The Schrodinger and the Heisenberg Operators

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Abstract

This brief article brings out the fact that if the partial derivative of the Heisenberg operator with respect to time is zero then the Heisenberg and the Schrodinger operators become identical.

Introduction

We first show that if the Schrodinger operator commutes with the Hamiltonian then $\left[e^{iHt}, \hat{O}_S\right] = 0$. That leads to the Schrodinger and the Heisenberg operators becoming identical. In the next stage it is shown that $\frac{\partial \hat{O}_H}{\partial t} = 0 \Rightarrow \left[H, \hat{O}_S\right] = 0$.Therefore we have $\frac{\partial \hat{O}_H}{\partial t} = 0 \Rightarrow \hat{O}_H = \hat{O}_S$

The Schrodinger Operator Commuting with the Hamiltonian

We start with the definition of the Heisenberg operator[1]

$$\hat{O}_H = e^{iHt/\hbar} \hat{O}_S e^{-iHt/\hbar} \quad (1)$$

Setting $\hbar$ to unity we have

$$\hat{O}_H = e^{iHt/\hbar} \hat{O}_S e^{-iHt/\hbar} \quad (1')$$

$\hat{O}_S$:Heisenberg operator

$\hat{O}_H$:Heisenberg operator

[H as suffix denotes 'Heisenberg' while H in the exponential term denotes 'Hamiltonian]then

Theorem 1

$$\left[H, \hat{O}_S\right] = 0 \Rightarrow \left[e^{iHt}, \hat{O}_S\right] = 0$$

Indeed,

$$e^{iHt} \hat{O}_S = \left[1 + \frac{iH}{1!} + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \frac{(iH)^4}{4!} + \cdots \ldots \ldots \right] \hat{O}_S$$

$$e^{iHt}\hat{O}_S = \hat{O}_S + \frac{iH}{1!}\hat{O}_S + \frac{(iH)^2}{2!}\hat{O}_S + \frac{(iH)^3}{3!}\hat{O}_S + \frac{(iH)^4}{4!}\hat{O}_S + \cdots \dots \dots$$

Since $[H, \hat{O}_S] = 0$ by our postulation

$$e^{iHt}\hat{O}_S = \hat{O}_S + \hat{O}_S\frac{iH}{1!} + \hat{O}_S\frac{(iH)^2}{2!} + \hat{O}_S\frac{(iH)^3}{3!} + \hat{O}_S\frac{(iH)^4}{4!} + \cdots \dots \dots$$

$$\Rightarrow e^{iHt}\hat{O}_S = \hat{O}_S\left[1 + \frac{iH}{1!} + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \frac{(iH)^4}{4!} + \cdots \dots \dots\right]$$

$$\Rightarrow e^{iHt}\hat{O}_S = \hat{O}_S e^{iHt}$$

$$[e^{iHt}, \hat{O}_S] = 0 \quad (2)$$

Subject to the validity of (2) we have

$$\hat{O}_H = e^{iHt}\hat{O}_S e^{-iHt} = \hat{O}_S e^{iHt}e^{-iHt} = \hat{O}_S \Rightarrow \hat{O}_H = \hat{O}_S$$

Theorem 2

Subject to the validity of equation (2) that is if $[e^{iHt}, \hat{O}_S] = 0$ then

$$\hat{O}_H = \hat{O}_S \quad (3)$$

Vanishing Time (Partial) Derivative of the Heisenberg Operator

Next we consider the Heisenberg Equation of motion[2][3]

$$\frac{d\hat{O}_H}{dt} = \frac{i}{\hbar}[H, \hat{O}_H] + \frac{\partial \hat{O}_H}{\partial t} \quad (4)$$

We consider a situation where $\frac{\partial \hat{O}_H}{\partial t} = 0$ that is we consider the equation of motion as:

$$\frac{d\hat{O}_H}{dt} = \frac{i}{\hbar}[H, \hat{O}_H] \; (5)$$

Setting $\hbar$ to unity we have

$$\frac{d\hat{O}_H}{dt} = i[H, \hat{O}_H] \; (6)$$

For such a situation we partial differentiate $\hat{O}_H$ with respect to time[$\hbar = 1$], keeping in mind that the Schrodinger operator is independent of time, we obtain:

$$\frac{\partial \hat{O}_H}{\partial t} = e^{iHt} iH\hat{O}_S e^{-iHt} + e^{iHt}\hat{O}_S e^{-iHt}(-iH)$$

$$\frac{\partial \hat{O}_H}{\partial t} = ie^{iHt} H\hat{O}_S e^{-iHt} - ie^{iHt}\hat{O}_S e^{-iHt} H$$

$$\frac{\partial \hat{O}_H}{\partial t} = ie^{iHt}\left(H\hat{O}_S - \hat{O}_S H\right)e^{-iHt}$$

$$\frac{\partial \hat{O}_H}{\partial t} = ie^{iHt}\left[H, \hat{O}_S\right]e^{-iHt}$$

Since by our initial postulation $\frac{\partial \hat{O}_H}{\partial t} = 0$ we have

$$ie^{iHt}\left[H, \hat{O}_S\right]e^{-iHt} = 0 \quad (7)$$

Since $'t'$ is arbitrary we have $\left[H, \hat{O}_S\right] = 0$

$$\frac{\partial \hat{O}_H}{\partial t} = 0 \Rightarrow \left[H, \hat{O}_S\right] = 0 \quad (8)$$

By theorem 1, $\left[H, \hat{O}_S\right] = 0 \Rightarrow \left[e^{iHt}, \hat{O}_S\right] = 0$ and by theorem 2, $\left[e^{iHt}, \hat{O}_S\right] = 0 \Rightarrow \hat{O}_H = \hat{O}_S$

But

$$\frac{\partial \hat{O}_H}{\partial t} = 0 \Rightarrow \left[H, \hat{O}_S\right] = 0$$

Therefore

$$\frac{\partial \hat{O}_H}{\partial t} = 0 \Rightarrow \hat{O}_H = \hat{O}_S \quad (9)$$

Thus

$$\frac{d\hat{O}_H}{dt} = \left[H, \hat{O}_H\right] \Rightarrow \hat{O}_H = \hat{O}_S \quad (10)$$

Conclusion

As claimed we have deduced the fact that vanishing of the partial derivative of the Heisenberg operator with respect to time leads to the Schrodinger and the Heisenberg operators becoming identical

References

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3. Wikipedia, Heisenberg Picture, Link: https://en.wikipedia.org/wiki/Heisenberg_picture, Accessed on 21st December, 2019.