Two Body Motion: A Rudimentary Inspection

Anamitra Palit

Freelancer Physicist

palit.anamitra@gmail.com

Cell:91 9163892336

Abstract

The basic laws of mechanics, conservation of linear momentum and angular momentum have been considered along with a third equation—a new one-- to bring out conflicting features in the subject by considering two body motion.

Keywords: Two Body Motion, Linear Momentum, Angular Momentum, Torque ,Center of Mass

MSC: 00A79

Introduction

Two body motion[1] has been considered in view of conservation of linear and angular momentum along with a third new equation. The equations have been considered in the laboratory frame as well as in the center of mass[COM] frame in order to have a view of certain conflicting features.

We consider a two body motion with mutual interaction between $m_1$and m_2 proceeding in accordance with Newton's third law[strong form is necessary for our work]

Two Body Motion and the Basic Equations

The mass m_1 lies at the location $\vec{r}_1$and the mass m_2 lies at the location $\vec{r}_2$.The position vectors$\vec{r}_1$ and $\vec{r}_2$ are reckoned with respect to some inertial frame of reference. Total torque[2] is given by:

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$$

$\vec{\tau}_1$ and $\vec{\tau}_2$ are the torques on the masses m_1 and m_2 respectively

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

$\vec{F}_{12}$: Force on $m_1$due to m_2

$\vec{F}_{21}$: Force on m_2 due to m_1

Now

$$\vec{F}_{21} = -\vec{F}_{12}$$

We have also assumed Newton's third law in the strong form that is the two mentioned forces act along the straight line joining the two masses m_2 and m_2

Therefore,

$$\vec{\tau} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12}$$

$$= (\vec{r}_1 - \vec{r}_2) \times f(|\vec{r}_1 - \vec{r}_2|)\hat{r}_{12} = 0; \hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$\vec{\tau} = 0$$

The above implies the conservation of angular momentum[3].

The equation $\vec{\tau} = 0$ holds true in all inertial frames of reference for a two particle system. Force in general and hence the force of mutual interaction for our case is invariant in all inertial frames .

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

$$\vec{\tau} = \vec{r}_1 \times m_1 \frac{d^2\vec{r}_1}{dt^2} + \vec{r}_2 \times m_2 \frac{d^2\vec{r}_1}{dt^2}$$

$$\vec{\tau} = \frac{d}{dt}\left(\vec{r}_1 \times m_1 \frac{d\vec{r}_1}{dt}\right) + \frac{d}{dt}\left(\vec{r}_2 \times m_2 \frac{d\vec{r}_2}{dt}\right)$$

$$\vec{\tau} = 0 \Rightarrow \frac{d}{dt}\left(\vec{r}_1 \times m_1 \frac{d\vec{r}_1}{dt} + \vec{r}_2 \times m_2 \frac{d\vec{r}_2}{dt}\right) = 0$$

$$\vec{r}_1 \times m_1 \frac{d\vec{r}_1}{dt} + \vec{r}_2 \times m_2 \frac{d\vec{r}_2}{dt} = \vec{L}$$

$$\vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L}(1)$$

The above should hold in all inertial frames of reference for an isolated two particle system.

Next we calculate

$$\vec{r}_{12} \times \frac{d^2\vec{r}_{12}}{dt^2} = (\vec{r}_1 - \vec{r}_2) \times \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \left(\frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2}\right)$$

$$= (\vec{r}_1 - \vec{r}_2) \times \left(\frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2}\right)$$

$$= (\vec{r}_1 - \vec{r}_2) \times \left(\frac{\vec{F}_1}{m_1} + \frac{\vec{F}_1}{m_2}\right)$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) = 0; [\vec{F}_1 \parallel (\vec{r}_1 - \vec{r}_2)]$$

Therefore,

$$\vec{r}_{12} \times \frac{d^2\vec{r}_{12}}{dt^2} = 0$$

The above holds true for all inertial frames of reference since $\vec{F}_1 \parallel (\vec{r}_1 - \vec{r}_2)$ is valid for all inertial frames

$$\frac{d}{dt}\left(\vec{r}_{12} \times \frac{d\vec{r}_{12}}{dt}\right) = 0$$

$$\vec{r}_{12} \times \frac{d\vec{r}_{12}}{dt} = const$$

$$(\vec{r}_1 - \vec{r}_2) \times \left(\frac{d\vec{r}_1}{dt} - \frac{d\vec{r}_2}{dt}\right) = \vec{C}$$

$$(\vec{r}_1 - \vec{r}_2) \times \left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right) = \vec{C}\ (2)$$

The above equation is invariant in form in regard of all inertial frames of reference.

To sum up equations (1),(2) and (3) are invariant in form in the inertial frames of reference.

Earlier we had

$$\vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L}$$

From the conservation of linear momentum[4][5],

$$\vec{p}_1 + \vec{p}_2 = \vec{p}\ (3)$$

$\vec{p}$,$\vec{L}$ and $\vec{C}$ are constant vectors. Interestingly equation (1) embodies the fact that the interaction acts along the straight line joining the two masses and consequently the fact that change of momentum takes place only in such a direction. If Newton's third law was not valid in the strong form net torque could not have been zero and we could not have written equation (1).Equation (3) on its own does not take care of the fact that momentum can change only along the line joining the two masses. Equation (1) imposes the condition that accelerations can take place only along the direction on which the two masses lie.

To take note of the fact that equations(1) ,(2) and (3) retain their forms on all inertial frames of reference including the center of mass[6] frame of reference.

Center of Mass Frame Considerations

In the center of mass frame[COM frame] $\vec{p} = 0$. Therefore (2) reduces to,

$$(\vec{r}_1-\vec{r}_2)\times\vec{p}_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)=\vec{C}\ \ (4)$$

and (1) reduces to

$$(\vec{r}_1-\vec{r}_2)\times\vec{p}_1=\vec{L}\ (5)$$

Equations(4) and (5) are identical in the COM frame with $\vec{L}=\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}\vec{C}$ .Therefore equation[(1) and (3)] $\equiv$ [(2)and (3)] in both the COM frame and in the lab frame.

Deriving the Conflict

Equations (1) and (3) give us

$$(\vec{r}_1-\vec{r}_2)\times\vec{p}_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-(\vec{r}_1-\vec{r}_2)\times\frac{\vec{p}}{m_2}=\vec{C}\ (6)$$

Equations (2) and (3)lead to

$$\vec{r}_1\times\vec{p}_1+\vec{r}_2\times(\vec{p}-\vec{p}_1)=\vec{L}\ (7)$$

Equations (6)and (7) expected to be identical in the laboratory frame. Indeed by setting $\vec{p}=0$ we obtain identical equations for the COM frame. On transforming back to the lab frame we should have (6) and (7) as identical equations. That implies that in the lab frame

$$(\vec{r}_1-\vec{r}_2)\times\vec{p}_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-(\vec{r}_1-\vec{r}_2)\times\frac{\vec{p}}{m_2}=\lambda\big((\vec{r}_1-\vec{r}_2)\times\vec{p}_1+\vec{r}_2\times\vec{p}\big)=\vec{C}\ (8)$$

$$(\vec{r}_1-\vec{r}_2)\times\left[\vec{p}_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)-\frac{\vec{p}}{m_2}\right]=\lambda\big((\vec{r}_1-\vec{r}_2)\times\vec{p}_1+\vec{r}_2\times\vec{p}\big)=\vec{C}$$

$$(\vec{r}_1-\vec{r}_2).\left[(\vec{r}_1-\vec{r}_2)\times\left[\vec{p}_1\left(\frac{1}{m_1}-\frac{1}{m_2}\right)-\frac{\vec{p}}{m_2}\right]\right]=\lambda(\vec{r}_1-\vec{r}_2).\big((\vec{r}_1-\vec{r}_2)\times\vec{p}_1+\vec{r}_2\times\vec{p}\big)\ (9)$$

$$0=\lambda(\vec{r}_1-\vec{r}_2).(\vec{r}_2\times\vec{p})$$

$$(\vec{r}_1-\vec{r}_2).(\vec{p}\times\vec{r}_2)=0$$

$$\vec{p}.\big(\vec{r}_2\times(\vec{r}_1-\vec{r}_2)\big)=0$$

$$\vec{p}.(\vec{r}_2\times\vec{r}_1)=0\ (10)$$

Equation (10) will be valid if $\vec{p}$ lies in the plane containing $\vec{r}_1$and $\vec{r}_2$.The total momentum always has to lie in the plane containing $\vec{r}_1$and $\vec{r}_2$ [irrespective of the initial conditions].We may shift the position of the origin keeping it at rest with its original position. In the process we may change the plane containing

$\vec{r}_1$and $\vec{r}_2$ without changing the total momentum $\vec{p}$. If in the initial position of the origin equation (10) was valid , it will not be valid in the final position of the origin.

We recall the following equations relabeling them with the letters of the alphabet

$$\vec{p}_1 + \vec{p}_2 = 0 \ (A)$$

$$\vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L} \ (B)$$

$$(\vec{r}_1 - \vec{r}_2) \times \left(\frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}\right) = \vec{C} \ (C)$$

$[(A)$and $(B)] \equiv [(A)$and $(C)]$

But $(B) \not\equiv (C)$

Similarly

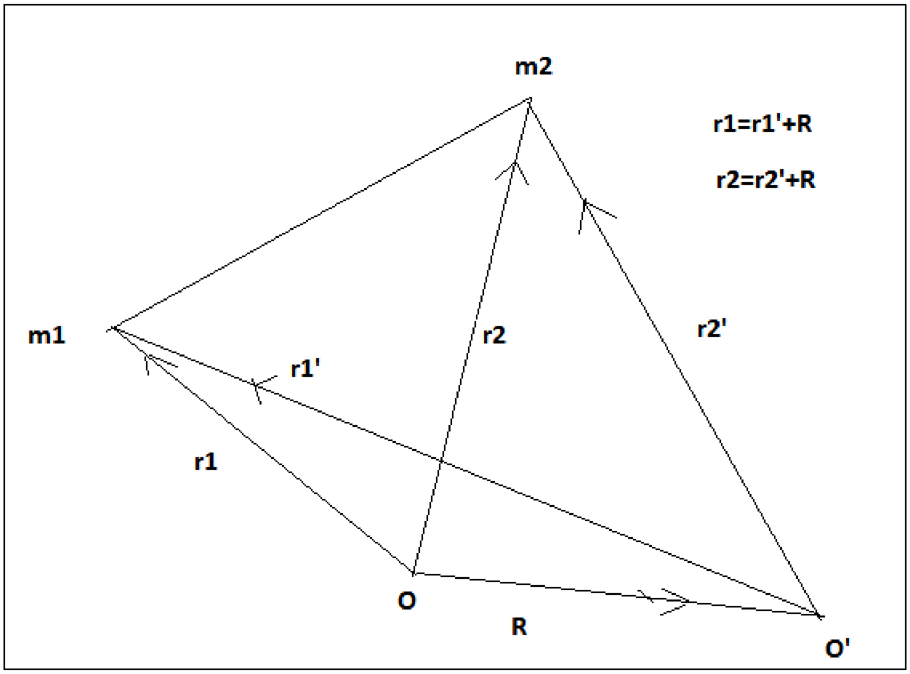
$[(3)$and $(1)] \equiv [(2)$and $(1)] \nRightarrow (3) \equiv (2)$

Conservation of Angular momentum equation, a Closer Look

We recall equation (1)

$$\vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L}$$

From O[inertial] we transform to a new inertial origin:



$$\vec{r}_1 = \vec{r}_1' + \vec{R}$$

$$\vec{r}_2 = \vec{r}_2' + \vec{R}$$

$$\frac{d\vec{r}_1}{dt} = \frac{d\vec{r}_1{}'}{dt} + \frac{d\vec{R}}{dt} \Rightarrow \vec{v}_1 = \vec{v}_1' + \frac{d\vec{R}}{dt} \Rightarrow m_1\vec{v}_1 = m_1\vec{v}_1' + m_1\frac{d\vec{R}}{dt} \Rightarrow \vec{p}_1 = \vec{p}_1' + \frac{d\vec{R}}{dt}$$

$$\frac{d\vec{r}_2}{dt} = \frac{d\vec{r}_2{}'}{dt} + \frac{d\vec{R}}{dt} \Rightarrow \vec{v}_2 = \vec{v}_2' + \frac{d\vec{R}}{dt} \Rightarrow m_2\vec{v}_2 = m_2\vec{v}_2' + m_2\frac{d\vec{R}}{dt} \Rightarrow \vec{p}_2 = \vec{p}_2' + \frac{d\vec{R}}{dt}$$

$$\left(\vec{r}_1' + \vec{R}\right) \times \left(\vec{p}_1' + m_1\frac{d\vec{R}}{dt}\right) + \left(\vec{r}_2' + \vec{R}\right) \times \left(\vec{p}_2' + m_2\frac{d\vec{R}}{dt}\right) = \vec{L}\ (11)$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{R} \times \vec{p}_1' + m_1\vec{r}_1' \times \frac{d\vec{R}}{dt} + m_1\vec{R} \times \frac{d\vec{R}}{dt} + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times \vec{p}_2' + m_2\vec{r}_2' \times \frac{d\vec{R}}{dt} + m_2\vec{R} \times \frac{d\vec{R}}{dt} = \vec{L}$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times (\vec{p}_1' + \vec{p}_2') + (m_1\vec{r}_1' + m_2\vec{r}_2') \times \frac{d\vec{R}}{dt} + (m_1 + m_2)\vec{R} \times \frac{d\vec{R}}{dt} = \vec{L}$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times (\vec{p}_1' + \vec{p}_2') + (m_1 + m_2)\vec{R}_{COM}' \times \frac{d\vec{R}}{dt} + (m_1 + m_2)\vec{R} \times \frac{d\vec{R}}{dt} = \vec{L}$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times \vec{p}' + (m_1 + m_2)\vec{R}'_{COM} \times \frac{d\vec{R}}{dt} + (m_1 + m_2)\vec{R} \times \frac{d\vec{R}}{dt} = \vec{L}$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times \vec{p}' + (m_1 + m_2)\vec{R}'_{COM} \times \vec{a} + (m_1 + m_2)\vec{R} \times \vec{a} = \vec{L}$$

[$\vec{p}$ and $\vec{a}$ are time independent vectors]

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' + \vec{R} \times \vec{p}' + (m_1 + m_2)\left(\vec{R}_{COM} + \vec{R}\right) \times \vec{a} = \vec{L}$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' = \vec{L} - \left(\vec{R} \times \vec{p}' + (m_1 + m_2)\left(\vec{R}_{COM} + \vec{R}\right) \times \vec{a}\right) = \vec{L}'$$

$$\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2' = \vec{L}'(12)$$

$\left(\vec{R} \times \vec{p}' + (m_1 + m_2)\left(\vec{R}_{COM} + \vec{R}\right) \times \vec{a}\right)$ should be time independent. Else equation (12) is not invariant in form. But we have seen in the derivation of equation (1) that net torque is zero for two body motion in all inertial frames of motion that this equation[equation (1) should be invariant in all, inertial frames.

Laboratory Frame Considerations[Independent of Transformations]

Without concerning ourselves with transformations[and the associated inconsistencies] we have the following:

Equations (1),(2), and (3) are independent. Using transformations we find that the equations in the COM frame corresponding to (1),(2) and (3) are the equations (4),(5) and (6). But there is some problematic issue ---contradictions----associated with the results ensuing from the transformations, as indicated earlier . Ignoring the transformations though such denial is hypothetical since physics is inevitably associated with transformations, we have the following…

In three dimensions , with equations (1) ,(2) and (3), we have nine independent scalar equations. For any given pair of locations given by $\vec{r}_1$ and $\vec{r}_2$ we have six unknowns the three components for each of $\vec{p}_1$ and $\vec{p}_2$. We have an over determined set.If $\vec{r}_1$ is specified we have nine scalar equations and nine unknowns three components of each $\vec{p}_1, \vec{p}_2$ and $\vec{r}_2$. These components are fixed up irrespective of initial conditions relating individually to $\vec{p}_1$ and $\vec{p}_2$. If the conservation of energy is taken into account then we have only two independent degrees of freedom, any two components of $\vec{r}_1$.

In two dimensions , with equations (1) ,(2) and (3), we have six independent scalar equations. For any given pair of locations given by $\vec{r}_1$ and $\vec{r}_2$ we have four unknowns: the two components for each of $\vec{p}_1$ and $\vec{p}_2$. We have an over determined set. If $\vec{r}_1$ is specified we have six scalar equations and six unknowns two components for of each $\vec{p}_1, \vec{p}_2$ and $\vec{r}_2$. These components are fixed up irrespective of initial conditions relating individually to$\vec{p}_1$ and $\vec{p}_2$ . . If the conservation of energy is taken into account then we have only one independent degree of freedom, any one component of $\vec{r}_1$

The Conservation of Energy Equation

The force equations in a two body[isolated two body] interaction

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = f(|\vec{r}_1 - \vec{r}_2|; m_1, m_2)(\vec{r}_1 - \vec{r}_2) \; (13.1)$$

Considering Newton's third law we have

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = -f(|\vec{r}_1 - \vec{r}_2|; m_1, m_2)(\vec{r}_1 - \vec{r}_2) \; (13.2)$$

Therefore,

$$\frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = f(|\vec{r}_1 - \vec{r}_2|; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(\vec{r}_1 - \vec{r}_2)$$

Let $\vec{r} = \vec{r}_1 - \vec{r}_2; \vec{v} = \vec{v}_1 - \vec{v}_2$

Now we have,

$$\frac{d^2\vec{r}}{dt^2} = f(|\vec{r}|; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{r} \; (14)$$

$$\frac{d\vec{r}}{dt}\frac{d^2\vec{r}}{dt^2} = f(r; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{r}\frac{d\vec{r}}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = f(r; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{1}{2}\frac{d(\vec{r}.\vec{r})}{dt}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = f(r; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{dr^2}{dt}$$

$$\frac{dv^2}{dt} = f(r; m_1, m_2)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{dr^2}{dt}$$

$$\mu dv^2 = 2f(r; m_1, m_2)rdr; \mu = \frac{m_1 m_2}{m_1 + m_2} \;\; (15)$$

$$\frac{1}{2}\mu \int_i^f dv^2 = \int_i^f f(r; m_1, m_2)rdr$$

i: initial; f: initial

Let $\int f(r; m_1, m_2)rdr = \Phi(r, m_1, m_2); \Phi(\mathrm{r}, m_1, m_2) = -\mathrm{U}(\mathrm{r}, m_1, m_2)$

We now have

$$\frac{1}{2}\mu {v_f}^2 - \frac{1}{2}\mu {v_i}^2 = \Phi(\mathrm{r_f}, m_1, m_2) - \Phi(\mathrm{r_i}, m_1, m_2) \; (16)$$

$$\frac{1}{2}\mu {v_f}^2 - \frac{1}{2}\mu {v_i}^2 = -\mathrm{U}(\mathrm{r_f}, m_1, m_2) + U(\mathrm{r_i}, m_1, m_2)$$

$$\vec{v}_i = \vec{v}_{1i} - \vec{v}_{2i}$$

$$\vec{v}_f = \vec{v}_{1f} - \vec{v}_{2f}$$

The Conservation of Energy equation:

$$\frac{1}{2}\mu {v_i}^2 + U(\mathrm{r_i}, m_1, m_2) = \frac{1}{2}\mu {v_f}^2 + \mathrm{U}(\mathrm{r_f}, m_1, m_2) = \mathrm{E} \;\; (17)$$

$$\frac{1}{2}\mu(\vec{v}_{1i} - \vec{v}_{2i})^2 + U(\mathrm{r_i}, m_1, m_2) = \frac{1}{2}\mu\left(\vec{v}_{1f} - \vec{v}_{2f}\right)^2 + \mathrm{U}(\mathrm{r_f}, m_1, m_2) = \mathrm{E}$$

$$\frac{1}{2}\mu({v_{1i}}^2 + {v_{2i}}^2 + 2\vec{v}_{1i}.\vec{v}_{2i}) + U(\mathrm{r_i}, m_1, m_2) = \frac{1}{2}\mu({v_{2i}}^2 + {v_{2i}}^2 + 2\vec{v}_{1i}.\vec{v}_{2i}) + \mathrm{U}(\mathrm{r_f}, m_1, m_2) = \mathrm{E}$$

$$\frac{1}{2}\mu\left(\frac{{p_{1i}}^2}{{m_1}^2} + \frac{{p_{2i}}^2}{{m_2}^2} + 2\frac{\vec{p}_{1i}.\vec{p}_{2i}}{m_1 m_2}\right) + U(\mathrm{r_i}, m_1, m_2) = \frac{1}{2}\mu\left(\frac{{p_{1f}}^2}{{m_1}^2} + \frac{{p_{2f}}^2}{{m_2}^2} + 2\frac{\vec{p}_{fi}.\vec{p}_{fi}}{m_1 m_2}\right) + U(\mathrm{r_i}, m_1, m_2) = \mathrm{E}\ (18)$$

Conclusion

We have analyzed two body motion in the laboratory frame as well as in the center of mass frame. The conflicting issue as asserted at the beginning has been brought out in the analysis.

References

1. Wikipedia,Two Body Problem https://en.wikipedia.org/wiki/Two-body_problem, accessed on 13.11.2019

2.Wikipedia, Torque Link https://en.wikipedia.org/wiki/Torque accessed on 13.11.2019

3. Wikipedia, Angular Momentum

Link: https://en.wikipedia.org/wiki/Angular_momentum#Conservation_of_angular_momentum,

accessed on 13.11.2019

4. Halliday D, Resnick R, Walker J, Fundamentals of Physics,John Wiley and Sons, 9th Edition, 2011, page 211-216

5. Wikipedia, Linear Momentum, Link: https://en.wikipedia.org/wiki/Momentum#Conservation, accessed on 13.11.2019

6. Halliday D, Resnick R, Walker J, Fundamentals of Physics,John Wiley and Sons, 9th Edition, 2011, p 201-203