the zeros of a surgical representation. Recall that the zeros of any given function are the points in the support for which the function vanishes. We prove that we can transfer the notion of the zeros of any function discontinous on it's support to the zeros of the surgical representations.

Theorem 5.1. *Let $f:[a,b]\subset \mathbb{R}\longrightarrow \mathbb{R}$ and let f accepts surgery. Then $f(x_1)=0$ for $x_1\in [a,b]$ if and only if $\widehat{f}(x_1)=0$.*

Proof. Let $f:[a,b]\subset \mathbb{R}\longrightarrow \mathbb{R}$ and suppose f accepts surgery. Then we can write

$$\widehat{f}=\tilde{f}_{[a,b]\setminus \cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}\prod_{i=1}^{n}h_{[x_i-\epsilon,x_i+\epsilon]}.$$

Then we have that $\widehat{f}(x) = \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}\prod_{i=1}^{n} h_{[x_i-\epsilon,x_i+\epsilon]}(x)$ for any $x \in [a,b]$. In the case x is in the support of the structured part of the representation, then it follows by definition 2.1 that

$$\widehat{f}(x) = \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}(x).$$

Since $\tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}$ is a continous body of f, it follows that

$$\tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}(x) = 0.$$

In the case x lies in the support of the delicate part, then it must be in the support of some component $h_{[x_i-\epsilon,x_i+\epsilon]}$ for some fixed $1 \leq i \leq n$. In that case, we claim that $h_{[x_i-\epsilon,x_i+\epsilon]}(x) = 0$. By definition 2.1, we have

$$h_{[x_i-\epsilon,x_i+\epsilon]}(x) \leq Kf(x) = 0$$

for $K > 0$. Similarly, we have that $h_{[x_i-\epsilon,x_i+\epsilon]}(x) \geq Af(x) = 0$ for some $A > 0$, and it follows that $|h_{[x_i-\epsilon,x_i+\epsilon]}(x)| = 0$. Thus $h_{[x_i-\epsilon,x_i+\epsilon]}(x) = 0$, and the first part of the argument is complete. Conversely, suppose $\widehat{f}(x) = 0$ for some $x \in [a,b]$. Then since the surgical representation exist, it follows that f accepts surgery and that

$$\widehat{f} = \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}\prod_{i=1}^{n} h_{[x_i-\epsilon,x_i+\epsilon]}.$$

If x lies in the support of $\tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}$, then by definition 2.1

$$\widehat{f}(x) = \tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}(x) = 0.$$

Since $\tilde{f}_{[a,b]\setminus\cup_{i=1}^{n}[x_i-\epsilon,x_i+\epsilon]}$ is still the body of f, it follows that $f(x) = 0$. Otherwise, x must lie in the support of some component of the delicate part. That is, $h_{[x_i-\epsilon,x_i+\epsilon]}(x) = 0$ for some $1 \leq i \leq n$. Using the fact that $h_{[x_i-\epsilon,x_i+\epsilon]} \asymp f$ on it's support, it follows that $f(x) = 0$. This completes the proof. $\square$

Corollary 5.1. Let $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$. If there exist some $x_0 \in [a,b]$ such that $f(x_0) = 0$, then $\frac{1}{\widehat{f}(x)}$ is discontinous on $[a,b]$.

Proof. The result follows from Theorem 5.1. $\square$

Remark 5.2. It is important to notice that the converse of Corollary 5.1 also holds on continous functions.

6. The space $\mathcal{SA}_{[a,b]}$ of functions

In this section we introduce the space of discontinuous functions that accepts surgery; in other words, the space of functions whose surgical representation exists. We launch the following terminology in that regard.

Definition 6.1. Let $\mathcal{D}_{[a,b]}$ denote the space of functions supported on $[a,b]$ and having conductors on the reals $\mathbb{R}$. Then we set

$$\mathcal{SA}_{[a,b]} := \left\{ f \in \mathcal{D}_{[a,b]} : \widehat{f} \quad \textbf{exist} \right\}.$$

Proposition 6.1. *The set $\mathcal{SA}_{[a,b]}$ is a subspace of the function space $\mathcal{D}_{[a,b]}$.*

Proof. It suffices to verify the axioms of a set to be a subspace in this setting. We focus more on discontinuous functions on the support $[a,b]$ since if a function is continuous, then by neccessity it follows from the inequality in the foregone discussion the surgical representation certainly exists and coincides with f, that is $f = \widehat{f}$. Since the space of continuous functions is a vector space the result holds. Let $\lambda \in \mathbb{R}$ and $f \in \mathcal{SA}_{[a,b]}$, then it follows that $\widehat{f}$ exist. Thus by linearity it follows that $\widehat{\lambda f} = \lambda \widehat{f}$ exist, and it follows that $\lambda f \in \mathcal{SA}_{[a,b]}$. Again pick $f, g \in \mathcal{SA}_{[a,b]}$, then it follows that $\widehat{f}$ and $\widehat{g}$ both exists. Thus by linearity we have $\widehat{f+g} = \widehat{f} + \widehat{g}$ and it follows that $\widehat{f+g}$ also exists. Thus $f + g \in \mathcal{SA}_{[a,b]}$. This completes the claim that $\mathcal{SA}_{[a,b]}$ is a subspace of the function space with elements supported on $[a,b]$. □

Remark 6.2. Next we prove a result about the convergence of any function in the space $\mathcal{SA}_{[a,b]}$ in the weak norm.

Theorem 6.3. *Every sequence in the space $\mathcal{SA}_{[a,b]}$ converges in the weak norm.*

Proof. Let $f_n \in \mathcal{SA}_{[a,b]}$ be a sequence. Then we have

$$\begin{aligned}
||\widehat{f}_n - \widehat{f}|| &= ||\widehat{f_n - f}|| \\
&= \sup\left\{ |(f_n \tilde{-} f)(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon] \right\} \\
&= \sup\{ |\tilde{f}_n(x) - \tilde{f}(x)| : x \in [a,b] \setminus [x_i - \epsilon, x_i + \epsilon] \} \\
&< \delta
\end{aligned}$$

for $\delta > 0$ and $\delta > 0$ can be made arbitrarily small by taking n sufficiently large, since $\tilde{f}_n$ is continuous on its support. This proves the claim. □

[1].

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Department of Mathematics, African Institute for Mathematical science, Ghana
E-mail address: `theophilus@aims.edu.gh/emperordagama@yahoo.com`

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