FROM PERIODS TO ANABELIAN GEOMETRY AND QUANTUM AMPLITUDES

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ABSTRACT. To better understand and investigate Kontsevich-Zagier conjecture on abstract periods, we consider the case of algebraic Riemann Surfaces representable by Belyi maps.

The category of morphisms of Belyi ramified maps and Dessins D'Enfant, will be investigated in search of an analog for periods, of the Ramification Theory for decomposition of primes in field extensions, controlled by theirs respective algebraic Galois groups.

This suggests a relation between the theory of (cohomological, Betti-de Rham) periods and Grothendieck's Anabelian Geometry [1] (homotopical/ local systems), towards perhaps an algebraic analog of Hurwitz Theorem, relating the the algebraic de Rham cohomology and algebraic fundamental group, both pioneered by A. Grothendieck.

There seem to be good prospects of better understanding the role of absolute Galois group in the physics context of scattering amplitudes and Multiple Zeta Values, with their incarnation as Chen integrals on moduli spaces, as studied by Francis Brown, since the latter are a homotopical analog of de Rham Theory.

The research will be placed in the larger context of the ADE-correspondence, since, for example, orbifolds of finite groups of rotations have crepant resolutions relevant in String Theory, while via Cartan-Killing Theory and exceptional Lie algebras, they relate to TOEs.

Relations with the author's reformulation of cohomology of cyclic groups as a discrete analog of de Rham cohomology [2] and the *arithmetic Galois Theory* [3] will provide a purely algebraic toy-model of the said algebraic homology/homotopy group theory of Grothendieck as part of *Anabelian Geometry*. It will allow an elementary investigation of the main concepts defining periods and algebraic fundamental group, together with their conceptual relation to algebraic numbers and Galois groups.

The Riemann surfaces with Platonic tessellations, especially the Hurwitz surfaces, are related to the finite Hopf sub-bundles with symmetries the "exceptional" Galois groups $PSL_2(F_p), p = 5, 7, 11$.

The corresponding *Platonic Trinity* $5, 7, 11/TOI/E678$ leads to connections with ADE-correspondence, and beyond, e.g. TOEs and ADEX-Theory [4].

Quantizing "everything" (cyclotomic quantum phase and finite Platonic-Hurwitz geometry of qubits/baryons) could perhaps be "The Eightfold (Petrie polygon) Way" to finally understand what quark flavors and fermion generations really are.

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1. Introduction

The proposed research[1] is a part of, and contributes to the long-term goal of understanding the relation between *periods*, as coefficients of the period isomorphism [5], and *scattering amplitudes* [6, 7, 8, 9, 10], as stated in a previous proposal [11], followed by research prior and during the previous visit to IHES [12, 13], as well as during the follow up investigations [14, 15]:

Galois Theory	*Scattering Amplitudes*	*Dessins D'Enfants*
Abelian	Periods and Motives	*Anabelian*
$Betti - deRham/Hodge$	$\ldots$"$HurwitzTheorem$"$\ldots$	$Chen\ It.\ Int./KZ\ eq.$

The ideas and specific problems to be studied are presented below.

The main goal is to conduct a study of Riemann surfaces over $\bar{Q}$, that admit a Belyi map, and to understand the relations between abstract periods and motives via *Dessins D'Enfants* §2.2.2.

Further speculations regarding important problems in Physics are included for future reference.

This "a priori" research plan is flexible, and the received guidance during an upcoming visit will take precedence.

Since the research proposal grew considerably, additional help will be requested via a proposal for a Summer School at IHES centered on "Periods, Motives and Applications to Physics", to be submitted shortly.

2. On Kontsevich Conjecture Regarding Abstract Periods

The author intends to investigate **Kontsevich conjecture on abstract periods**, and its relation to Grothendieck's Conjecture [16], by considering the special case of *Riemann Surfaces representable by Belyi maps* [17], which started Grothendieck on his long march on Galois Theory [18, 19].

It is remarkable its deep relation to Galois Theory [20], hence with algebraic numbers, as "classical periods", as well as a plethora of other "essential ingredients" (Dessins d'Enfants, cartographic maps, matrix integrals, moduli spaces, Kontsevich model, chord diagrams etc., conform Lando and Zvonkin book [21]).

[1]Publishing this GP is in the spirit of modern open research methodology gaining popularity for its benefits to society. A distributed research, like the Great Mersenne Search has yielded notable results. It is also in the spirit of Klein's belief that it is " " ... justifiable to publish connective observations of this kind ...", whenever the subject "... which is after all one in substance, has been only too much broken ..." [43], p.216. This not a rely race, but may the reader take the baton further on.

2.1. **Trends and State-of-the-Art of Research on Periods.** An update (for the author) of the research on periods was in order. Some points to be investigated emerge.

2.1.1. *Period identities, Hilbert's 3rd problem and Dehn Invariants.* In the concrete direction for studying preiods, the work of Juan Viu-Sos for example [23], reduces periods $\int_D \omega$ to geometric volumes $\int_{Semi-Alg} Vol$ of semi-algebraic sets (see also [24], in an analysis in the sense of *measure theory*, to better understand the limit process from finite-additive to σ-additive. In some sense this is going "back-in-time" to Lebesque and Borel, but notably under the guidance of Hilbert (3rd Problem).

It is notable, and worth investigate how Dhen surgery and invariants enter the picture. The reason is, that Dehn surgery provide (perhaps) an alternative description of how to build / glue a manifold, capturing Betti homology in a homotopical way. Does this refine the *homological* period isomorphism?

Additional references to be investigated are listed in [24], notably (for the author), presenting work by Waldshmidt and Yushinava.

The difficulty of this line of investigation lies on the "forgetting structure" when going from a categorical point of view to a "Cauchy/numebrical-methods/algorithmic" approach for the real numbers: $\mathbf{R}/\bar{Q}$ is a difficult to understand object, conform with Grothendieck's Conjecture ([16], p.2):

$$Transcendental\ Degree[Per(X)/Q] = Dim(Motivic\ Galois\ group).$$

2.1.2. *... and Grothendieck's Conjecture.* In the conceptual direction, Grothendieck's initial work on periods, he introduced (first?) *algebraic de Rham cohomology* to "organise" periods into an algebraic structure, and then the work evolved into *Anabelian Geometry*, around the *algebraic fundamental group*, and a general "Galois Theory":

$$Motivic\ Galois\ Groups \leftrightarrow Algebraic\ Fundamental\ Groups?$$

with Chen Iterated Integrals as a homotopic version of de Rham Theory, and hince the well-known connections with MZVs, Feynman Integrals etc. (but not so well understood).

2.1.3. *Why Belyi ramified covers.* It is natural to look at *Belyi ramified covers of the Rieman sphere with a Mobius homological mark-up* ($SL_2(C)$ / "conformal group base point"): this (Belyi's Theorem) *was* Grothendieck "turning moment" (letter to Faltings).

Ajub's geometric conjecture (loc. cit. §5, Th. 40, p.7) seems to compare the algebraic fundamental group of a field extension and the relative motivic group. What the author takes from this, is the relevance of the homologic/homotopic algebraic de Rham/ Chen framework, probably subject to a "Hurewicz Theorem" ...

2.2. **Is there a Ramification Theory of Periods?** In view of the tight connections bewteen algebraic numbers and periods, it is worth strentening the analogy: "Is there a *Ramification Theory for Periods*?".

2.2.1. *Periods: "Numbers" or "Functions"?* Note that "numbers", like the algebraic i, have *geometric interpretations* (a la Klein geometry), e.g. rotation by $\frac{1}{4} \cdot 2\pi$, the fundamental period 2π; i.e. the representation point of view is more lucrative, hence Galois Theory, complementing Archimedes' Cauchy-like approximation of π.

On this dichotomy (numbers vs. functions), see also [25].

2.2.2. *The Category of Ramified Belyi Covers.* Consider Belyi maps for Riemann surfaces defined over the rationals, in analogy to covering maps and their deck transformations, or field extensions and Galois groups.

A morphism of Belyi maps is a morphism of Riemann surfaces compatible with the corresponding ramified covers defined by the Belyi maps B_i:

$$\begin{array}{ccc} (X_1, D_1) & \xrightarrow{f} & (X_2, D_2) \\ \downarrow{\scriptstyle B_2} & & \downarrow{\scriptstyle B_1} \\ (CP^1, D) & \xrightarrow{Mobius\ Tr} & (CP^1, D) \end{array} \qquad \begin{array}{c} \Gamma \to PSL_2(Z) \\ \downarrow \\ Aut(CP^1) = PSL_2(C) \end{array}$$

Here $D = \{0, 1\}$ is the standard divisor on the Riemann sphere $S^2 = CP^1$, with standard chain, the cut $\gamma = [0, 1]$. Associated to these, on the Riemann sphere side, we have the standard period 2π and logarithm $\log(z)$.

On the "other side" of the ramified cover X, we have the corresponding Dessins D'Enfant $\gamma_i = B_i^{-1}(\gamma), i = 1, 2$, with their boundaries, the divisors $D_i = \partial\gamma_i, i = 1, 2$ [21].

The study of how the Dessins D'Enfant determines (relates to) the periods of Riemann surfaces, would be a starting point. The morphisms (via Hurwitz Theorem) are an analog of covering maps / field extensions, and could lead to a *Galois Theory for Belyi Ramified Covers*, Jacobian varieties, period isomorphisms for X_i:

$$Dessins\ D'Enfant \quad \& \quad Jacobian\ Periods?$$

Remark 2.1. More generally, one can consider a category of epi's / mono's, the torsor of its associated subgroupoid, and a paitr of adjoint functors, playing the role of a Galois Connection, in order to derive the "absolute theory" at Categorical Theory level, as a "tool-box". It may lead to connections between motives and the theory of generalized cohomology theories (P. Hilton [26]), via triples and spectra.

2.2.3. *The relation with KZ-moves.* Linearity and Stokes Theorem are captured by considering the period isomorphism. The "change of variables" (diffeo/biholomorphic) is built in the formalism of differential forms.

Hence, it seems that the essential part of the KZ-moves, modulo the torsor structure due to equivalence via isomorphisms, is the way ramification ramifies under a ramified cover.

For covering maps this would correspond to the lattice structure of the fundamental group of the base space, via its universal covering map:

$$\begin{array}{ccccc} \textit{Deck transformations} & & (Paths(X;pt),\pi_1(X)) & \longrightarrow & (Y_1, Deck(Y_1/X)) \\ \textit{and} & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_2} \\ \textit{Covering Maps} & & (X,pt) & \longrightarrow & (X,pt) \end{array}$$

On the other hand a differential form, e.g. a 1-form in our case, defines a monodromy, and therefore a ramified cover via path integral lifting. How all these relate to periods remains to be seen ...

2.2.4. *Prime Decomposition and Ramification of Dessins D'Enfant?* The ramification process (and theory) should go parallel to the ramification of primes under field extensions:

$$Alg.\ N.T.: e,f,g\ I \to D \to Gal(F_{p^f}/F_p) \quad \overset{?}{\leftrightarrow} \quad Mor: (X, Belyi_1) \to (X_2, Belyi_2)$$

Remark 2.2. It is about *Spec* of $Z - Mod$ and $Spec(Z)$ (primes) ...

The decomposition of primes is controlled by the structure of the Galois group (e.g. Abelian case/Cyclotomic $K = Q(\zeta_n)$: $Gal \cong Z/n^\times$ and orbit decomposition of the "space" Z/n; ramification: multiplier by p dividing n, i.e. quotients/ resonance / substructure).

Note that restricting to 3-ramified points (Belyi maps), restricts the degrees of freedom of the lifting process ("flat connections"/local systems case/ representations of the fundamental group).

2.2.5. *Rigidity vs. Continuous parameters.* Specifically, consider the periods represented by quadruples (X, D, ω, γ) defined on Riemann surfaces having an algebraic numbers model over $\bar{Q}$, with a Belyi map [21], p.79:

$$Belyi\ Map: \quad f: X \hookrightarrow CP^1 \quad (3 - points\ ramified\ covers).$$

This case exhibits a *rigidity* (lack of a continuous parameter) [21], p.76), which hints towards a connection with braids representations, KZ-equations, MZVs etc. [22], essential concepts from the "Number Theory Side" of Feynman Integrals as amplitudes.

The case of higher number of ramification points probably corresponds to families of periods indexed by parameters.

2.2.6. *Homological vs. Homotopical.* An investigation of the Conjecture will start from understanding the relation between these periods and the "discrete DATA" (Dessins D'Enfants), as a (homotopical?) analog of the Hodge structure characterizing the Betti-de Rham *homological* period isomorphism.

The pertaining goal is to identify a more tangible (combinatorial) structure that corresponds to periods, invariant under a different kind of "moves" (e.g. Pachner moves, Rademeister moves, chord diagrams relations etc.) and allow for a correspondence with the 3-moves of Kontsevich's Conjecture.

Byproduct of the study would be a better understanding of the relation between "homological" and "homotopical" periods, as intuitively corresponding to the "Abelian vs. Anabelian" case.

Indeed, Galois groups controlling algebraic numbers, as a special case of algebraic fundamental groups (homotopy), are special cases of periods, controlled by the algebraic de Rham cohomology. But these two algebraic theories should be related by an analog of *Hurewicz Theorem*, as intuitively "hoped" by the present author in the IHES talks [13].

2.2.7. *A Study of Periods of Elliptic Curves.* A specific study which could yield results [2], and a better understanding of the basic concepts and relations between them is that of elliptic curves. In this case the periods are related by the *Legendre relation*:

$$det[\int_{\gamma_i} \omega_j]_{i,j=1..2} = 2\pi i.$$

References to be consulted during the upcoming research: [27, 29, 28, 30].

2.2.8. *... and beyond: Hurwitz Surfaces.* The case of Hurwitz Surface $X \to S^2$, maximizing the automorphism group is even more interesting to study, as it corresponds to Belyi maps with 3-ramification points of orders $2, 3$ and 7 [31], and has applications to *Finite String Theory*: modeling baryons as finite Hopf fibrations (finite qubits), with Platonic tesselations (to be explained later on).

The "higher genus Platonic solids":

$$Genus\ g\ Platonic\ Riemann\ Surfaces: \quad M(g;n) = \mathcal{H}_+/\Gamma,\ \Gamma\ (2,k,l)-triangle\ group,$$

some of which attain their Hurwitz bound $84(g-1) = 2 \cdot |\xi(M)||S_4|$, are good candidates for symmetry of fundamental stringy states, provide a mathematical framework where "(Finite) String Theory meets the Standard Model on the Quantum Computing ground".

More specifically, the notable cases of automorphism groups $PSL_2(F_p), p = 5, 7, 11$ (Dodecahedron $g = 0$, Klein quintic $g = 3$ etc.) is important in view of crepant

[2] Conform Polya's advise [42]: "If you can't solve a problem, there is an easier one you can't solve; find it!".

resolutions of orbifolds C^2/Γ, for finite subgroups of rotations Γ (Mckay/ADE correspondence etc.).

2.2.9. *Periods of the Klein quartic and Belyi, Galois, Gauss etc.* Klein's quartic is a very good example to see how the geometry (group theory aspects) relates to algebra (Galois action) and the analytic (Jacobian) [33].

The maps from Klein's quartic to tori (Hodge cycles as factors of the Jacobian?):

$$\begin{array}{ccccc} (Klein, D_1)_{|F_7} & \xrightarrow{\psi_i} & (Torus, D_2)_{|F_7} & & \textit{Divisors vs. groups cosets} \\ \downarrow{\scriptstyle Bel_1} & & \downarrow{\scriptstyle Bel_2} & & \\ S^2_{|O=S_4} & \longrightarrow & S^2_{|O=S_4} & & \textit{Hypermaps \& Dessins D'Enfant} \end{array}$$

will be related to Belyi maps presentations (loc. cit. §4.2) in terms of edges of tesselations or cuts with their associated divisors, providing a better understanding of the *geometric aspects*, coming from the Platonic tesselations, and the analytic aspects (1-forms, Betti bases and periods). [3]

Other good examples, include two genus 2 RS with Platonic tesselations (cyclotomic over $Q(\zeta_5)$). For example, the basis for the lattice of periods computed loc. cit. p.44:

$$1 + \zeta^2 - \zeta^3 - \zeta^4, \quad -1 + \zeta^3 + \zeta^4 - \zeta^5,$$

should be related to Gauss / Kummer periods (see Klein's map).

The case of Fermat surfaces $x^k + y^k + z^k$, also Platonic, invite to a study the connections with Weyl Conjectures, zeros and the corresponding Riemann Hypothesis (now a Theorem).

The *pair of pants decomposition* of Riemann surfaces [33] §5, could be related to the "Ramification Theory", alluded to earlier, when related to Belyi maps. It is reminiscent of coproducts, as for instance in *Frobenius algebras* characterizing 2D-TQFTs, and of CFTs.

2.3. **Klein Quartic, String Theory and Elementary Particles.** These Riemann-Platonic surfaces, with tesselations reminiscent of lattices playing a crucial role in Algebraic Number Theory (lattice models of finite fields), and Algebraic Geometry (Hodge theory: Betti vs. de Rham structures), not clear how related to Belyi (hyper)maps, have symmetry groups which are related to the *finite Hopf sub-bundles*, derived from $S^1 = U(1)$ circle or $S^3 = SU(2)$ monopole bundles.

[3] This particular article of Karcher and Weber seems to be a perfect source of info relevant for the present research goals of the PI.

2.3.1. *Applications to the Standard Model.* Taking the Klein quartic as an example of a Platonic-Riemann Surface that is Hurwitz, defined by the fundamental invariant polynomial $\Phi = xy^3 + yz^3 + zx^3 = 0$ (usually denoted by Φ_4[32]):

$$\begin{array}{ccc} Z/7 \to M(3;7) & U(1) \to S^3 & Finite\ G \to SO(3)\ \&\ SU(2) \\ \downarrow{\scriptstyle Bel_1} & \downarrow{\scriptstyle Hopf} & \downarrow{\scriptstyle Orbifolds\ etc} \\ Dodecahedron \to S^2 & S^2 \quad \cong SO(3)/SO(2) & Platonic\ orbits\ \&\ TOIs. \end{array}$$

opens the "right" way to understand elementary Particles and their mass:

$$The\ Eightfold\ (Petrie)\ Way!$$

with $SU(3)$ playing the role of a Galois group, and quiver representations refining Gell-Man's 8-fold way via Gabriel's Theorem [34, 35].

3. From Periods to Anabelian Geometry

This suggests a relation between the theory of (homological) periods and Grothendieck's Anabelian Geometry, towards perhaps of an algebraic analog of Hurwitz Theorem, relating the the algebraic de Rham cohomology and algebraic fundamental group, both pioneered by A. Grothendieck in *Esquisse d'un program* [1].

Indeed, as early as during the previous visit, the author had a strong intuition of a "Hurwitz Theorem" larger framework surrounding the theory of abstract periods, motives and Galois Theory, as presented in his talk [13]:

$$\begin{array}{cc} From\ Feynman\ Diagrams... & to\ Line\ Integral : (\gamma, \omega) \to \int_\gamma \omega \\ Chen\ Iterated\ Int. \to de\ Rham\ Period\ Iso & \pi(X;p)/NC \to H^1(X) \end{array}$$

The investigation will be along the following lines: think of γ as a cobordism, ω as a propagator and $\int_\gamma \omega$ as an amplitude (work/circulation); $\pi_1(X)$ as a groupoid makes sense and a "physical form" of Hurewich Theorem seams to "refine" the period isomorphism.

According to the philosophy of Anabelian Geometry, "What is being represented here as space with this algebraic fundamental group?" (Compare with 1-forms defining connections whose monodromy define a representation of the fundamental group). What is the underlying "Representation Theorem"? (Models: Pontryagin, Tannaka-Krein, Yoneda etc.)

3.1. **A Physics Interpretation of Period, and Montonen-Olive/T-Duality.**

Periods $\int_\gamma \omega$ can be interpreted along the following lines, as follows; we will be specific: 1D-case of Riemann surfaces.

3.1.1. *Closed vs. Open Periods; Electric vs. Magnetic.* The flow of the 1-form, which mathematically measures the geometric object γ with boundary $D = \partial\gamma$, can be viewed as the *work/flux* of a "probe" in the dynamics defined by the 1-form.

Its poles and zeros have *charges*: electric charges are the *residues* $\int_{S^1(p)} \omega$ (Gauss Law; circle around point p), while the *periods are magnetic charges* $\int_\gamma \omega$, in a homological basis γ_i.

There are relations between charges, and Riemann-Roch Theorem restricts the possible dynamics.

3.1.2. *Helmhodtz/Hodge and "Maxwell's Eq."* The Hodge duality corresponds to Helmholtz Decomposition. It hints to the fact that it reflects the structure of the *group of symmetries of the space* X ("Gauge group"):

$$translations/grad,\ rotations/curl,\ similarities/divergence.$$

The local group is conformal, with its polar decomposition; the "global aspects" are captured by the fundamental group $\pi_1(X)$ ("Galois Group").

Correspondingly there is an underlying "gauge theory" with connection 1-form ω, and hence a *Motonen-Olive Duality* between "electric" and "magnetic", which in String Theory corresponds to *T-Duality*.

The point is that the physical interpretation complements a "purely Grothendieck" approach for understanding periods, paving the road towards understanding Feynman Integrals, and more importantly, *intrinsic scattering amplitudes* (to be made precisse in view of the new methods for computing MZV amplitudes: [36]).

3.2. **... and Quantum Physics Amplitudes (beyond Veneziano).** There seems to be good prospects of better understanding the role of absolute Galois group in the physics context of scattering amplitudes and Multiple Zeta Values, with their incarnation as Chen integrals on moduli spaces, as studied by Francis Brown, since the latter are a homotopical analog of de Rham Theory:

$$Quantum\ Amplitudes \leftrightarrow Integrals\ on\ Moduli\ Spaces.$$

The fact that *maximally helicity violating* (MHV) 3-point amplitudes ressamble the cross-ratios on the Riemann sphere (essentially the unique Lorentz invariant), with logarithm a hyperbolic metric, and that these "structure constants" determine in a recursive manner the n-point amplitudes, ressembling the antipode relation from Connes-Kreimer Hopf algebra approach to renormalization, is an indication that the quantum amplitudes are periods belonging to this framework: periods and motives [4]

[4]Physics is Mathematics discovered experimentally ...

3.3. **Arithmetic Galois Theory and Anabelian Geometry.** Specifically, the author's reformulation of cohomology of cyclic groups as a discrete analog of de Rham cohomology [2] and the associated analog period isomorphism, will be related with the *arithmetic Galois Theory* [3], again as a discrete, purely algebraic toy-model of the said algebraic homology/homotopy group theory of Grothendieck. It will allow an elementary investigation of the main concepts defining periods and algebraic fundamental group, together with their conceptual relation to algebraic numbers and Galois groups.

4. Research Ramifications to TOEs and ADEX-Theory

The research will be placed in the larger context of the ADE-correspondence, since, for example, orbifolds of finite groups of rotations have crepant resolutions relevant in String Theory, while via Cartan-Killing Theory and exceptional Lie algebras, they relate to TOEs and VOAs.

The applications to ADE-correspondence, and beyond, e.g. toe ADEX-Theory [4], is an exciting R&D opportunity to perhaps finally understand what quark flavors and generations really are.

4.1. **The Platonic Trinity.** Arnold's trinities [37] refer to Platonic symmetry groups T, O, I, related to exceptional Lie algebras $E6, E7, E8$ via the double of the first as Weyl groups. It also includes their invariant polynomials, and the Hopf bundles, but without reference to Klein quartic and the Galois groups $PSL_2(F_p), p = 5, 7, 11$.

What is remarkable (on top of other things), is that these "exceptional" $PSL's$ are *cyclotomic fibrations* over the Platonic groups of symmetry (TOI) of the Platonic tesselations of the Riemann sphere, as if they are finite qubit / Hopf bundles (see Galois' last letter [38]):

$$\begin{array}{ccc} Z/5 \to PSL_2(F_5) & Z/7 \to PSL_2(F_7) & Z/11 \to PSL_2(F_{11}) \\ \downarrow & \downarrow & \downarrow \\ A_4 = T & S_4 = O & S_5 = I \end{array}$$

Remark 4.1. These seem to be related to the genus 2 (Fermat surface), 3 (Klein quartic), as described in [33] (What about $p = 11$?).

Remark 4.2. The comments from §2.3.1 extend to this "coincidence": is a locally finite model of the Universe, based on finite qubits, "*The Way*" to "quantize everything!? ... justifying even the fundamental "postulates", e.g. quantization of angular momentum!

On the other hand, how do they relate to the higher genus Platonic tesselations via Belyi ramified covers, e.g. Klein's quartic as a Hurwitz surface!? [5].

[5]... and further, with the Lie algebras, McKay and Gabriel reps theory, via ADE-correspondence.

4.2. **Role of the Exceptional Lie algebra.** Note that the role played by the exceptional Lie algebra, in this *Finite Qubit Model* (or *Finite String Theory model*) is *not* that of containing the SM gauge groups $U(1) \times SU(2)^L \times SU(3)$, since that would not be a true unification of interactions, and would not lead to a deeper understanding of masses of elementary particles etc., but rather as a kind of "loop groups" of paths due to multiple reflexions in the virtual Weyl mirrors, as if the baryons (modeled by these rank 3 root systems) are some kind of *kaleidoscopic beam splitters* [39]!

5. Conclusions

The main thread for the proposed research, is to study *algebraic de Rham cohomology* AND *algebraic fundamental groups* together, in order to understand why Feynman integrals (or scattering amplitudes in general, independent on a particular method of computation), are related to "homological periods" (algebraic de Rham isomorphism) on one hand, but are related to Chen iterated integrals in the Number Theory side, which is a *homotopical de Rham Theory*, hence to be studied in the algebraic context of Anabelian geometry.

This leads to the rich area of graphs embedded on surfaces, i.e. *Dessins D'Enfant*, as a sort of generalization of a lattice embedded in a vector space, and Hodge structures controlling how Betti homology "sits" inside de Rham cohomology.

A study of the category of morphisms of Belyi maps, which capture both the ramification data, but also divisor data and homotopy groups (Anabelian Galois groups), could in principle help clarify the role of the "change of variables" Kontsevich-Zagier relation, beyond the torsor due to isomorphism equivalence.

The analogy with the theory of decomposition of primes, corresponding to the structure of the Galois group (inertia, decomposition and degree), supports the belief that such a study could yield new results and a better understanding of the structure of the "absolute" ring of periods.

At the "elementary" level of Algebraic Number Theory, the representation point of view used to study the *arithmetic Galois Theory* (s.e.s. of Abelian groups and their $Aut_{Ab}()$ symmetries), functorially corresponding to (algebraic) Galois Theory, can be thought of as an analog of covering spaces and deck transformations.

This provides another example of "Anabelian Geometry", "a la Grothendieck", together with, and corresponding to, the "main example" of *algebraic fundamental groups*, namely the Galois groups.

This homotopical theory aspects of Galois Theory in the non-commutative case will be studied elsewhere, together with the homological aspects, via the relation between the discrete de Rham cohomology [2] and algebraic de Rham cohomology of Grothendieck [14].

Applications to physics are proposed via the special case of Riemann Surfaces with Platonic tesselations, and the study of the role of the Hurwitz surfaces, i.e. those with maximal symmetry. For example, Klein quartic is instrumental in String Theory. It is explained how this study could be a bridge between the Standard Model and String Theory, via a qubit model (Hopf bundle) interpretation.

There are of course rich connections with ADE-correspondence, Klein singularities and orbifolds, RS as crepant rsolutions etc. [40], and also with TOEs (e.g. Lisi's [41]) and ADEX Theory [4], emphasizing the roles of the exceptional Lie groups in fundamental physics.

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