Metric Conventions and the FLRW Metric

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Abstract

We investigate the flat space time Friedmann Lemaitre Robertson Walker model of cosmology in the light of two metric signatures:$(-,+,+,+)$ and $(+,-,-,-)$. We arrive at the interesting fact that with the $(+,-,-,-)$signature both the Ricci scalar and the energy component of the stress energy tensor are negative for all points of cosmological time. With this component we do have a positive pressure favoring the expansion of the universe.

Introduction

The flat space time Friedmann Lemaitre Robertson Walker model of cosmology has been explored in the light of two metric signatures:$(-,+,+,+)$ and $(+,-,-,-)$. We arrive at the interesting fact that with the $(+,-,-,-)$ signature both the Ricci scalar and the energy component of the stress energy tensor are negative for all points no cosmological time.

The Metric and some Associated Properties

Metric[general form]

$$ds^2 = g_{tt}c^2dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2 \quad (1)$$

The coefficients g_{xx}, g_{yy} and g_{zz} always have the same sign. This sign is opposite to that of g_{tt}

We also have

$$g^{\alpha i}g_{i\beta} = \delta^{\alpha}{}_{\beta} \quad (2)$$

The above implies that for the orthogonal systems

$$g^{\alpha i}g_{i\beta} = \delta^{\alpha}{}_{\beta}$$

$$g^{\alpha\alpha}g_{\alpha\alpha} = 1$$

with no summation on α

$$g^{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}} \quad (3)$$

With that information we proceed with the conventions

We have by definition[1]

$$g^{\alpha\beta} = \frac{Cofactor\ of\ g_{\alpha\beta}}{\det g} \quad (4)$$

where matrix $g = \left[g_{\alpha\beta}\right]_{4\times4}$

From (4) by applying the rules for determinants we can prove (2)

There is also another important observation that we might make. If the sign of each $g_{\alpha\beta}$ reverses the sign of $\det g$ will not change since g is a 4×4 matrix. We may infer that if the sign of each $g_{\alpha\beta}$ reverses then the sign of cofactor of $g_{\alpha\beta}$ will reverse since this cofactor corresponds to a 3×3 matrix. We will make use of this property later.

Regarding the metric we consider two conventions and their impact on FLRW metric

With Signature 1

Signature:$(-,+,+,+)$

$$ds^2 = -|g_{tt}|c^2dt^2 + |g_{xx}|dx^2 + \left|g_{yy}\right|dy^2 + |g_{zz}|dz^2 \quad (5)$$

$ds^2 = -c^2d\tau^2$;if $ds^2 = c^2d\tau^2$then proper time interval becomes imaginary for time like paths

FLRW metric[2] for the flat space time model of cosmology in this convention reads

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad (6)$$

we have the following results[3]

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = c^{-2}(a\ddot{a} + 2\dot{a}^2)\ (7.1)$$

We set c=1

Therefore

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = (a\ddot{a} + 2\dot{a}^2) \quad (7.2)$$

$$R = g^{\mu\mu} R_{\mu\mu} \ (8)$$

$$R = g^{tt} R_{tt} + g^{xx} R_{xx} + g^{yy} R_{yy} + g^{zz} R_{zz}$$

$$R = -R_{tt} + \frac{1}{a^2}\left(R_{xx} + R_{yy} + R_{zz}\right)$$

$$R = -\left(-3\frac{\ddot{a}}{a}\right) + \frac{3}{a^2}(a\ddot{a} + 2\dot{a}^2)$$

The Ricci scalar reads

$$R = \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \ (9)$$

$$R_{tt} - \frac{1}{2} R g_{tt} = 8\pi G T_{tt}; c = 1$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2} 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)(-1) = 8\pi G T_{tt}$$

We have the Friedmann equation

$$3\frac{\dot{a}^2}{a^2} = 8\pi G T_{tt}$$

$$T_{tt} > 0 \ (10)$$

Again

$$R_{xx} - \frac{1}{2} R g_{xx} = 8\pi G T_{xx}$$

$$(a\ddot{a} + 2\dot{a}^2) - \frac{1}{2}\left(6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)a^2\right) = 8\pi G T_{xx}$$

$$(a\ddot{a} + 2\dot{a}^2) - 3\big((a\ddot{a} + \dot{a}^2)\big) = 8\pi G T_{xx}$$

We now have a standard result:

$$-2a\ddot{a} - \dot{a}^2 = 8\pi G T_{xx}$$

[To take note of the fact that we working are in the Cartesian system and not in the spherical system of coordinates]

$$T_{xx} = T_{yy} = T_{zz} < 0 \ (11)$$

We have a negative pressure with this convention.

With Signature2

Signature:$(+,-,-,-)$

$$ds^2 = |g_{tt}|c^2dt^2 - |g_{xx}|dx^2 - |g_{yy}|dy^2 - |g_{zz}|dz^2 \ (12)$$

$ds^2 = c^2d\tau^2;$

Christoffel symbols

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha s}\left[\frac{\partial g_{\beta s}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha s}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{s}}\right] \ (13)$$

If convention changes then the sign of each $g_{\alpha\beta}$ and $g^{\alpha\beta}$ reverses . This has been already discussed. The sign of $\Gamma^{\alpha}{}_{\beta\gamma}$ does not change. The Riemann tensor and the Ricci tensor remain unaltered by convention . But what about the sign of the Ricci scalar and the stress energy tensor? Let us check

Friedmann metric in this convention

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \ \ (14)$$

Since the Ricci tensor components remain unchanged[discussed earlier] we have

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = c^{-2}(a\ddot{a} + 2\dot{a}^2) \ \ (15.1)$$

Setting c to unity

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = (a\ddot{a} + 2\dot{a}^2) \ \ (15.2)$$

$$R = g^{\mu\mu}R_{\mu\mu}$$

$$R = g^{tt}R_{tt} + g^{xx}R_{xx} + g^{yy}R_{yy} + g^{zz}R_{zz}$$

$$R = R_{tt} - \frac{1}{a^2}(R_{xx} + R_{yy} + R_{zz})$$

$$R = \left(-3\frac{\ddot{a}}{a}\right) - \frac{3}{a^2}(a\ddot{a} + 2\dot{a}^2)$$

then

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \quad (16)$$

$$R < 0 \quad (17)$$

From then field equations

$$R_{tt} - \frac{1}{2}Rg_{tt} = 8\pi G T_{tt}$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 8\pi G T_{tt}$$

$$T_{tt} < 0 \quad (18)$$

Convention changes the sign and the value of T_{tt}

$$R_{xx} - \frac{1}{2}Rg_{xx} = 8\pi G T_{xx}$$

$$(a\ddot{a} + 2\dot{a}^2) - \frac{1}{2}\left(-6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)a^2\right) = 8\pi G T_{xx}$$

$$(a\ddot{a} + 2\dot{a}^2) + 3\big((a\ddot{a} + \dot{a}^2)\big) = 8\pi G T_{xx} \quad (19)$$

$$T_{xx} = T_{yy} = T_{zz} > 0 \quad (20)$$

[To take note of the fact that we working are in the Cartesian system and not in the spherical system of coordinates]

We have positive pressure

We consider models for which $\frac{\dot{a}}{a}$ increases with time, for example

$$a = Cexp\left(\frac{k}{1 + 1/n}t^{\frac{1}{n}+1}\right)$$

where k is a very small positive number and n is a very large positive number. C has to be a very small constant close to zero so that for t t=0 ‘a’ is very close to zero.

$$\dot{a} = Cexp\left(\frac{k}{1 + 1/n}t^{\frac{1}{n}+1}\right) \times kt^{1/n}$$

$$\frac{\dot{a}}{a} = \frac{Cexp\left(\frac{k}{1+1/n}t^{1/n}\right) \times kt^{1/n}}{Cexp\left(\frac{k}{1+1/n}t^{1/n}\right)}$$

$$\frac{\dot{a}}{a} = kt^{1/n}$$

Again

$$\ddot{a} = Ck^2t^{2/n}exp\left(\frac{k}{1+1/n}t^{\frac{1}{n}+1}\right)$$

$$\frac{\ddot{a}}{a} = \frac{Ck^2t^{2/n}exp\left(\frac{k}{1+1/n}t^{1/n}\right)}{Cexp\left(\frac{k}{1+1/n}t^{1/n}\right)}$$

$$\frac{\ddot{a}}{a} = k^2t^{2/n}$$

eg: $k = 10^{-30}, n = 10^{40}$

With the signature $(+,-,-,-)$ of the metric increase of $\frac{\dot{a}}{a}$ and $\frac{\ddot{a}}{a}$with time will lead to the accumulation of negative energy in space. To conserve energy the celestial bodies have to move faster as is observed with the accelerated expansion the universe. This is consistent with the increase of positive pressure which falls in line with the accelerated expansion of the universe. We must keep in mind that the time component T_{tt} corresponds to the field energy while $T^{ti}; i = x, y, z$ represent the three components of spatial field momenta. Increase of positive pressure is in keeping with the increasing spatial momenta.

We know that convention should not alter the content of physics. Therefore the conclusions from the metric signature $(-,+,+,+)$ is expected to be of an equivalent nature. But the interpretations from the signature $(+,-,-,-)$ is closer to the interpretation of our senses in that an accelerated expansion of the universe is an observed fact. We also do have a visible agreement with the conservation of energy principle.

One may think of models where $\frac{\dot{a}}{a}$ falls off to zero or becomes a small steady value after some stage. That does not provide us with any explanation for the accelerating expansion of the universe.

Conclusion

With the signature $(+,-,-,-)$ the Ricci scalar and T_{tt} are negative while pressure[from field momentum] is positive With the signature, $(-,+,+,+)$,the Ricci scalar and T_{tt} are positive. But pressure is negative. The two conventions relating to the two signatures should be equivalent. Nevertheless the inferences from the signature $(+,-,-,-)$ are palpable to our senses with the acceptance of negative energy state which are not uncommon in physics. All this is with the denial of difficulties with metrics in a general manner as pointed out in the article " On the metric Coefficients"[4]

References

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