The Schrodinger and the Heisenberg Operators

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Abstract

This brief article in the first section brings out the fact that the Heisenberg and the Schrodinger operators become identical if the operators correspond to a conserved quantity. In Section II solutions we show that for time independent operators A_S that satisfy $[A_S, H] = 0$ where H satisfies the Schrodinger equation $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ we have $A_S = a$ where 'a' is an eigen value of the stated equation. It is independent of space and time coordinates.

Introduction

In Section I we first show that if the Schrodinger operator commutes with the Hamiltonian then $\left[e^{iHt}, \hat{O}_S\right] = 0$. That leads to the Schrodinger and the Heisenberg operators becoming identical. In Section II solutions we show that for time independent operators A_S that satisfy $[A_S, H] = 0$ where H satisfies the Schrodinger equation $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ we have $A_S = a$ where 'a' is an eigen value of the stated equation. It is independent of space and time coordinates.

Section I

The Schrodinger Operator Commuting with the Hamiltonian

We start with the definition of the Heisenberg operator[1]

$$\hat{O}_H = e^{iHt/\hbar}\hat{O}_S e^{-iHt/\hbar} \quad (1)$$

Setting $\hbar$ to unity we have

$$\hat{O}_H = e^{iHt/\hbar}\hat{O}_S e^{-iHt/\hbar} \quad (1')$$

$\hat{O}_S$:Heisenberg operator

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[H as suffix denotes 'Heisenberg' while H in the exponential term denotes 'Hamiltonian]then

Theorem 1

$$[H, \hat{O}_S] = 0 \Rightarrow [e^{iHt}, \hat{O}_S] = 0$$

Indeed,

$$e^{iHt}\hat{O}_S = \left[1 + \frac{iHt}{1!} + \frac{(iHt)^2}{2!} + \frac{(iHt)^3}{3!} + \frac{(iHt)^4}{4!} + \cdots \ldots \ldots .\right]\hat{O}_S$$

$$e^{iHt}\hat{O}_S = \hat{O}_S + \frac{iHt}{1!}\hat{O}_S + \frac{(iHt)^2}{2!}\hat{O}_S + \frac{(iHt)^3}{3!}\hat{O}_S + \frac{(iH)^4}{4!}\hat{O}_S + \cdots \ldots \ldots$$

Since $[H, \hat{O}_S] = 0$ by our postulation

$$e^{iHt}\hat{O}_S = \hat{O}_S + \hat{O}_S\frac{iHt}{1!} + \hat{O}_S\frac{(iHt)^2}{2!} + \hat{O}_S\frac{(iHt)^3}{3!} + \hat{O}_S\frac{(iHt)^4}{4!} + \cdots \ldots \ldots$$

$$\Rightarrow e^{iHt}\hat{O}_S = \hat{O}_S\left[1 + \frac{iHt}{1!} + \frac{(iHt)^2}{2!} + \frac{(iHt)^3}{3!} + \frac{(iHt)^4}{4!} + \cdots \ldots \ldots .\right]$$

$$\Rightarrow e^{iHt}\hat{O}_S = \hat{O}_S e^{iHt}$$

$$[e^{iHt}, \hat{O}_S] = 0 \quad (2)$$

Subject to the validity of (2) we have

$$\hat{O}_H = e^{iHt}\hat{O}_S e^{-iHt} = \hat{O}_S e^{iHt}e^{-iHt} = \hat{O}_S \Rightarrow \hat{O}_H = \hat{O}_S$$

Theorem 2

Subject to the validity of equation (2) that is if $[e^{iHt}, \hat{O}_S] = 0$ then

$$\hat{O}_H = \hat{O}_S \quad (3)$$

For any conserved quantity $[H, \hat{O}_S] = 0 \Rightarrow [e^{iHt}, \hat{O}_S] = 0 \quad \Rightarrow \hat{O}_H = \hat{O}_S$

For such a conserved quantity ,since $\hat{O}_H = \hat{O}_S$,we have,

$$\hat{O}_H|\psi_H> = \hat{O}_S|\psi_H> = \qquad (4)$$

But ψ_H on the left side and hence on the right side is independent of time: the Heisenberg state function does not evolve with time . On the right side $\hat{O}_S$ being a Schrodinger operator does not evolve with time . Therefore $\hat{O}_H|\psi_H>$ itself does not evolve with time , not just the expectation value. $\hat{O}_H(t)|\psi_H>$ does not evolve with time.

Section II

We consider the Schrodinger equation[2]

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \ \ (5)$$

H represents the operator Hamiltonian

We right multiply both sides of (5) by an arbitrary linear operator A_S that does not evolve with time. Therefore (5) implies

$$i\hbar\frac{\partial\psi}{\partial t}A_S = H\psi A_S \ \ \Rightarrow i\hbar\frac{\partial(\psi A_S)}{\partial t} = H\psi A_S$$

Equation (6) is an operator equation

$$i\hbar\frac{\partial(\psi A_S)}{\partial t} = H\psi A_S \ (6)$$

Equation (6) is an operator equation

Let $\psi A_S = \phi$ ≡Operator

$$i\hbar\frac{\partial\phi}{\partial t} - H\phi = 0 \ (7)$$

Applying the Integrating factor $e^{\frac{iHt}{\hbar}}$ as per standard technique

$$i\hbar e^{\frac{iHt}{\hbar}}\frac{\partial\phi}{\partial t} - e^{\frac{iHt}{\hbar}}H\phi = 0$$

$$e^{\frac{iHt}{\hbar}}\frac{\partial\phi}{\partial t} + \frac{i}{i\hbar}e^{\frac{iHt}{\hbar}}H\phi = 0$$

$$\frac{\partial\left(e^{\frac{iHt}{\hbar}}\phi\right)}{\partial t} = 0$$

$$e^{\frac{iHt}{\hbar}}\phi = C'(x,y,z)$$

$$e^{\frac{iHt}{\hbar}}\psi A_S = C'(x,y,z) \equiv \text{Time independent operator (8)}$$

$$\psi A_S = e^{\frac{-iHt}{\hbar}} C'(x,y,z) \ (9)$$

Applying the same technique[3] we may solve (5) to obtain

$$\psi = e^{\frac{-iHt}{\hbar}} D(x,y,z) \quad (10)$$

$$A_S\psi = A_S e^{\frac{-iHt}{\hbar}} D(x,y,z) \ \ (11)$$

Left multiplying (9) by $A_S$we obtain

$$A_S\psi A_S = A_S e^{\frac{-iHt}{\hbar}} C'(x,y,z) \ \ (12)$$

Using (11) with (16) we 0obtain

$$A_S e^{\frac{-iHt}{\hbar}} D(x,y,z) A_S = A_S e^{\frac{-iHt}{\hbar}} C'(x,y,z)$$

$$DA_S = C' \ 13)$$

Relation (13) is independent of the fact as to whether H commutes with $A_S$or not

If $[H, A_S] = 0$

$$i\hbar A_S \frac{\partial\psi}{\partial t} = A_S H\psi$$

$$\Rightarrow i\hbar \frac{\partial(A_S\psi)}{\partial t} = HA_S\psi \quad (14)$$

Using the substitution$\phi = A_S\psi$

$$i\hbar \frac{\partial\phi}{\partial t} = H\phi$$

$$\frac{\partial\phi}{\partial t} + \frac{i}{\hbar} H\phi = 0$$

Integrating factor: $e^{\frac{iHt}{\hbar}}$

$$e^{\frac{iHt}{\hbar}} \left(\frac{\partial\phi}{\partial t} + \frac{i}{\hbar} H\phi \right) = 0$$

$$\frac{\partial\left(e^{\frac{iHt}{\hbar}}\phi\right)}{\partial t} = 0$$

$$\phi = e^{\frac{-iHt}{\hbar}} C(x,y,z)$$

$$A_S\psi = e^{\frac{-iHt}{\hbar}}C(x,y,z) \quad (15)$$

Left multiplying (9) by $A_S$we obtain

For $[H, A_S] = 0$ both (11) and (15) hold. We have

$$A_S e^{\frac{-iHt}{\hbar}}D(x,y,z) = e^{\frac{-iHt}{\hbar}}C(x,y,z) \quad (16)$$

But

$$[H, A_S] = 0 \Rightarrow \left[A_S, e^{\frac{-iHt}{\hbar}}\right] = 0$$

Therefore

$$e^{\frac{-iHt}{\hbar}}A_S D(x,y,z) = e^{\frac{-iHt}{\hbar}}C(x,y,z)$$

$$A_S D = C \quad (17)$$

Points to Observe

1. $DA_S = C' \Rightarrow DA_S\psi = C'\psi \Rightarrow A_S\psi = \frac{1}{D}C'\psi$ since D is a multiplicative function. If ψ is a common eigenstate of H and A_S that is if $[H, A_S] = 0$ which is true of operators A_S that stand for conserved quantities, then $\frac{1}{D}C'$ is an eigen value of ψ: $\frac{1}{D}C'$ =a['a' is independent of space and timer coordinates]. The time independent operator $C' = aD$ is a time independent multiplicative function of spatial coordinates. Thus the operator $C'$is a time independent multiplicative function.
2. $[H, A_S] = 0 \Rightarrow A_S\psi = a\psi$ provided ψ is an eigenfunction of H.Again

   $$\psi = e^{\frac{-iHt}{\hbar}}D(x,y,z) \Rightarrow A_S\psi = ae^{\frac{-iHt}{\hbar}}D(x,y,z).$$

   But $A_S\psi = e^{\frac{-iHt}{\hbar}}C(x,y,z)$ for $[H, A_S] = 0$

   Therefore

   $$ae^{\frac{-iHt}{\hbar}}D(x,y,z) = e^{\frac{-iHt}{\hbar}}C(x,y,z) \Rightarrow aD(x,y,z) = C$$

   Since by (17) $A_S D = C$ for $[H, A_S] = 0$ we have $aD = A_S D \Rightarrow A_S = a$. This means that A_S is a multiplicative factor independent of the spatial as well as of the time coordinates

Conclusion

We conclude that for time independent operators A_S that satisfy $[A_S, H] = 0$ where H satisfies the equation $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ we have $A_S = a$ where 'a' is an eigen value of the stated equation. It is independent of space and time coordinates. In physics $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ is the Schrodinger equation while the operator $A_S$represents a conserved observable whose expectation value does not evolve with time.

References

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