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Exponential Factorization and Polar Decomposition of Multivectors in $C(p,q)$, $p+q \leq 3$

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Abstract. In this paper we consider general multivector elements of Clifford algebras $C(p,q)$, $p+q \leq 3$, and study multivector equivalents of polar decompositions and factorization into products of exponentials, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar).

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1. Introduction

The polar decomposition can be thought of as related via matrix isomorphisms to a factorization into an orthogonal matrix (the reverse gives the inverse of this factor) and a symmetric matrix (the corresponding multivector is self reverse, i.e. invariant under reversion).

Motivated by previous research into the polar decomposition of complexified quaternions and octonions [18], we look for exponential factorizations of general multivectors in Clifford algebras $m \in C(p,q)$, $p+q \leq 3$, $m = RS$, with the inverse of the first factor R^{-1} usually being given by reversion, and the second factor $S = \widetilde{S}$ being identical to its own reverse (corresponding to a symmetric matrix). We aim to split each factor further into elementary exponential functions of simple blades, usually resulting in exponentials with blades of all grades zero (scalars) to n (pseudoscalar) in the exponents. We further gained motivation and insight from [4], where also a form of polar decomposition is applied to rotors in $Cl(3,0)$ and $Cl(4,1)$. Since the factorizations obtained, easily allow to express the multivector inverse (compare

Dedicated to Rev. Ralph A. Smith (Mitaka, Tokyo, Japan) on the occasion of his 70th birthday.

also [13]) by reversing factor order and signs in the exponentials, we also provide these.

We treat the factorization in order of increasing dimension for Clifford algebras $Cl(p,q)$, $p+q=1,2$, and 3. The interesting cases of multivectors in $Cl(3,0)$ and $Cl(0,3)$ have important applications in the real three-dimensional world (e.g. in physics and robotics, etc.) and in Clifford analysis[5], and they are simpler than the case of mixed signature $Cl(2,1)$. Then we treat the case $Cl(1,2)$ via the isomorphism $Cl(1,2) \cong Cl(3,0)$. The case $Cl(2,1)$ follows last, since the mixed signature, and the presence of null blades and idempotents needs more case distinctions. An appendix features more technical details of $Cl(2,1)$ regarding the isomorphism $Cl(1,2) \cong Cl(2,0) \otimes Cl(1,0)$ and the use of idempotents, and multivector norms in $Cl(2,1)$.

Apart from the standard involutions of main (or grade) involution $\widehat{m}$, reverse $\widetilde{m}$, and Clifford conjugation $\overline{m}$, we also refer to principal involution (or transposition anti-involution) [1, 2, 3, 14] in real Clifford algebras) $\mathrm{pi}(m)$, which is a combination of reversion and negating the sign of every basis vector with negative square.

The paper is structured as follows. Section 2 begins with an elementary demonstration of this factorization in $Cl(1,0)$ and $Cl(0,1)$. Section 3 expands this to the case of $Cl(2,0)$. Applying the isomorphism $Cl(1,1) \cong Cl(2,0)$, Section 4 treats $Cl(1,1)$, while Section 5 deals with $Cl(0,2) \cong \mathbb{H}$ in analogy to quaternions but also introducing an exponential factorization with blades of grades zero, one and two in the exponents. Then we treat the factorizations of multivectors in $Cl(3,0)$ and $Cl(0,3)$ in Section 6. Next, Section 7 deals with the factorization in $Cl(1,2)$ by applying the isomorphism $Cl(1,2) \cong Cl(0,3)$. Section 8 explicitly treats multivector factorization in $Cl(2,1)$, making use of the isomorphism $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$. Finally, appendix A provides details of the isomorphism $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$, including a look at the role of idempotents in $Cl(2,1)$ and norm definitions for multivectors in $Cl(2,1)$.

2. Factorization in $Cl(1,0)$ and $Cl(0,1)$

The algebra $Cl(1,0)$ is isomorphic to hyperbolic numbers. A general element $m \in Cl(1,0)$ is given by

$$m = m_0 + m_1 e_1, \quad e_1^2 = 1, \quad m_0, m_1 \in \mathbb{R}. \tag{2.1}$$

It can be expressed in the form

$$m = (\beta + \alpha e_1) h_1(e_1), \tag{2.2}$$

such that $\beta > |\alpha| \geq 0$, and $h_1(e_1) = \pm 1$ or $h_1(e_1) = \pm e_1$. This allows to write m as *product of exponentials*

$$\begin{aligned} m &= e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_1 e_1} h_1(e_1), \\ \alpha_0 &= \tfrac{1}{2} \ln(\beta^2 - \alpha^2), \quad \alpha_1 = \operatorname{atanh}(\alpha/\beta). \end{aligned} \tag{2.3}$$

Regarding reversion symmetry we have

$$\widetilde{m} = m. \tag{2.4}$$

The inverse can be variously expressed as

$$m^{-1} = e^{-\alpha_0}\,\widehat{m'} = e^{-\alpha_0}\,\overline{m'}. \tag{2.5}$$

The algebra $Cl(0,1)$ is isomorphic to complex numbers $\mathbb{C}$. A general element $m \in Cl(0,1)$ is given by

$$m = m_0 + m_1 e_1, \quad e_1^2 = -1, \quad m_0, m_1 \in \mathbb{R}. \tag{2.6}$$

It can therefore be represented in the polar form of complex numbers

$$m = e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_1 e_1}, \quad \alpha = \tfrac{1}{2}\ln(m_0^2 + m_1^2), \quad \alpha_1 = \operatorname{atan2}(m_1, m_0). \tag{2.7}$$

The multivector is again self reverse $\widetilde{m} = m$, and the inverse can be written in three different ways as

$$m^{-1} = e^{-\alpha_0}\,\widehat{m'} = e^{-\alpha_0}\,\overline{m'} = e^{-\alpha_0}\,\mathrm{pi}(m'). \tag{2.8}$$

3. Factorization in $Cl(2,0)$

A general element $m \in Cl(2,0)$ can be represented as

$$\begin{aligned} &m = m_0 + m_1 e_1 + m_2 e_2 + m_{12} e_{12}, \\ &m_0, m_1, m_2, m_{12} \in \mathbb{R}, \quad e_{12}^2 = -1. \end{aligned} \tag{3.1}$$

We can rewrite m as

$$\begin{aligned} &m = m_1 e_1 + m_2 e_2 + m_0 + m_{12} e_{12} = a u' + b R, \\ &a = \sqrt{m_1^2 + m_2^2}, \quad b = \sqrt{m_0^2 + m_{12}^2}, \\ &u' = (m_1 e_1 + m_2 e_2)/a, \quad R = (m_0 + m_{12} e_{12})/b, \quad u'^2 = R\widetilde{R} = 1. \end{aligned} \tag{3.2}$$

If $m_1 e_1 + m_2 e_2 = 0$ or $m_0 + m_{12} e_{12} = 0$, then the factorization is already complete in the form

$$\begin{aligned} &m = bR = e^{\alpha_0} e^{\alpha_2 e_{12}}, \qquad \alpha_0 = \ln(b), \qquad \alpha_2 = \operatorname{atan2}(m_{12}, m_0), \\ &S = b = e^{\alpha_0}, \quad \widetilde{S} = S, \\ &R = e^{\alpha_2 e_{12}}, \quad R^{-1} = e^{-\alpha_2 e_{12}} = \widetilde{R} = \overline{R} = \mathrm{pi}(R), \quad m^{-1} = e^{-\alpha_0}\widetilde{R}, \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} &m = a u' = e^{\alpha_0'} u', \quad \alpha_0' = \ln(a), \quad \widetilde{m} = m, \\ &u'^{-1} = u' = \widetilde{u'} = \mathrm{pi}(u'), \quad m^{-1} = e^{-\alpha_0'} u'. \end{aligned} \tag{3.4}$$

We therefore assume from now on that both $m_1 e_1 + m_2 e_2 \neq 0$ and $m_0 e_2 + m_{12} e_1 \neq 0$, and compute

$$\begin{aligned} &m = a u' + b R = (a u' R^{-1} + b) R = (a u + b) R, \\ &u = u' R^{-1} = R u', \qquad u^2 = u u = \widetilde{u} u = u' \widetilde{R} R u' = 1 \end{aligned} \tag{3.5}$$

We can therefore always rewrite $m \in Cl(2,0)$ as

$$m = (\beta + \alpha u)h_1(u)R, \tag{3.6}$$

such that $\beta > \alpha \geq 0$ and $h_1(u) = 1$ or $h_1(u) = u$. This leads to the general factorization (with R from (3.2))

$$\begin{aligned}
m &= e^{\alpha_0} e^{\alpha_1 u} h_1(u) e^{\alpha_2 e_{12}} = SR, \\
\alpha_0 &= \tfrac{1}{2}\ln(\beta^2 - \alpha^2), \quad \alpha_1 = \operatorname{atanh}(\alpha/\beta), \quad \alpha_2 = \operatorname{atan2}(\langle R\rangle_2 e_{12}^{-1}, \langle R\rangle_0), \\
S &= e^{\alpha_0} e^{\alpha_1 u} h_1(u), \quad R = e^{\alpha_2 e_{12}} = S^{-1} m, \\
S^{-1} &= e^{-\alpha_0} e^{-\alpha_1 u} h_1(u), \quad e^{-\alpha_1 u} = \widehat{e^{-\alpha_1 u}} = \overline{e^{-\alpha_1 u}}, \\
\widetilde{S} &= S, \quad R^{-1} = e^{-\alpha_2 e_{12}} = \widetilde{R} = \overline{R} = \operatorname{pi}(R), \quad m^{-1} = R^{-1} S^{-1}.
\end{aligned} \tag{3.7}$$

4. Factorizing $Cl(1,1)$

The isomorphism $Cl(2,0) \cong Cl(1,1)$ with

$$1 = 1, \quad E_1 = e_1, \quad E_2 = e_{12}, \quad E_{12} = e_2, \tag{4.1}$$

where $\{e_1, e_2\}$ is the orthonormal basis of $\mathbb{R}^2$, and $\{E_1, E_2\}$ is the orthonormal basis of $\mathbb{R}^{1,1}$, allows to factorize $m \in Cl(1,1)$ by first isomorphically mapping it to $Cl(2,0)$, factorizing it there (as shown above in Section 3), and map the factorized result back to $Cl(1,1)$.

We get

$$m = m_0 + m_1 E_1 + m_2 E_2 + m_{12} E_{12} \overset{(4.1)}{=} m_0 + m_1 e_1 + m_2 e_{12} + m_{12} e_2. \tag{4.2}$$

To factorize this multivector $m_0 + m_1 e_1 + m_2 e_{12} + m_{12} e_2$ in $Cl(2,0)$, we simply exchange the places of m_2 and m_{12} in (3.1) to (3.7). And finally we map the factorization obtained back to $Cl(1,1)$ with (4.1). The inverse will be with $m = e^{\alpha_0} m' : m^{-1} = e^{\alpha_0} \overline{m'}$. Viewed strictly in $Cl(1,1)$, the exponential corresponding to $e^{\alpha_1 u}$ will no longer have a single grade one vector as argument, but a sum of vector plus bivector.

5. Factorization of $Cl(0,2)$

Because of the isomorphism to quaternions $Cl(0,2) \cong \mathbb{H}$ the result is straight forward

$$\begin{aligned}
m &= m_0 + m_1 e_1 + m_2 e_2 + m_{12} e_{12} = |m| e^{\alpha_2' i'} = e^{\alpha_0} e^{\alpha_2' i'}, \\
|m|^2 &= m\overline{m} = m_0^2 + m_1^2 + m_2^2 + m_{12}^2, \quad \alpha_0 = \ln(|m|), \\
i' &= \frac{m_1 e_1 + m_2 e_2 + m_{12} e_{12}}{\sqrt{m_1^2 + m_2^2 + m_{12}^2}}, \\
\alpha_2' &= \operatorname{atan2}(\sqrt{m_1^2 + m_2^2 + m_{12}^2}/|m|,\, m_0/|m|), \\
m^{-1} &= \overline{m}/|m| = e^{-\alpha_0} e^{-\alpha_2' i'}.
\end{aligned} \tag{5.1}$$

Indeed any factorization known for quaternions $\mathbb{H}$ can be realized via the isomorphism $Cl(0,2) \cong \mathbb{H}$ in $Cl(0,2)$ as well.

Furthermore, we can factorize $m \in Cl(0,2)$ in exponentials specified by grade

$$\begin{aligned} m &= e^{\alpha_0} e^{\alpha_1 u} e^{\alpha_2 e_{12}} \\ &= e^{\alpha_0}(\cos\alpha_1 \cos\alpha_2 + u \sin\alpha_1 \cos\alpha_2 + u\, e_{12} \sin\alpha_1 \sin\alpha_2 + e_{12} \cos\alpha_1 \sin\alpha_2), \\ u^2 &= -1, \qquad u \in \mathbb{R}^{0,2}, \end{aligned} \tag{5.2}$$

by computing

$$\begin{aligned} \alpha_0 &= \ln(|m|), \qquad \alpha_1 = \arccos\left(\frac{\sqrt{m_0^2 + m_{12}^2}}{|m|}\right), \\ \alpha_2 &= \operatorname{atan}(m_{12}/m_0), \qquad u = \frac{m_1 e_1 + m_2 e_2}{|m| \sin\alpha_1}\, e^{-\alpha_2 e_{12}}\,. \end{aligned} \tag{5.3}$$

For the factors S and R and for the inverse of m we then have, respectively,

$$\begin{aligned} S &= e^{\alpha_0} e^{\alpha_1 u}, \quad \widetilde{S} = S, \quad S^{-1} = e^{-\alpha_0} e^{-\alpha_1 u}, \\ R &= e^{\alpha_2 e_{12}}, \quad R^{-1} = \widetilde{R} = \overline{R} = \mathrm{pi}(R) = e^{-\alpha_2 e_{12}}, \\ m^{-1} &= R^{-1} S^{-1}. \end{aligned} \tag{5.4}$$

And no matter what factorization we choose (via isomorphism to quaternions, or directly as in (5.2)), after defining $m' = e^{-\alpha_0} m$, we can always express the inverse multivector by

$$m^{-1} = \overline{m}/|m| = e^{-\alpha_0}\, m'^{-1}, \quad m'^{-1} = \overline{m'} = \mathrm{pi}(m'). \tag{5.5}$$

6. Factorization in $Cl(3,0)$ and $Cl(0,3)$

Unit vectors u, unit bivectors i_2, and the central unit pseudoscalar $i = e_{123}$ in $Cl(3,0)$ square to

$$u^2 = +1, \qquad i_2^2 = -1, \qquad i^2 = -1. \tag{6.1}$$

Unit vectors u, unit bivectors i_2, and the central unit pseudoscalar $i = e_{123}$ in $Cl(0,3)$ square to

$$u^2 = -1, \qquad i_2^2 = -1, \qquad i^2 = +1. \tag{6.2}$$

The even subalgebras of both $Cl(3,0)$ and $Cl(0,3)$ are isomorphic to quaternions $\mathbb{H}$: $Cl_2(3,0) \cong Cl_2(0,3) \cong \mathbb{H}$. That means general multivectors m in $Cl(3,0)$ and $Cl(0,3)$ can always be represented as complex ($i^2 = -1$) or hyperbolic ($i^2 = +1$) (bi)quaternions:

$$m = p + iq, \tag{6.3}$$

where in both cases p and q are (isomorphic to) quaternions

$$p = a_p e^{\alpha_p i_p}, \quad q = a_q e^{\alpha_q i_q}, \quad a_p, a_q \in \mathbb{R}_0^+, \quad i_p^2 = i_q^2 = -1, \tag{6.4}$$

with bivectors $i_p, i_q \in Cl_2(3,0)$ or $\in Cl_2(0,3)$.

Remark 6.1. *Note that for $a_q = 0$ or $a_p = 0$ the factorization is already achieved in the form of*

$$m = a_p e^{\alpha_p i_p} = e^{\alpha_0} e^{\alpha_p i_p}, \;\; \alpha_0 = \ln a_p,$$
$$\textit{or} \quad m = i a_q e^{\alpha_q i_q} = i e^{\alpha_0'} e^{\alpha_q i_q}, \;\; \alpha_0' = \ln a_q. \tag{6.5}$$

In the rest of this Section, we therefore assume that both $a_p \neq 0$ and $a_q \neq 0$.

Clifford conjugation maps

$$i_p \to -i_p, \qquad i_q \to -i_q, \qquad i \to i. \tag{6.6}$$

Clifford conjugation applied to (6.4) is equivalent to quaternion conjugation. Therefore we obtain

$$\begin{aligned} m\overline{m} &= (p+iq)(\overline{p}+i\overline{q}) = p\overline{p} + i^2 q\overline{q} + i2\tfrac{1}{2}(p\overline{q}+q\overline{p}) \\ &= a_p^2 + i^2 a_q^2 + i2a_p a_q \cos(p,q) = r_0 + ir_3 \in \mathbb{R} + i\mathbb{R}, \end{aligned} \tag{6.7}$$

and $\cos(p,q)$ being the cosine of the four-dimensional (4D) angle between quaternions p, q, because $\frac{1}{2}(p\overline{q}+q\overline{p})$ expresses the inner (or scalar) product in four dimensions for quaternions. We thus have $|\cos(p,q)| \leq 1$, which means for the hyperbolic case ($i^2 = +1$): $r_0 = a_p^2 + a_q^2 \geq |2a_p a_q| \geq |r_3| = 2|a_p a_q \cos(p,q)|$.

We can always factorize $m\overline{m}$ and compute its square root as

$$m\overline{m} = e^{2\alpha_0} e^{2\alpha_3 i}, \qquad \sqrt{m\overline{m}} = e^{\alpha_0} e^{\alpha_3 i}, \tag{6.8}$$

with

$$e^{\alpha_0} = (r_0^2 - i^2 r_3^2)^{\frac{1}{4}}, \qquad \alpha_0 = \tfrac{1}{4}\ln(r_0^2 - i^2 r_3^2), \tag{6.9}$$

and

$$\alpha_3 = \frac{1}{2}\left\{\begin{array}{lll} \operatorname{atan2}(r_3, r_0) & \text{for} & m \in Cl(3,0) \\ \operatorname{atanh}(r_3/r_0) & \text{for} & m \in Cl(0,3) \end{array}\right. . \tag{6.10}$$

Next, we devide m by the central square root $\sqrt{m\overline{m}}$ and obtain the normed multivector

$$M = \frac{m}{\sqrt{m\overline{m}}} = m e^{-\alpha_0} e^{-\alpha_3 i}, \tag{6.11}$$

with unit norm

$$M\,\overline{M} = 1. \tag{6.12}$$

The resulting form of M will therefore be (similar to (6.3) and (6.4))

$$M = P + Qi = a_P e^{\alpha_P i_P} + i a_Q e^{\alpha_Q i_Q} = e^{\alpha_P i_P}(a_P + i a_Q e^{-\alpha_P i_P} e^{\alpha_Q i_Q}), \tag{6.13}$$

with

$$M\,\overline{M} = 1 = a_P^2 + i^2 a_Q^2. \tag{6.14}$$

The two quaternions P and Q can be computed explicitly as

$$P = \langle M \rangle_{\text{even}}, \qquad Q = \langle M \rangle_{\text{odd}}\, i^{-1}, \tag{6.15}$$

with amplitudes

$$a_P = \sqrt{P\overline{P}}, \qquad a_Q = \sqrt{Q\overline{Q}}, \tag{6.16}$$

unit bivectors

$$|\langle P\rangle_2| = \sqrt{-\langle P\rangle_2^2}, \qquad i_P = \frac{\langle P\rangle_2}{|\langle P\rangle_2|}, \tag{6.17}$$

$$|\langle Q\rangle_2| = \sqrt{-\langle Q\rangle_2^2}, \qquad i_Q = \frac{\langle Q\rangle_2}{|\langle Q\rangle_2|}, \tag{6.18}$$

and phase angles

$$\alpha_P = \operatorname{atan2}(\langle P\rangle_2 i_P^{-1}, \langle P\rangle_0), \qquad \alpha_Q = \operatorname{atan2}(\langle Q\rangle_2 i_Q^{-1}, \langle Q\rangle_0), \tag{6.19}$$

Computation of $M\overline{M}$ yields

$$\begin{aligned} M\overline{M} &= e^{\alpha_P i_P}(a_P + i a_Q e^{-\alpha_P i_P} e^{\alpha_Q i_Q})(a_P + i a_Q e^{-\alpha_Q i_Q} e^{\alpha_P i_P})e^{-\alpha_P i_P} \\ &= e^{\alpha_P i_P}\big(a_P^2 + i^2 a_Q^2 + i a_P a_Q (e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q})e^{\alpha_P i_P}\big)e^{-\alpha_P i_P} \\ &= a_P^2 + i^2 a_Q^2 + i a_P a_Q e^{\alpha_P i_P}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q} e^{\alpha_P i_P})e^{-\alpha_P i_P}. \end{aligned} \tag{6.20}$$

Because $M\overline{M} = a_P^2 + i^2 a_Q^2 = 1$ we must have the second term in round brackets to be zero

$$e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q} e^{\alpha_P i_P} = e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim} = 0. \tag{6.21}$$

We now analyze M further

$$\begin{aligned} M &= a_P e^{\alpha_P i_P} + i a_Q e^{\alpha_Q i_Q} = e^{\alpha_P i_P}(a_P + a_Q i(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - 0)) \\ &\overset{(6.21)}{=} e^{\alpha_P i_P}\big(a_P + a_Q i(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - \tfrac{1}{2} e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - \tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim})\big) \\ &= e^{\alpha_P i_P}\big(a_P + a_Q i \tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim})\big), \end{aligned} \tag{6.22}$$

where the term

$$\tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim}) = \langle e^{-\alpha_P i_P} e^{\alpha_Q i_Q}\rangle_2 \tag{6.23}$$

is a pure bivector. Multiplied with trivector i we get a vector with length ω and unit direction u, $u^2 = 1$ for $Cl(3,0)$, and $u^2 = -1$ for $Cl(0,3)$,

$$\omega u = i\langle e^{-\alpha_P i_P} e^{\alpha_Q i_Q}\rangle_2 = i\,\frac{a_P}{a_Q}\,\langle P^{-1}Q\rangle_2. \tag{6.24}$$

Thus in full generality, the multivector M can be represented by

$$M = e^{\alpha_P i_P}(a_P + a_Q\,\omega u) = (a_P + a_Q\,\omega u')e^{\alpha_P i_P}, \quad u' = e^{\alpha_P i_P} u e^{-\alpha_P i_P}. \tag{6.25}$$

Note that unit vector u', is simply a rotated version of u. Computing

$$M\overline{M} = e^{\alpha_P i_P}(a_P + a_Q\omega u)(a_P - a_Q\omega u)e^{-\alpha_P i_P} = \ldots = a_P^2 - u^2 a_Q^2 \omega^2, \tag{6.26}$$

shows by comparison with (6.14), that $\omega^2 = 1$, i.e. $\omega = 1$. Without restriction of generality, we can therefore express

$$\begin{aligned} M &= e^{\alpha_P i_P}(a_P + a_Q u) = (a_P + a_Q u')e^{\alpha_P i_P}, \\ M\overline{M} &= a_P^2 - u^2 a_Q^2 = a_P^2 - u'^2 a_Q^2 = 1, \end{aligned} \tag{6.27}$$

TABLE 1. Multiplication table of $Cl(3,0)$.

	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
1	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	1	e_{123}	$-e_2$	e_1	e_{12}
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	e_{31}	-1	$-e_{12}$	$-e_1$
e_{31}	e_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	e_{12}	-1	$-e_2$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_1$	$-e_2$	-1

We thus end up with

$$\begin{aligned} &M = e^{\alpha_2 i_2} e^{\alpha_1 u} = e^{\alpha_1 u'} e^{\alpha_2 i_2}, \\ &\alpha_1 = \left\{ \begin{array}{l} \operatorname{atanh}(a_Q/a_P) \ \text{ for } \ Cl(3,0) \\ \operatorname{atan2}(a_Q, a_P) \ \text{ for } \ Cl(0,3) \end{array} \right\}, \quad \alpha_2 = \alpha_P, \qquad i_2 = i_P. \end{aligned} \tag{6.28}$$

In total, we therefore have

$$\begin{aligned} m &= M\sqrt{m\overline{m}} = e^{\alpha_2 i_2} e^{\alpha_1 u} e^{\alpha_0} e^{\alpha_3 i} \\ &= e^{\alpha_0} e^{\alpha_2 i_2} e^{\alpha_1 u} e^{\alpha_3 i} = e^{\alpha_0} e^{\alpha_1 u'} e^{\alpha_2 i_2} e^{\alpha_3 i}. \end{aligned} \tag{6.29}$$

We can further write

$$\begin{aligned} &m = SR, \quad S = \widetilde{S} = e^{\alpha_0} e^{\alpha_1 u'}, \quad S^{-1} = e^{-\alpha_0} e^{-\alpha_1 u'}, \\ &R = e^{\alpha_2 i_2} e^{\alpha_3 i}, \quad R^{-1} = \widetilde{R} = e^{-\alpha_2 i_2} e^{-\alpha_3 i}, \\ &m^{-1} = R^{-1} S^{-1}. \end{aligned} \tag{6.30}$$

7. Factorization of $Cl(1,2)$

The results of the previous section lend themselves to factorize multivectors in $Cl(1,2) \cong Cl(3,0)$, based on the isomorphism $Cl(1,2) \cong Cl(3,0)$. We list the multiplication tables, Table 1 for $Cl(3,0)$ and Table 2 for $Cl(1,2)$. $Cl(1,2) \cong Cl(3,0)$ can be verified from Tables 1 and 2, which can be brought into agreement by indentifying

$$\begin{aligned} 1 = 1, E_1 = e_1, E_2 = e_{12}, E_3 = e_{31}, \\ E_{12} = e_2, E_{23} = e_{23}, E_{31} = e_3, E_{123} = e_{123}, \end{aligned} \tag{7.1}$$

where $\{E_1, E_2, E_3\}$ is the orthonormal vector basis of $\mathbb{R}^{1,2}$ generating $Cl(1,2)$, and $\{e_1, e_2, e_3\}$ is the orthonormal vector basis of $\mathbb{R}^3$ generating $Cl(3,0)$.

Factorization of multivectors $m \in Cl(1,2)$ can be achieved by mapping m via the isomorphism (7.1) to its isomorphic counterpart $m' \in Cl(3,0)$, then factorize m' in $Cl(3,0)$, and finally map the factorized form back with applying (7.1) again in reverse. In particular the unit vector $u \in Cl(3,0)$ and

TABLE 2. Multiplication table of $Cl(1,2) \cong Cl(3,0)$.

	1	E_1	E_{12}	E_{31}	E_2	E_{23}	E_3	E_{123}
1	1	E_1	E_{12}	E_{31}	E_2	E_{23}	E_3	E_{123}
E_1	E_1	1	E_2	$-E_3$	e_2	E_{123}	$-E_{31}$	E_{23}
E_{12}	E_{12}	$-E_2$	1	E_{23}	$-E_1$	E_{31}	E_{123}	E_3
E_{31}	E_{31}	E_3	$-E_{23}$	1	E_{123}	$-E_{12}$	E_1	E_2
E_2	E_2	$-E_{12}$	E_1	E_{123}	-1	$-E_3$	E_{23}	$-E_{31}$
E_{23}	E_{23}	E_{123}	$-E_{31}$	E_{12}	E_3	-1	$-E_2$	$-E_1$
E_3	E_3	E_{31}	E_{123}	$-E_1$	$-E_{23}$	E_2	-1	$-E_{12}$
E_{123}	E_{123}	E_{23}	E_3	E_2	$-E_{31}$	$-E_1$	$-E_{12}$	-1

the unit bivector i_2 in (6.29) become

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3 = u_1 E_1 + u_2 E_{12} + u_3 E_{31}, \tag{7.2}$$

$$i_2 = b_{12} e_{12} + b_{23} e_{23} + b_{31} e_{31} = b_{12} E_2 + b_{23} E_{23} + b_{31} E_3, \tag{7.3}$$

with $u_1^2 + u_2^2 + u_3^2 = 1$, and $b_{12}^2 + b_{23}^2 + b_{31}^2 = 1$.

Viewed strictly in $Cl(1,2)$, the exponentials corresponding to $e^{\alpha_1 u}$ and $e^{\alpha_2 i_2}$ will therefore no longer have a single grade one vector or a single grade two bivector as respective arguments, but in both cases a sum of vector plus bivector will appear as arguments.

8. Explicit approach for Cl(2,1)

See Appendix A for the details of the isomorphism $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$ and the role of idempotents in $Cl(2,1)$.

Let $m \in Cl(2,1)$, be a general multivector. We assume a factorization into two factors $m = RS$ with distinct symmetries under the reverse involution

$$\widetilde{R} = R^{-1}, \widetilde{R}R = R\widetilde{R} = 1, \qquad \widetilde{S} = S. \tag{8.1}$$

We *intend* to write R and S in terms of exponential functions as

$$R = e^{\alpha_2 i_2} h_2(i_2) e^{\alpha_3 i} h_3(i), \qquad S = e^{\alpha_0} e^{\alpha_1 u} h_1(u), \tag{8.2}$$

with four real phase angles $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \in \mathbb{R}$, unit vector $u \in \mathbb{R}^{p,q}$, unit bivector $i_2 \in Cl_2(p,q)$, and unit pseudoscalar $i = e_{123}$, $i^2 = 1$. The four phase angles, the vector u and the bivector i_2 can be determined as explained below.

Remark 8.1. *Because hyperbolic elements with positive square, like $u = e_1$, $i_2 = e_{23}$ or $i = e_{123}$ may occur, the factorization may require additional factors $\{\pm 1, \pm u, \pm i_2, \pm i\}$, encoded in $\{h_1(u), h_2(i_2), h_3(i)\}$, $h_1(u)^2 = h_2(i_2)^2 = h_3(i)^2 = 1$, $h_1(u)^{-1} = h_1(u)$, $h_2(i_2)^{-1} = h_2(i_2)$, $h_3(i)^{-1} = h_3(i)$, needed to cover all four segments of the respective hyperbolic plane, divided by the asymptotic lines through the origin in the directions $1 \pm u$, $1 \pm i_2$, and $1 \pm i$.*

Remark 8.2. *After establishing the explicit forms of R and S in* (8.2) *as detailed below, the computation of the inverse of m is straightforward, as*

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long as m is not itself a divisor of zero.

$$m^{-1} = S^{-1}R^{-1}, \quad R^{-1} = e^{-\alpha_2 i_2}h_2(i_2)e^{-\alpha_3 i}h_3(i) = e^{-\alpha_3 i}h_3(i)e^{-\alpha_2 i_2}h_2(i_2),$$
$$S^{-1} = e^{-\alpha_0}e^{-\alpha_1 u}h_1(u). \tag{8.3}$$

First, we compute the product of the reverse of m with m

$$\widetilde{m}m = \widetilde{(RS)}RS = \widetilde{S}\widetilde{R}RS = \widetilde{S}S = S^2 = a + bu, \quad u^2 = \begin{cases} -1, \\ 0, \\ +1. \end{cases}, \tag{8.4}$$

where $a, b \in \mathbb{R}$. We assume that $S^2 \neq 0$. But note that there are multivectors with zero square, e.g. e_1+e_3, $(e_1+e_3)^\sim = e_1+e_3$, $(e_1+e_3)^\sim(e_1+e_3) = e_1^2+e_3^2 = 1-1 = 0$. Therefore for $S^2 = 0$, m will have null vectors like e_1+e_3 as factors. These could be removed by contraction with $(e_1 - e_3)/2 = \mathrm{pi}(e_1 + e_3)/2$, because $(e_1 + e_3)\rfloor(e_1 + e_3)/2 = (e_1^2 - e_3^2)/2 = (1 + 1)/2 = 1$.

Note that $\widetilde{m}m = S^2$ is self reverse. That means it must be composed of scalars and vectors. The vector part $bu = \langle S^2\rangle_1$ of S^2 yields for $(bu)^2 \neq 0$,

$$u = \frac{\langle S^2\rangle_1}{|\langle S^2\rangle_1|}. \tag{8.5}$$

For unit vector u, $u^2 = -1$, we compute

$$e^{4\alpha_0} = a^2 + b^2, \quad \alpha_0 = \tfrac{1}{4}\ln(a^2 + b^2), \quad \alpha_1 = \tfrac{1}{2}\mathrm{atan2}(b, a), \tag{8.6}$$
$$S^2 = e^{2\alpha_0}e^{2\alpha_1 u} = e^{2\alpha_0}(\cos(2\alpha_1) + u\sin(2\alpha_1)), \quad S = e^{\alpha_0}e^{\alpha_1 u}, \tag{8.7}$$
$$R = mS^{-1} = me^{-\alpha_0}e^{-\alpha_1 u}, \quad \widetilde{R}R = 1. \tag{8.8}$$

The combination[1] $\alpha_1 u$ for $u^2 = 0$ can be defined as

$$\alpha_1 u = \tfrac{1}{2}\frac{bu}{a}, \quad \alpha_0 = \tfrac{1}{2}\ln(a), \quad S^2 = e^{2\alpha_0}e^{2\alpha_1 u} = e^{2\alpha_0}(1 + 2\alpha_1 u), \tag{8.9}$$
$$S = e^{\alpha_0}e^{\alpha_1 u}, \quad R = mS^{-1} = me^{-\alpha_0}e^{-\alpha_1 u}, \quad \widetilde{R}R = 1. \tag{8.10}$$

For $u^2 = 1$ we can always rewrite $S^2 \in \mathbb{R} + u\mathbb{R}$ in one of four forms

$$S^2 = \pm(\beta + \alpha u) \quad \text{or} \quad S^2 = \pm(\beta + \alpha u)u, \tag{8.11}$$

such that $\beta > |\alpha| \geq 0$. Then we can determine

$$e^{4\alpha_0} = \beta^2 - \alpha^2, \quad \alpha_0 = \tfrac{1}{4}\ln(\beta^2 - \alpha^2), \quad \alpha_1 = \tfrac{1}{2}\mathrm{atanh}(\alpha/\beta), \tag{8.12}$$
$$S^2 = e^{2\alpha_0}e^{2\alpha_1 u}h_1(u), \quad h_1(u) = \pm 1 \text{ or } \pm u. \tag{8.13}$$

We will divide m only by $e^{\alpha_0}e^{\alpha_1 u}$, which results in

$$R = me^{-\alpha_0}e^{-\alpha_1 u}, \quad \widetilde{R}R = h_1(u). \tag{8.14}$$

---

[1] We can exclude the case of $S = \alpha + a$, $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^{p,q}$, $S^2 = \langle S^2\rangle_1 \neq 0$, $S^4 = 0$, because $S^2 = (\alpha + a)^2 = \alpha^2 + 2\alpha a + a^2 = 2\alpha a$ means $\alpha^2 = -a^2$. And if a would square to zero (necessary for $S^4 = 0$), then also the real scalar α would have zero square, therefore α itself would be zero. But then $S = a$, and $S^2 = 0$, contrary to the assumption $S^2 \neq 0$.

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The R thus obtained can be expanded in the following way[2], abbreviating ch = cosh, sh = sinh,

$$\begin{aligned} R &= e^{\alpha_3 i} h_3(i) e^{\alpha_2 i_2} h_2(i_2) \\ &= (\mathrm{ch}\alpha_3 + i\,\mathrm{sh}\alpha_3) h_3(i) \begin{cases} \cos(\alpha_2) + i_2 \sin(\alpha_2), \;\; h_2(i_2) = 1, & \langle R \rangle_2^2 < 0, \\ 1 + \alpha_2 i_2, \;\; h_2(i_2) = 1, & \langle R \rangle_2^2 = 0, \\ \big(\mathrm{ch}(\alpha_2) + i_2\,\mathrm{sh}(\alpha_2)\big) h_2(i_2), & \langle R \rangle_2^2 > 0, \end{cases} \end{aligned} \tag{8.15}$$

because $i^2 = +1$ in $Cl(2,1)$. For brevity, we also use the notation $c_3 = \cosh(\alpha_3)$ and $s_3 = \sinh(\alpha_3)$. Further expanding R gives (ordered by grade from 0 to 3) for $\langle R \rangle_2^2 < 0$

$$R = \big(c_3 \cos\alpha_2 + s_3 \sin(\alpha_2)\, i i_2 + c_3 \sin(\alpha_2) i_2 + s_3 \cos(\alpha_2) i\big) h_3(i), \tag{8.16}$$

for $\langle R \rangle_2^2 > 0$

$$R = \big(c_3\,\mathrm{ch}\alpha_2 + s_3\,\mathrm{sh}(\alpha_2)\, i i_2 + c_3\,\mathrm{sh}(\alpha_2) i_2 + s_3\,\mathrm{ch}(\alpha_2) i\big) h_3(i) h_2(i_2), \tag{8.17}$$

and for $\langle R \rangle_2^2 = 0$

$$R = \mathrm{ch}(\alpha_3) + \mathrm{sh}(\alpha_3)\,\alpha_2\, i_2\, i + \mathrm{ch}(\alpha_3)\,\alpha_2\, i_2 + i\,\mathrm{sh}(\alpha_3). \tag{8.18}$$

For $\langle R \rangle_2^2 \neq 0$ we can compute

$$i_2 = \frac{\langle R \rangle_2}{|\langle R \rangle_2|}. \tag{8.19}$$

This allows us to compute for $\langle R \rangle_2^2 < 0$

$$\tan\alpha_2 = \frac{\langle R \rangle_2 i_2^{-1}}{\langle R \rangle_0}, \qquad \alpha_2 = \mathrm{atan2}\big(\langle R \rangle_2 i_2^{-1}, \langle R \rangle_0\big). \tag{8.20}$$

We can immediately divide R by $e^{\alpha_2 i_2}$ to obtain a linear combination of scalar and trivector, that can be expressed as

$$R e^{-\alpha_2 i_2} = \big\{ b + ai \quad \text{or} \quad (b + ai) i \big\} = (b + ai) h_3(i), \tag{8.21}$$

such that $b > |a| \geq 0$, and $h_3(i) = 1$ or $h_3(i) = i$. Note that an overall sign will already be accounted for in $e^{\alpha_2 i_2}$, $i_2^2 = -1$. Next we compute

$$\alpha_3 = \mathrm{atanh}(a/b), \tag{8.22}$$

and finally obtain the total factorization

$$R = e^{\alpha_2 i_2} \left\{ \begin{array}{ll} e^{\alpha_3 i} & \text{or} \\ e^{\alpha_3 i}\, i & \end{array} \right\} = e^{\alpha_2 i_2}\, e^{\alpha_3 i}\, h_3(i)\,. \tag{8.23}$$

For $\langle R \rangle_2^2 = 0$ it is easier to first analyze the combination $\langle R \rangle_0 + \langle R \rangle_3$ in the form of

$$\langle R \rangle_0 + \langle R \rangle_3 = \left\{ \begin{array}{ll} \pm(\beta + \alpha i) & \text{or} \\ \pm(\beta + \alpha i) i & \end{array} \right\} = (\beta + \alpha i) h_3(i), \tag{8.24}$$

[2]The case of $\langle R \rangle_3 = 0$ can only occur for $\alpha_3 = 0$, and then simply $\mathrm{ch}\alpha_3 = \cosh 0 = 1$, $\mathrm{sh}\alpha_3 = \sinh 0 = 0$.

such that $\beta > |\alpha| \geq 0$, and four possible values for $h_3(i) = \pm 1$ or $h_3(i) = \pm i$. We then obtain

$$\alpha_3 = \operatorname{atanh}(\alpha/\beta), \quad \langle R\rangle_0 + \langle R\rangle_3 = \left\{ \begin{array}{cc} \pm\, e^{\alpha_3 i} & \text{or} \\ \pm i\, e^{\alpha_3 i} & \end{array} \right\} = e^{\alpha_3 i} h_3(i). \tag{8.25}$$

Next, we divide R by the central factor $\langle R\rangle_0 + \langle R\rangle_3$ to obtain

$$R\big(\langle R\rangle_0 + \langle R\rangle_3\big)^{-1} = 1 + \alpha_2 i_2, \qquad \alpha_2 i_2 = \big\langle R(\langle R\rangle_0 + \langle R\rangle_3)^{-1}\big\rangle_2. \tag{8.26}$$

Therefore for $\langle R\rangle_2^2 = 0$ we obtain the total factorization of R as

$$\begin{aligned} R &= (1 + \alpha_2 i_2) \left\{ \begin{array}{cc} \pm\, e^{\alpha_3 i} & \text{or} \\ \pm i\, e^{\alpha_3 i} & \end{array} \right\} \\ &= (1 + \alpha_2 i_2) e^{\alpha_3 i} h_3(i) = e^{\alpha_2 i_2} e^{\alpha_3 i} h_3(i). \end{aligned} \tag{8.27}$$

For $\langle R\rangle_2^2 > 0$ we can ignore a total sign common to $\langle R\rangle_0$ and $\langle R\rangle_2$, which can be taken into account later in connection with the $e^{\alpha_3 i}$ factor. So we assume without loss of generality, that we can bring $\langle R\rangle_0 + \langle R\rangle_2$ in the following form

$$\langle R\rangle_0 + \langle R\rangle_2 = \big\{a + b i_2 \quad \text{or} \quad (a + b i_2) i_2\big\} = (a + b i_2) h_2(i_2), \tag{8.28}$$

with $a > |b| \geq 0$, and $h_2(i_2) = 1$ or $h_2(i_2) = i_2$. We can then compute the $\cosh \alpha_3$-factor common to $\langle R\rangle_0$ and $\langle R\rangle_2$ by

$$\cosh \alpha_3 = \sqrt{a^2 - b^2}. \tag{8.29}$$

Furthermore we get

$$\alpha_2 = \operatorname{atanh}(b/a), \tag{8.30}$$

and hence

$$\langle R\rangle_0 + \langle R\rangle_2 = \pm \cosh \alpha_3 \left\{ \begin{array}{l} e^{\alpha_2 i_2}, \\ e^{\alpha_2 i_2}\, i_2\,. \end{array} \right\} = \pm \cosh \alpha_3\, e^{\alpha_2 i_2}\, h_2(i_2). \tag{8.31}$$

Next we divide by $e^{\alpha_2 i_2} h_2(i_2)$ to obtain

$$R e^{-\alpha_2 i_2} h_2(i_2) = \left\{ \begin{array}{cc} \pm(\beta + \alpha i), & \text{or} \\ \pm(\beta + \alpha i)\, i & \end{array} \right\} = (\beta + \alpha i) h_3(i), \tag{8.32}$$

such that $\beta > |\alpha| \geq 0$, and $h_3(i) = \pm 1$ or $h_3(i) = \pm i$. Using α, β and $h_3(i)$ we compute

$$\alpha_3 = \operatorname{atanh}(\alpha/\beta), \qquad (\beta + \alpha i) h_3(i) = e^{\alpha_3 i} h_3(i). \tag{8.33}$$

Thus the full factorization of R for $\langle R\rangle_2^2 > 0$ results in

$$R = e^{\alpha_2 i_2} e^{\alpha_3 i}\, h_2(i_2) h_3(i). \tag{8.34}$$

In this way we have been able to explicitly compute the four real phase angles $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \in \mathbb{R}$, unit vector $u \in \mathbb{R}^{p,q}$, unit bivector $i_2 \in Cl_2(p,q)$. For the special cases of a null vector $u \neq 0$, $u^2 = 0$, we only obtain the

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product combination $\alpha_1 u$. Similarly for a null bivector $i_2 \neq 0$, $i_2^2 = 0$, we only obtain the product combination $\alpha_2 i_2$.

$$\begin{aligned} m &= e^{\alpha_2 i_2} e^{\alpha_3 i}\, h_2(i_2) h_3(i) e^{\alpha_0} e^{\alpha_1 u}\, h_1(u) \\ &= e^{\alpha_0}\, e^{\alpha_2 i_2}\, h_2(i_2)\, e^{\alpha_1 u}\, h_1(u)\, e^{\alpha_3 i}\, h_3(i). \end{aligned} \tag{8.35}$$

9. Conclusion

In this paper we have considered general elements of all Clifford algebras $C(p,q)$, $p+q \leq 3$, and studied multivector equivalents of polar decompositions and factorization into products of exponentials, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar). Depending on the algebra, we used methods of direct computation or applied several isomorphisms, to simplify the computation at hand or make use of known results in isomorpic representations. It may be possible in the future to extend this approach to even higher dimensional Clifford algebras. The present work can e.g. be applied in the study of Lipschitz versors, see e.g. E.4.2 in [19], pinor and spinor groups, etc.

Acknowledgments

EH wishes to thank God: *In the beginning God created the heavens and the earth* [6].

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Appendix A. Properties of $Cl(2,1)$

A.1. The isomorphism $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$

The multiplication tables for $Cl(2,1)$ (Table 3) and tensor product $Cl(2,0) \otimes Cl(1,0)$ (Table 4) are seen to be isomorphic by indentifying

$$\begin{aligned}(1,1) = 1, (E_1,1) = e_1, (E_2,1) = e_2, (-E_{12},h) = e_3,\\ (E_{12},1) = e_{12}, (E_1,h) = e_{23}, (E_2,h) = e_{31}, (1,h) = e_{123},\end{aligned} \tag{A.1}$$

TABLE 3. Multiplication table of $Cl(2,1)$.

	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
1	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	-1	e_{123}	e_2	$-e_1$	$-e_{12}$
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	e_{23}	e_{123}	$-e_3$	$-e_2$	e_{31}	1	e_{12}	e_1
e_{31}	e_{31}	e_3	e_{123}	e_1	$-e_{23}$	$-e_{12}$	1	e_2
e_{123}	e_{123}	e_{23}	e_{31}	$-e_{12}$	$-e_3$	e_1	e_2	1

TABLE 4. Multiplication table of $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$.

	$(1,1)$	$(E_1,1)$	$(E_2,1)$	$(-E_{12},h)$	$(E_{12},1)$	(E_1,h)	(E_2,h)	$(1,h)$
$(1,1)$	$(1,1)$	$(E_1,1)$	$(E_2,1)$	$(-E_{12},h)$	$(E_{12},1)$	(E_1,h)	(E_2,h)	$(1,h)$
$(E_1,1)$	$(E_1,1)$	$(1,1)$	$(E_{12},1)$	$(-E_2,h)$	$(E_2,1)$	$(1,h)$	(E_{12},h)	(E_1,h)
$(E_2,1)$	$(E_2,1)$	$(-E_{12},1)$	$(1,1)$	(E_1,h)	$(-E_1,1)$	$(-E_{12},h)$	$(1,h)$	(E_2,h)
$(-E_{12},h)$	$(-E_{12},h)$	(E_2,h)	$(-E_1,h)$	$(-1,1)$	$(1,h)$	$(E_2,1)$	$(-E_1,1)$	$(-E_{12},1)$
$(E_{12},1)$	$(E_{12},1)$	$(-E_2,1)$	$(E_1,1)$	$(1,h)$	$(-1,1)$	$(-E_2,h)$	(E_1,h)	(E_{12},h)
(E_1,h)	(E_1,h)	$(1,h)$	(E_{12},h)	$(-E_2,1)$	(E_2,h)	$(1,1)$	$(E_{12},1)$	$(E_1,1)$
(E_2,h)	(E_2,h)	$(-E_{12},h)$	$(1,h)$	$(E_1,1)$	$(-E_1,h)$	$(-E_{12},1)$	$(1,1)$	$(E_2,1)$
$(1,h)$	$(1,h)$	(E_1,h)	(E_2,h)	$(-E_{12},1)$	(E_{12},h)	$(E_1,1)$	$(E_2,1)$	$(1,1)$

where $\{e_1, e_2, e_3\} \in \mathbb{R}^{2,1}$ generate $Cl(2,1)$, $\{E_1, E_2\} \in \mathbb{R}^2$ generate $Cl(2,0)$, and $h \in \mathbb{R}^1$ generates $Cl(1,0)$.

That means a multivector in $Cl(2,1)$ can be expressed via the isomorphism as

$$\begin{aligned} m &= m_0 + m_1e_1 + m_2e_2 + m_3e_3 + m_{12}e_{12} + m_{23}e_{23} + m_{31}e_{31} + m_{123}e_{123} \\ &= m_0 + m_1e_1 + m_2e_2 + m_{12}e_{12} + (m_{123} + m_{23}e_1 + m_{31}e_2 - m_3e_{12})e_{123} \\ &\to \\ m' &= m_0(1,1) + m_1(E_1,1) + m_2(E_2,1) + m_{12}(E_{12},1) \\ &\qquad + m_{123}(1,h) + m_{23}(E_1,h) + m_{31}(E_2,h) + m_3(-E_{12},h) \\ &= (m_0 + m_1E_1 + m_2E_2 + m_{12}E_{12}, 1) \\ &\qquad + (m_{123} + m_{23}E_1 + m_{31}E_2 - m_3E_{12}, h) \\ &= (p,1) + (q,h), \end{aligned} \tag{A.2}$$

with $p, q \in Cl(2,0)$

$$\begin{aligned} p &= m_0 + m_1E_1 + m_2E_2 + m_{12}E_{12}, \\ q &= m_{123} + m_{23}E_1 + m_{31}E_2 - m_3E_{12}. \end{aligned} \tag{A.3}$$

A.2. Taking idempotents of $Cl(2,1)$ into account and multivector norms

Finally, we add some considerations that take the idempotent structure of $Cl(2,1)$ into account. Using Clifford conjugation (combining reversion and grade involution) allows to compute, see (6.2) in [13],

$$m\overline{m} = r_0 + ir_3 \in \mathbb{R} + i\mathbb{R}. \tag{A.4}$$

For $Cl(2,1)$ the pseudoscalar squares to $i^2 = +1$, and computation of the square root of $m\overline{m}$ requires $r_0 > 0$ and $|r_3| \leq r_0$. The idempotent representation

$$\begin{aligned} &id_+ = \frac{1+i}{2}, \quad id_- = \frac{1-i}{2}, \quad id_+ + id_- = 1, \quad id_+ - id_- = i, \\ &id_+^2 = id_+, \quad id_-^2 = id_-, \quad id_+ id_- = id_- id_+ = 0, \end{aligned} \tag{A.5}$$

leads to

$$\begin{aligned} r_0 + ir_3 &= r_0(id_+ + id_-) + r_3(id_+ - id_-) = (r_0 + r_3)id_+ + (r_0 - r_3)id_- \\ &= \sqrt{r_0 + r_3}^2 id_+ + \sqrt{r_0 - r_3}^2 id_- \\ &= (\sqrt{r_0 + r_3}id_+ + \sqrt{r_0 - r_3}id_-)^2 = \sqrt{r_0 + ir_3}^2. \end{aligned} \tag{A.6}$$

And the two coefficient square roots $\sqrt{r_0 + r_3}$, $\sqrt{r_0 - r_3}$, can only be computed if $r_0 + r_3 \geq 0$ and $r_0 - r_3 \geq 0$, that means $r_0 \geq |r_3| \geq 0$. Graphically, in the r_0, r_3-plane, the square roots of $r_0 + ir_3$ can be computed in the cone (segment of the plane, symmetric domain around r_0-axis, $r_0 > 0$) with $r_0 \geq |r_3|$, if we want to avoid the use of complex square roots.

Assuming for $Cl(2,1)$, that $r_0 \geq |r_3|$ is fulfilled, we can compute

$$\begin{aligned} \sqrt{m\overline{m}} &= \sqrt{r_0 + r_3}id_+ + \sqrt{r_0 - r_3}id_- \\ &= \tfrac{1}{2}(\sqrt{r_0 + r_3} + \sqrt{r_0 - r_3}) + \tfrac{1}{2}(\sqrt{r_0 + r_3} - \sqrt{r_0 - r_3})i. \end{aligned} \tag{A.7}$$

This can be represented as exponential by computing

$$e^{\alpha_0} = \left(r_0^2 - r_3^2\right)^{\frac{1}{4}}, \qquad \alpha_0 = \tfrac{1}{4}\ln\left(r_0^2 - r_3^2\right), \tag{A.8}$$

$$\alpha_3 = \tfrac{1}{2}\operatorname{atanh}(r_3/r_0), \qquad \sqrt{m\overline{m}} = e^{\alpha_0}e^{\alpha_3 i}. \tag{A.9}$$

Since $r_0 \geq |r_3|$ is restrictive to only one of four segments in the hyperbolic r_0, r_3-plane, it is of interest to investigate the other three segments of the r_0, r_3-plane. We now show, that even for $i^2 = +1$, the linear combination $r_0 + ir_3$ always can be cast into exponential form and additional factors ± 1 or $\pm i$ as follows.

For $r_3 = 0$, we will either have $r_0 = 0$ (origin of the r_0, r_3-plane, then m has zero determinant, and also no inverse), or we can represent the real scalar (on the r_0-axis) as $m\overline{m} = r_0 = \pm e^{2\alpha_0}$.

The more interesting case $r_3 \neq 0$, assuming $0 < \alpha < \beta$, leads to one of the following five situations

- $m\overline{m} = \pm\alpha i$ (here $r_0 = 0$)
- $m\overline{m} = \beta \pm \alpha i$
- $m\overline{m} = -\beta \mp \alpha i = -(\beta \pm \alpha i)$
- $m\overline{m} = \pm\alpha + \beta i = (\beta \pm \alpha i)i$
- $m\overline{m} = \mp\alpha - \beta i = -(\beta \pm \alpha i)i$

For the first item, we can compute $e^{2\alpha_0} = \alpha$, $\alpha_0 = \frac{1}{2}\ln(\alpha)$. For the remaining four items we can always compute $\alpha_3 = \frac{1}{2}\operatorname{atanh}(\pm\alpha/\beta)$ and $\alpha_0 = \frac{1}{4}\ln(\beta^2 - \alpha^2)$. We can therefore represent $m\overline{m}$, in one of the following three forms

$$m\overline{m} = \pm i\, e^{2\alpha_0} \quad \text{or} \tag{A.10}$$

$$m\overline{m} = \pm\, e^{2\alpha_0}(\cosh(2\alpha_3) + i\sinh(2\alpha_3)) = \pm\, e^{2\alpha_0} e^{2\alpha_3 i} \quad \text{or} \tag{A.11}$$

$$m\overline{m} = \pm i\, e^{2\alpha_0}(\cosh(2\alpha_3) + i\sinh(2\alpha_3)) = \pm i\, e^{2\alpha_0} e^{2\alpha_3 i}. \tag{A.12}$$

Since i with $i^2 = +1$ has no square root, only the second line allows to directly compute the square root of $m\overline{m}$. Note, that the negative sign in $-e^{2\alpha_0}e^{2\alpha_3 i}$ would necessitate the use of the complex imaginary unit $I \in \mathbb{C}$, $I^2 = -1$, which would necessarily drop out in the total factorization, because m is assumed to be real from the very beginning. We observe, that the third line subsumes the first when $\alpha_3 = 0$.

The square root of the exponential factors $e^{2\alpha_0}e^{2\alpha_3 i}$ can always be taken, using α_0 and α_3. We may therefore define for $i^2 = +1$ the special norm

$$|m|_+ = \sqrt{e^{2\alpha_0}e^{2\alpha_3 i}} = e^{\alpha_0}e^{\alpha_3 i}. \tag{A.13}$$

Division of m by this special norm leads to

$$M_+ = \frac{m}{|m|_+}, \qquad M_+\,\overline{M_+} = h_3(i) = \left\{ \begin{array}{l} \pm 1, \\ \pm i. \end{array} \right. \tag{A.14}$$

Finally, some further insights, which may be worth noting, can be gained by changing again to the isomorphism view point $Cl(2,1) \cong Cl(2,0) \otimes Cl(1,0)$. Every $m \in Cl(2,1)$ can be written as

$$\begin{aligned} m &= m_0 + m_1e_1 + m_2e_2 + m_3e_3 + m_{12}e_{12} + m_{23}e_{23} + m_{31}e_{31} + m_{123}e_{123} \\ &= m_0 + m_1e_1 + m_2e_2 + m_{12}e_{12} + (m_{123} + m_{23}e_1 + m_{31}e_2 - m_3e_{12})e_{123}, \end{aligned} \tag{A.15}$$

or with $p, q \in Cl(2,0)$,

$$p = m_0 + m_1e_1 + m_2e_2 + m_{12}e_{12}, \tag{A.16}$$

$$q = m_{123} + m_{23}e_1 + m_{31}e_2 - m_3e_{12}, \tag{A.17}$$

as hyperbolic ($i^2 = 1$) linear combination of two multivectors in $Cl(2,0)$ with

$$m = p + iq. \tag{A.18}$$

The product $m\overline{m}$ gives

$$m\overline{m} = p\overline{p} + q\overline{q} + 2i\tfrac{1}{2}(p\overline{q} + q\overline{p}), \tag{A.19}$$

where

$$p\overline{p} = p_0^2 + p_{12}^2 - p_1^2 - p_2^2 = \overline{p}p, \quad q\overline{q} = \overline{q}q, \tag{A.20}$$

$$\tfrac{1}{2}(p\overline{q} + q\overline{p}) = p_0q_0 + p_{12}q_{12} - p_1q_1 - p_2q_2 = \tfrac{1}{2}(\overline{p}q + \overline{q}p). \tag{A.21}$$

Therefore we also have

$$m\overline{m} = \overline{p}p + \overline{q}q + 2i\tfrac{1}{2}(\overline{p}q + \overline{q}p) = \overline{m}m. \tag{A.22}$$

A non-zero product $m\overline{m}$ can always be represented as

$$m\overline{m} = (\beta + \alpha i)\left\{ \begin{array}{l} \pm 1 \;\text{ or} \\ \pm i \end{array} \right\} = (\beta + \alpha i)h_3(i), \tag{A.23}$$

with $\beta > |\alpha| \geq 0$, and $h_3(i) = \pm$ or $h_3(i) = \pm i$.

For $h_3(i) = 1$, we have

$$m\overline{m} = e^{2\alpha_0} e^{2\alpha_3 i}, \qquad \sqrt{m\overline{m}} = e^{\alpha_0} e^{\alpha_3 i}, \tag{A.24}$$

and obtain the multivector factor M of unit norm (using Clifford conjugation)

$$M = \frac{m}{\sqrt{m\overline{m}}}, \qquad M\overline{M} = 1. \tag{A.25}$$

We can express M as

$$M = P + Qi, \tag{A.26}$$

with

$$P = M_0 + M_1 e_1 + M_2 e_2 + M_{12} e_{12}, \tag{A.27}$$

$$Q = M_{123} + M_{23} e_1 + M_{31} e_2 - M_3 e_{12}, \tag{A.28}$$

and because of $M\overline{M} = \overline{M}M = 1$ have

$$P\overline{P} + Q\overline{Q} = 1, \quad \tfrac{1}{2}(P\overline{Q} + Q\overline{P}) = \tfrac{1}{2}(\overline{P}Q + \overline{Q}P) = 0, \tag{A.29}$$

that means

$$P\overline{Q} + \overline{P\overline{Q}} = \overline{P}Q + \overline{\overline{P}Q} = 0. \tag{A.30}$$

Assuming that $P\overline{P}$ is not zero (i.e. P is invertible) we can rewrite

$$\begin{aligned} M &= P(1 + iP^{-1}Q) = P(1 + i\frac{1}{P\overline{P}}\overline{P}Q) = P(1 + i\frac{1}{P\overline{P}}(\overline{P}Q - 0)) \\ &= P(1 + i\frac{1}{P\overline{P}}(\overline{P}Q - \tfrac{1}{2}\overline{P}Q - \tfrac{1}{2}\overline{Q}P)) \\ &= P(1 + i\frac{1}{P\overline{P}}\tfrac{1}{2}(\overline{P}Q - \overline{Q}P)). \end{aligned} \tag{A.31}$$

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