On Four Velocity and Four Momentum

Anamitra Palit

Free lancer,Physicist

P154 Motijheel Avenue, Flat C4, Kolkata 700074, India

palit.anamitra@gmail.com

Cell: +919163892336

Abstract

We derive in this article that the four dot product between two arbitrary velocities is less than the square of the speed of light in vacuum and the product of two four momenta is less than the product of the two masses involved and the square of the speed of light[in vacuuo].Based on these results we strive to show a contradiction. Flat spacetime has been considered.

Introduction

The writing concerns itself with a derivation that the four dot product of two arbitrary velocities is less than c^2, c being the speed of light in vacuum and that the four dot product of two arbitrary four momenta is less than $m_1 m_2 c^2$. Finally we arrive at acontradiction.We have considered flat space time in the article.

A Mathematical Result

First we consider the following result:

For arbitrary real numbers a_1, a_2, b_1 and b_2,

$$(a_1 b_1 - a_2 b_2)^2 \geq ({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) \quad (1)$$

Proof: $$(a_1 b_1 - a_2 b_2)^2 - ({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2)$$

$$= {a_1}^2 {b_1}^2 + {a_2}^2 {b_2}^2 - 2a_1 a_2 b_1 b_2 - \left({a_1}^2 {b_1}^2 + {a_2}^2 {b_2}^2 - {a_1}^2 {b_2}^2 - {a_2}^2 {b_1}^2\right)$$
$$= {a_1}^2 {b_2}^2 + {a_2}^2 {b_1}^2 - 2a_1 a_2 b_1 b_2$$

$$= (a_1 b_2 - a_2 b_1)^2 \geq 0$$

Therefore$(a_1 b_1 - a_2 b_2)^2 - ({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2) \geq 0$

$$(a_1 b_1 - a_2 b_2)^2 \geq ({a_1}^2 - {a_2}^2)({b_1}^2 - {b_2}^2)$$

Or,

Either $(a_1b_1 - a_2b_2) \geq \sqrt{a_1{}^2 - a_2{}^2}\sqrt{b_1{}^2 - b_2{}^2}$ or $a_1b_1 - a_2b_2 \leq -\sqrt{a_1{}^2 - a_2{}^2}\sqrt{b_1{}^2 - b_2{}^2}$

Dot Product of Two Four Velocities

Using the above result we will prove that the four velocity[1] dot product[2],

$$v_1.v_2 >= c^2 \quad (2)$$

Proof: First we consider the relations

$$1) c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \geq \sqrt{c^2 v_{1t}{}^2 - |\vec{v}_1|^2}\sqrt{c^2 v_{12t}{}^2 - |\vec{v}_2|^2}$$

$$2) c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \leq -\sqrt{c^2 v_{1t}{}^2 - |\vec{v}_1|^2}\sqrt{c^2 v_{12t}{}^2 - |\vec{v}_2|^2}$$

The above two relations have been written from (1) where we have considered the following,

$a_1 = cv_{1t}, a_2 = |\vec{v}_1|, b_1 = cv_{2t}, , b_2 = |\vec{v}_2|$

[suffix 't' denotes the time component]

But $c^2 v_{1t}{}^2 - v_{1t}{}^2 v_{2t}{}^2 = c^2; c^2 v_{12t}{}^2 - v_{1t}{}^2 v_{2t}{}^2 = c^2$

The first inequality gives us

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \geq c^2$$

Where $\vec{v}_1 = (v_{1x}, v_{1y}, v_{1z}); \vec{v}_2 = (v_{2x}, v_{2y}, v_{2z})$ where v_{1i} and v_{2i} are the spatial components of proper velocity and not of coordinate velocity

$$v_1.v_2 = c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| Cos\theta$$

One should take cognizance of the fact that by Cauchy Schwarz inequality

$$(v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z})^2 \leq (v_{1x}{}^2 + v_{1y}{}^2 + v_{1z}{}^2)(v_{2x}{}^2 + v_{2y}{}^2 + v_{2z}{}^2)$$

$$\Rightarrow \frac{(v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z})^2}{(v_{1x}{}^2 + v_{1y}{}^2 + v_{1z}{}^2)(v_{2x}{}^2 + v_{2y}{}^2 + v_{2z}{}^2)} \leq 1$$

$$\Rightarrow -1 \leq \frac{v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}}{\sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}} \leq 1$$

Therefore with some suitable θ we may write

$$\frac{v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}}{\sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}} = Cos\theta$$

$$\Rightarrow v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \sqrt{{v_{1x}}^2 + {v_{1y}}^2 + {v_{1z}}^2}\sqrt{{v_{2x}}^2 + {v_{2y}}^2 + {v_{2z}}^2}\; Cos\theta$$

Therefore

$$\Rightarrow v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = |\vec{v}_1||\vec{v}_2|\; Cos\theta$$

We do have the above relation for some suitable θ[at any relativistic speed notwithstanding thefasct that we are not considering coordinate speed components but that we have taken the spatial part of proper acceleration[celertity]

If $0 < Cos\theta < 1$,

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| > c^2$$

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2$$

$$v_1.v_2 > c^2$$

If $-1 < Cos\theta < 0$,

$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| > c^2$[since $Cos\theta$ is negative]

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2|Cos\theta > c^2$$

$$v_1.v_2 > c^2$$

If $Cos\theta = 1$,

$$c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \geq c^2$$

Therefore in general:

$$v_1.v_2 > c^2 \; (3)$$

Let us consider the second inequality

$$2) c^2 v_{1t}.v_{2t} - |\vec{v}_1||\vec{v}_2| \leq -\sqrt{c^2 {v_{1t}}^2 - |\vec{v}_1|^2}\sqrt{c^2 {v_{12t}}^2 - |\vec{v}_2|^2}$$

If $v_1 = v_2$ this inequality reduces to $v^2 \leq -c^2$ which is not true. So we may dismiss the second inequality

Alternative considerations:

If possible let

$$v_1 . v_2 < c^2$$

Or,

$$c^2 v_{1t} . v_{2t} - \vec{v}_1 . \vec{v}_2 < c^2$$

We transform to a frame of reference where $\vec{v}_1 = 0$

We now have,

$$c^2 v_{1t} . v_{2t} < c^2$$

Now

$$v_{1t} = \frac{dt_1}{d\tau} = \gamma_1 ; v_{2t} = \frac{dt_2}{d\tau} = \gamma_2$$

Therefore

$$c^2 v_{1t} . v_{2t} = c^2 \gamma_1 \gamma_2$$

$$c^2 v_{1t} . v_{2t} < c^2 \Rightarrow c^2 \gamma_1 \gamma_2 < c^2$$

$$\gamma_1 \gamma_2 < 1$$

But $\gamma_1 = 1$since $\vec{v}_1 = 0$

$$\Rightarrow \gamma_2 < 1$$

The above relation is an impossible relation since γcannot be less than unity.

Therefore,

$$v_1 . v_2 \geq c^2$$

Dot Product of Two Four Moments

Next we pass on to the dot product of four momenta[3]

$$p_1 . p_2 = E_1 . E_1 - c^2 \vec{p}_1 . \vec{p}_2 \quad (4)$$

If possible let

$$p_1.p_2 < m_1 m_2 c^4$$

where m_1 and m_2 are the rest masses of the two particles.

We transform to a frame of reference where the velocity of the first particle, $\vec{v}_1 = 0 :\Rightarrow\ \vec{p}_1 = 0$

Therefore

$$E_1.E_1 < m_1 m_2 c^4$$

$$\Rightarrow m_1\gamma_1 c^2 m_2\gamma_2 c^2 < m_1 m_2 c^4$$

$$\Rightarrow \gamma_1\gamma_2 < 1$$

But the product of the gammas cannot be less than unity

Therefore

$$p_1.p_2 \nless {m_1}^2 c^4$$

Alternative considerations:

Let $a_1 = E_1; a_2 = c|\vec{p}_1|; b_1 = E_2; b_2 = c|\vec{p}_2|$

Considring (1) we have

$$\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right)^2 \geq \left({E_1}^2 - c^2|\vec{p}_1|\right)\left({E_2}^2 - c^2|\vec{p}_2|\right) = {m_1}^2c^4{m_2}^2c^4$$

Therefore

$\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right) \geq m_1m_2c^4$ or $\left(E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|\right) \leq -m_1m_2c^4$

If $p_1 = p_2$ we have for the second inequality

$${E_1}^2 - c^2|\vec{p}_1|^2 \leq -m_1m_2c^4$$

So we discard the second inequality

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2| \geq \sqrt{{E_1}^2 - c^2|\vec{p}_1|}\sqrt{{E_2}^2 - c^2|\vec{p}_2|} = m_1m_2c^4$$

We have

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2| \geq m_1m_2c^4$$

$$p_1.p_2 = E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|Cos\theta$$

$$E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|Cos\theta \geq E_1E_2 - c^2|\vec{p}_1||\vec{p}_2|$$

Therefore

$$p_{1.}p_2 \geq m_1m_2c^4$$

$$E_1E_2 - c^2\vec{p}_1.\vec{p}_2 \geq m_1m_2c^4 \; (6)$$

Further Consequences

From (5) we have

$$m_1\gamma_1c^2m_2\gamma_2c^2 - c^2m_1m_1\gamma_1\gamma_2\vec{v}_1.\vec{v}_2 \geq m_1m_2c^4$$

$$\gamma_1\gamma_2(c^2 - \vec{v}_1.\vec{v}_2) \geq c^2$$

$$c^2 - \vec{v}_1.\vec{v}_2 \geq \frac{c^2}{\gamma_1\gamma_2}$$

$$-\vec{v}_1.\vec{v}_2 \geq -c^2 + \frac{c^2}{\gamma_1\gamma_2}$$

Four dot product $v_1.v_2$ is given by

$$v_1.v_2 = c^2v_{t1}.v_{t2} - \vec{v}_1.\vec{v}_2 = c^2\gamma_1.\gamma_2 - \vec{v}_1.\vec{v}_2 \geq c^2\gamma_1.\gamma_2 - c^2 + \frac{c^2}{\gamma_1\gamma_2}$$

$$v_1.v_2 \geq c^2\left(\gamma_1\gamma_2 - 1 + \frac{1}{\gamma_1\gamma_2}\right)$$

If $\vec{v}_1 = \vec{v}_2$ then $\gamma_1 = \gamma_2$

$$v.v = c^2\left(\gamma^2 - 1 + \frac{1}{\gamma^2}\right) \; (7)$$

But the correct result is $v.v = c^2$

Thus we have arrived at a contradiction.

Conclusion

As claimed we have that the four dot product between two arbitrary velocities is less than the square of the speed of light in vacuum and the product of two four momenta is less than the product of the two masses involved and the square of the speed of light[in vacuuo]

References

1. Wikipedia, Four Velocity,https://en.wikipedia.org/wiki/Four-velocity
2. Janet C,MIT Course, https://uspas.fnal.gov/materials/12MSU/Conrad_4vector.pdf

3. Wikipedia, Four Momentum, https://en.wikipedia.org/wiki/Four-momentum