The Time Evolution Operator

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Abstract

In this article we have devised a simpler alternative solution to the operator equation for the usual time evolution operator. This is based on an interesting commutator relation which has been derived valid subject to a weak condition that two specific operators should not be simultaneously non invertible.

Introduction

The article considers an interesting commutator relation valid subject to a weak condition that two specific operators should not be simultaneously non invertible. Applying the stated relation we have devised a simpler alternative solution to the operator equation for the usual time evolution operator

Time evolution operator

Let us consider the operator function[1]

$$\hat{U}(t,t_0)=e^{i\hat{H}_0(t-t_0)}e^{-iH(t-t_0)}\ (1.1)$$

We would like to transform (1.1) to our advantage is done in standard treatment to obtain a form conducive to the construction of Feynman's diagrams.

From (1.1) we formulate the differential equation[2] and solve it subject to $\hat{U}(t_0,t_0)=1$:

$$i\frac{\partial\hat{U}(t,t_0)}{\partial t}=\hat{H}(t)\hat{U}(t,t_0)\quad(1.2)$$

Standard solution to (1.2) subject to $\hat{U}(t_0,t_0)=1$ is given by

$$\hat{U}(t,t_0)=T\left[exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\}\right]\ (3)$$

Where by definition[3] ,

$$T\left[exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\}\right]$$
$$= I + \frac{1}{1!}\int_{t_0}^{t}dt_1\hat{H}(t_1) + \frac{1}{2!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2 T\left[\hat{H}(t_1)\hat{H}(t_2)\right]$$
$$+\frac{1}{3!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\int_{t_0}^{t}dt_3\left[\hat{H}(t_1)\hat{H}(t_2)\hat{H}(t_3)\right] + \cdots (4)$$

The right side of (4) is conducive to the construction of Feynman's Diagrams

The Trial Solution and Subsequent Considerations

We consider the following trial solution:

$$\hat{U}(t,t_0) = exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\} \quad (5)$$

Solution given by (5) satisfies: $\hat{U}(t_0,t_0) = 1$

Partial differentiating the above with respect to 't' we have,

$$\frac{\partial\hat{U}(t,t_0)}{\partial t} = -iexp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\}\hat{H}(t) \quad (6)$$

We shall now prove that

$$exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\}\hat{H}(t) = \hat{H}(t)exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\} \quad (7)$$

that is

$$\left[exp\left\{-i\int_{t_0}^{t}dt'\hat{H}(t')\right\},\hat{H}(t)\right] = 0 \quad (8)$$

Proof of (8): We may first consider the relation

$$\hat{A}exp(-i\hat{A}) = exp(-i\hat{A})\hat{A} \quad (9)$$

which may be proved by direct expansion. Indeed

Left side of (7):

$$\hat{A}exp(-i\hat{A}) = \hat{A}\left[1 - \frac{i\hat{A}}{1!} + \frac{(i\hat{A})^2}{2!} - \frac{(i\hat{A})^3}{3!} + \cdots \ldots \ldots\right]$$

$$= \left[1 - \frac{i\hat{A}}{1!} + \frac{\left(i\hat{A}\right)^2}{2!} - \frac{\left(i\hat{A}\right)^3}{3!} + \cdots \ldots \ldots\right]\hat{A}$$

$$= exp\left(-i\hat{A}\right)\hat{A}$$

Let

$$\hat{A} = \int_{t_0}^{t} H(t')dt' \quad (10)$$

and

$$\hat{X} = \hat{A}exp\left(-i\hat{A}\right) = \left(\int_{t_0}^{t} H(t')dt'\right)exp\left(-i\int_{t_0}^{t} H(t')dt'\right) \ (11.1)$$

By applying (9) we have

$$\hat{X} = exp\left(-i\hat{A}\right)\hat{A} = exp\left(-i\int_{t_0}^{t} H(t')dt'\right)\left(\int_{t_0}^{t} H(t')dt'\right) \ (11.2)$$

Differentiating (11.1) with respect to time we have

$$\frac{\partial\hat{X}}{\partial t} = \left(\int_{t_0}^{t} H(t')dt'\right)H(t)exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right)exp\left(-i\int_{t_0}^{t} H(t')dt'\right)H(t) \ (12.1)$$

Differentiating (11.2) with respect to time we have

$$\frac{\partial\hat{X}}{\partial t} = -i\left(\int_{t_0}^{t} H(t')dt'\right)exp\left(-i\int_{t_0}^{t} H(t')dt'\right)H(t) + \left(\int_{t_0}^{t} H(t')dt'\right)H(t)exp\left(-i\int_{t_0}^{t} H(t')dt'\right) \ (12.2)$$

Since the left sides of (12.1) and (12.2) are identical the right sides will also be identical. This will hold if equation (8)[equivalently (7)] holds that is if we have $\left[exp\left\{-i\int_{t_0}^{t} dt'^{\hat{H}(t')}\right\}, \hat{H}(t)\right] = 0$. A relation like $\left[exp\left\{-i\int_{t_0}^{t} dt'^{\hat{H}(t')}\right\}, \hat{H}(t)\right] = b(t) \neq 0$ will upset the expected identicalness of (10.1) and (10.2)[we may consider different forms of $\hat{H}(t)$.].

$$\left[exp\left\{-i\int_{t_0}^{t} dt'^{\hat{H}(t')}\right\}, \hat{H}(t)\right] = b(t)$$

$$\Rightarrow exp\left\{-i\int_{t_0}^{t} dt'\hat{H}(t')\right\}\hat{H}(t) = b(t) + \hat{H}(t)exp\left\{-i\int_{t_0}^{t} dt'^{\hat{H}(t')}\right\} \ (13)$$

Using (13) with (12.1) we have,

$$\frac{\partial \hat{X}}{\partial t} = \left(\int_{t_0}^{t} H(t')dt'\right) H(t) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) H(t)$$

$$\frac{\partial \hat{X}}{\partial t} = \left(\int_{t_0}^{t} H(t')dt'\right) H(t) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right)\left[b(t) + \hat{H}(t) exp\left\{-i\int_{t_0}^{t} dt' \hat{H}(t')\right\}\right]$$

$$\frac{\partial \hat{X}}{\partial t} = \left(\int_{t_0}^{t} H(t')dt'\right) H(t) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right) b(t)$$
$$- i\left(\int_{t_0}^{t} H(t')dt'\right)\hat{H}(t) exp\left\{-i\int_{t_0}^{t} dt' \hat{H}(t')\right\} \;(14)$$

Equating the right sides of (12.2) and (14) we obtain

$$= \left(\int_{t_0}^{t} H(t')dt'\right) H(t) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) H(t)$$
$$= \left(\int_{t_0}^{t} H(t')dt'\right) H(t) exp\left(-i\int_{t_0}^{t} H(t')dt'\right) - i\left(\int_{t_0}^{t} H(t')dt'\right) b(t)$$
$$- i\left(\int_{t_0}^{t} H(t')dt'\right)\hat{H}(t) exp\left\{-i\int_{t_0}^{t} dt' \hat{H}(t')\right\}$$

We have the operator equation

$$\left(\int_{t_0}^{t} H(t')dt'\right) b(t) = 0 \;\; (15)$$

If the operator $b(t)$ has an inverse then

$$\left(\int_{t_0}^{t} H(t')dt'\right) b(t)[b(t)]^{-1} = 0$$

$$\int_{t_0}^{t} H(t')dt' = 0 \;\; (16)$$

Equation (16) cannot be entertained: we will not have any Feynman diagram as per conventional method

If the operator $\int_{t_0}^{t} H(t')dt'$has an inverse then

$$\left(\int_{t_0}^{t} H(t')dt'\right)^{-1}\left(\int_{t_0}^{t} H(t')dt'\right) b(t) = 0 \Rightarrow b(t) = 0 \;\; (17)$$

If $\int_{t_0}^{t} H(t')dt'$ and $b(t)$ are numbers then any one will be zero b=0 would be appropriate.

$b(t) = 0$ seems most plausible.

[If both the operators are expressible in matrix form it might happen both are non invertible at the same time??]

Unless both A and b are non invertible we have as follows

From equations(6) and (7) we obtain:

$$\frac{\partial \widehat{U}(t,t_0)}{\partial t} = -i\widehat{H}(t)exp\left\{-i\int_{t_0}^{t} dt'\widehat{H}(t')\right\} \quad (18)$$

Using (5) we have

$$\frac{\partial \widehat{U}(t,t_0)}{\partial t} = -iH(t)\widehat{U}(t,t_0)$$

$$\Rightarrow i\frac{\partial \widehat{U}(t,t_0)}{\partial t} = H(t)\widehat{U}(t,t_0)$$

In the above we have obtained (1.2). Our trial solution indeed satisfies(1.2)

Conclusion

As claimed an alternative solution has been considered against the existing one. This is in view of a commutator relation valid subject to a weak condition that two specific operators should not be simultaneously non invertible.

References

1. Peskin M E, Schroeder D V, Quantum Field theory, Chapter 4:Interacting fields and Feynman's Diagrams, Section 4.2: Perturbation Expansion of Correlation Functions,p84
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3. Peskin M E, Schroeder D V, Quantum Field theory, Chapter 4:Interacting fields and Feynman's Diagrams, Section 4.2: Perturbation Expansion of Correlation Functions,p85