Tutorial: Binary Star Masses from Newton's Laws

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Abstract

Newton's Law of Gravitational Force together with his Second and Third Laws of Motion implies that the inverse of the ratio of the acceleration magnitudes of the two stars of a binary system equals their mass ratio, and that the difference of their accelerations is proportional to their mass sum as well as being inversely proportional to the square of their separation, which causes their vector separation to trace out an elliptical orbit. The period and major axis length of that elliptical orbit yield the two stars' mass sum. The complete orbit's data isn't needed; five or more of the points which lie on an ellipse determine it, and the orbit sweeps out the area enclosed by that ellipse at a constant rate.

The Newtonian gravitational accelerations of the stars of a binary system

Newton's Law of Gravitational Force together with his Second Law of Motion implies the following pair of coupled equations of motion for the two stars of a binary system,

$$m_1\ddot{\mathbf{r}}_1 = -Gm_1m_2(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|^3 \quad \text{and} \quad m_2\ddot{\mathbf{r}}_2 = -Gm_2m_1(\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_2 - \mathbf{r}_1|^3, \tag{1a}$$

which, since the forces $-Gm_1m_2(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|^3$ and $-Gm_2m_1(\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_2 - \mathbf{r}_1|^3$ *are equal and opposite*, is consonant with Newton's Third Law of Motion. Therefore *summing* this pair of equations yields simply,

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = \mathbf{0} \quad \Rightarrow \quad (m_1/m_2)\ddot{\mathbf{r}}_1 = -\ddot{\mathbf{r}}_2 \quad \Rightarrow \quad (m_1/m_2) = (|\ddot{\mathbf{r}}_2|/|\ddot{\mathbf{r}}_1|), \tag{1b}$$

so *the ratio* (m_1/m_2) *of the masses* of the two stars of a binary system is equal *to the inverse* $(|\ddot{\mathbf{r}}_2|/|\ddot{\mathbf{r}}_1|)$ *of the ratio of the magnitudes* $|\ddot{\mathbf{r}}_1|$ *and* $|\ddot{\mathbf{r}}_2|$ *of their opposed-direction accelerations* $\ddot{\mathbf{r}}_1$ *and* $\ddot{\mathbf{r}}_2$.

The two Eq. (1a) coupled Newtonian equations of motion *can also be written*,

$$\ddot{\mathbf{r}}_1 = -Gm_2(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|^3 \quad \text{and} \quad \ddot{\mathbf{r}}_2 = -Gm_1(\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_2 - \mathbf{r}_1|^3, \tag{1c}$$

which shows that the acceleration of each star *is independent of its own mass*, in accord with the gravitational principle of equivalence. *Subtracting* this pair of equations yields the equation of motion *for the vector separation* $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$ *of the two stars*,

$$\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3, \text{ where } \mathbf{r} \stackrel{\text{def}}{=} (\mathbf{r}_1 - \mathbf{r}_2) \text{ and } M \stackrel{\text{def}}{=} (m_1 + m_2), \tag{1d}$$

which purely for reasons of familiarity of terminology, can also conveniently be presented as,

$$m\ddot{\mathbf{r}} = -GmM\mathbf{r}/|\mathbf{r}|^3, \text{ where } m \stackrel{\text{def}}{=} m_1m_2/(m_1 + m_2) = m_1m_2/M \text{ has the name "reduced mass".} \tag{1e}$$

Since in Eq. (1e), $mM = m_1m_2$ and $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$, the Eq. (1e) "force" $-GmM\mathbf{r}/|\mathbf{r}|^3$ is equal to the Eq. (1a) force $-Gm_1m_2(\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|^3$. The Eq. (1e) presentation of the Eq. (1d) vector-separation equation of motion sanctions the use of familiar terminology such as "force" for $-GmM\mathbf{r}/|\mathbf{r}|^3$, "angular momentum" $\mathbf{L}$ for $m(\mathbf{r} \times \dot{\mathbf{r}})$ and "energy" E for $m(|\dot{\mathbf{r}}|^2/2 - GM/|\mathbf{r}|)$.

In the next section we find, by applying Eq. (1d), that the vector separation $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$ of the two stars of a binary system traces out an elliptical orbit whose period and major-axis length yields, in conjunction with the universal gravitational constant G, *the sum* $(m_1 + m_2) = M$ *of the two stars' masses.*

The sum of the two stars' masses from their vector-separation elliptical orbit

The Eq. (1d) vector-separation equation of motion $\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3$ yields "angular momentum" conservation,

$$d(\mathbf{L}/m)/dt = d(\mathbf{r} \times \dot{\mathbf{r}})/dt = (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) + (\mathbf{r} \times \ddot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) - GM(\mathbf{r} \times \mathbf{r})/|\mathbf{r}|^3 = \mathbf{0}. \tag{2a}$$

Since $(\mathbf{L}/m) = (\mathbf{r} \times \dot{\mathbf{r}})$ *is a constant vector*, $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ *are always confined to the plane perpendicular to that constant vector*, i.e., $\mathbf{r}(t)$ *is planar*. Thus $|\mathbf{r} \times \dot{\mathbf{r}}| = (|\mathbf{L}|/m)$ *alone* is relevant, so we define $L \stackrel{\text{def}}{=} |\mathbf{L}|$.

The Eq. (1d) vector-separation equation of motion $\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3$ also yields "energy" conservation,

$$\begin{gathered} d(E/m)/dt = d\big(|\dot{\mathbf{r}}|^2/2 - GM/|\mathbf{r}|\big)/dt = \\ d\big((\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/2 - GM/(\mathbf{r} \cdot \mathbf{r})^{\frac{1}{2}}\big)/dt = (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) + GM\big((\dot{\mathbf{r}} \cdot \mathbf{r})/(\mathbf{r} \cdot \mathbf{r})^{\frac{3}{2}}\big) = \dot{\mathbf{r}} \cdot \big(\ddot{\mathbf{r}} + GM\mathbf{r}/|\mathbf{r}|^3\big) = 0. \end{gathered} \tag{2b}$$

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Having shown that the Eq. (1d) equation of motion $\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3$ for the vector-separation $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$ of the two stars implies the conservation relations $(\mathbf{r} \times \dot{\mathbf{r}}) = (\mathbf{L}/m)$ and $(|\dot{\mathbf{r}}|^2/2 - GM/|\mathbf{r}|) = (E/m)$, we would like *to solve these conservation relations* for the *locus* of that two-star vector-separation orbit, from which we in turn would like to obtain enough information to determine $M = (m_1 + m_2)$, the sum of the two stars' masses. Since we now know that this orbit is *planar*, the first thing we need to do is to express these conservation relations in *two-dimensional* polar coordinates, which have the following properties,

$$\mathbf{r} = (r\cos\theta,\, r\sin\theta), \quad |\mathbf{r}| = r,$$
$$\dot{\mathbf{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta,\, \dot{r}\sin\theta + r\dot{\theta}\cos\theta), \quad |\dot{\mathbf{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2, \quad |\mathbf{r}\times\dot{\mathbf{r}}| = r^2|\dot{\theta}|. \tag{3a}$$

Expressed in these two-dimensional polar coordinates $|\mathbf{r}\times\dot{\mathbf{r}}| = (L/m)$ becomes,

$$r^2|\dot{\theta}| = (L/m), \tag{3b}$$

and in these coordinates $(|\dot{\mathbf{r}}|^2/2 - GM/|\mathbf{r}|) = (E/m)$ becomes,

$$\big(\dot{r}^2 + r^2\dot{\theta}^2\big)/2 - GM/r = (E/m) \tag{3c}$$

We don't try to solve Eqs. (3c) *and* (3b) *for both $r(t)$ and $\theta(t)$*; we instead obtain *only the orbit's locus* $r(\theta)$ from these equations. We insert the relation $\dot{r}^2 = (dr/d\theta)^2\dot{\theta}^2$ into Eq. (3c), and then use Eq. (3b) to substitute $(L/m)^2 r^{-4}$ for $\dot{\theta}^2$ in the result, which yields the following differential equation for the locus $r(\theta)$,

$$(L/m)^2 r^{-4}\big((dr/d\theta)^2 + r^2\big)/2 - GM/r = (E/m). \tag{3d}$$

The disquieting factor r^{-4} in Eq. (3d) is eliminated upon changing the dependent variable from r to $u = (1/r)$ because $dr/d\theta = -u^{-2}(du/d\theta)$. After that change of dependent variable, Eq. (3d) reads,

$$(L/m)^2\big((du/d\theta)^2 + u^2\big)/2 - GMu = (E/m) \tag{3e}$$

Eq. (1d) *is satisfied by circular orbits $r(t) = \rho_0 > 0$ provided that $\dot{\theta}^2 = GM/\rho_0^3$*. Eq. (3e) *is correspondingly satisfied by circular loci $u(\theta) = 1/\rho_0$ provided that $(L/m)^2 = GM\rho_0$ and $(E/m) = -(GM/\rho_0)/2$*. Circular orbits *aren't general solutions of* Eq. (1d) *however, because $M \to 0$ suppresses the gravitational force, which allows* Eq. (1d) *to be satisfied by arbitrary constant-velocity straight-line trajectories* (Newton's First Law of Motion). In two-dimensional *rectangular* coordinates all straight-line loci have the form,

$$ax - by = \big(a^2 + b^2\big)^{\frac{1}{2}}\rho_0,$$

which is changed to *polar* coordinates *by substituting* $(r\cos\theta)$ *for x and* $(r\sin\theta)$ *for y*, with the result,

$$r\Big[\cos\theta\Big(a/\big(a^2+b^2\big)^{\frac{1}{2}}\Big) - \sin\theta\Big(b/\big(a^2+b^2\big)^{\frac{1}{2}}\Big)\Big] = \rho_0.$$

An angle δ that satisfies $\cos\delta = a/(a^2+b^2)^{\frac{1}{2}}$ and $\sin\delta = b/(a^2+b^2)^{\frac{1}{2}}$ of course *always exists*. That, together with the fact that $u = 1/r$, allows all straight-line loci in two dimensions to be written,

$$u(\theta) = \cos(\theta + \delta)/\rho_0. \tag{3f}$$

In *the $M \to 0$ limit of no gravitational force,* substitution of Eq. (3f) into Eq. (3e) yields,

$$(L/(m\rho_0))^2/2 = (E/m), \tag{3g}$$

which always entails *nonnegative energy E, as expected for no gravitational force and straight-line loci*.

We have now exhibited *two completely different special classes of solutions of the* Eq. (3e) *gravitational orbit locus differential equation*, namely the *bounded* circular loci $u(\theta) = 1/\rho_0$ and the *unbounded* straight-line loci $u(\theta) = \cos(\theta + \delta)/\rho_0$. It thus seems not unreasonable *to guess that the general solutions $u(\theta)$ of* Eq. (3e) *are linear-combination hybrids of these two opposite-character special solutions*, namely,

$$u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0. \tag{3h}$$

Inserting the Eq. (3h) circle/straight-line hybrid *guess* $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ into Eq. (3e) produces,

$$(L/(m\rho_0))^2\big(-2\beta\cos(\theta+\delta) + 1 + \beta^2\big)/2 - (GM/\rho_0)\big(-\beta\cos(\theta+\delta) + 1\big) = (E/m). \tag{3i}$$

Since in Eq. (3i) *the coefficient of* $\cos(\theta + \delta)$ *must vanish*, the *first consequence* of Eqs. (3h) and (3e) is that,

$$GM = \big((L/m)^2/\rho_0\big). \tag{3j}$$

Putting this result *back into* Eq. (3i) yields the *second consequence* of Eqs. (3h) and (3e),

$$(E/m) = (L/(m\rho_0))^2\big(\beta^2 - 1\big)/2 = (GM/\rho_0)\big(\beta^2 - 1\big)/2, \tag{3k}$$

from which we see that the energy E of this gravitational system *is negative only if* $\beta^2 < 1$, namely only if the Eq. (3h) circle/straight-line hybrid orbit locus $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ is more a circle than it is a straight line. Further on we deal with this orbit locus in more transparent detail *by converting it to two-dimensional rectangular coordinates*; it is difficult to become accustomed to the fact that very large values of $r(\theta)$ produce very small, innocuous-looking values of $u(\theta) = 1/r(\theta)$. In rectangular coordinates it is also *transparent* that this orbit locus *is always a conic section*. We already know that it is a *circle* when $\beta = 0$, but *a better overview* also shows that it is an *ellipse* when $0 < \beta^2 < 1$, a *parabola* when $\beta^2 = 1$ and a *hyperbola* when $\beta^2 > 1$. Thus *it is little wonder* that the Eq. (3k) *energy* of this gravitational system *is negative only if* $\beta^2 < 1$, *namely only if its orbit locus is a circle or ellipse.*

Eq. (3j) *is the key to obtaining* $M = m_1 + m_2$ from the universal gravitational constant G *and suitable input from the vector-separation orbit* $\mathbf{r}(t) = (\mathbf{r}_1(t) - \mathbf{r}_2(t))$ *of the two stars*. The *ingredients* which enter into the *right side* of Eq. (3j) are the parameter ρ_0 of the Eq. (3h) orbit locus $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ and the Eq. (3b) *conserved dynamical orbit entity* $(L/m) = r^2|\dot\theta|$. Regarding $r^2|\dot\theta|$, since the infinitesimal planar area $|dA|$ that corresponds to an infinitesimal angular arc $|d\theta|$ of the orbit is,

$$|dA| = \tfrac{1}{2}r(r|d\theta|) = r^2|d\theta|/2,$$

the planar area which the orbit sweeps out per unit time is,

$$|dA/dt| = r^2|\dot\theta|/2 = (L/m)/2. \tag{3l}$$

It was Johannes Kepler who first realized, in the course of studying the precise planetary-orbit observations of Tycho Brahe, *that* $|dA/dt|$ *is a conserved dynamical orbit entity*. Thus it has been *routine* for around 400 years *for astronomers to read off* $(L/m) = 2|dA/dt|$ *from orbital data. Alternatively, if the period* T *of the orbit has been observed*, then since $|dA/dt|$ is constant in time,

$$(L/m) = 2|dA/dt| = 2A/T, \tag{3m}$$

where A *is the area enclosed by the complete elliptical orbit locus* $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ *with* $\beta^2 < 1$. In terms of ρ_0 and β (where $\beta^2 < 1$), the area A enclosed by the complete elliptical orbit locus is,

$$A = \pi\,\rho_0^2/\big(1 - \beta^2\big)^{\frac{3}{2}} = \pi\,r_0R_0,\ \text{where } r_0 \stackrel{\text{def}}{=} \rho_0/(1 - \beta^2)^{\frac{1}{2}} \text{ and } R_0 \stackrel{\text{def}}{=} \rho_0/(1 - \beta^2). \tag{3n}$$

This r_0 *and* R_0 *are the respective half-lengths of the minor and major ellipse axes*. From Eqs. (3m) and (3n),

$$(L/m) = 2A/T = (2\pi/T)\,\rho_0^2/\big(1 - \beta^2\big)^{\frac{3}{2}} = (2\pi/T)\,r_0R_0. \tag{3o}$$

Insertion of Eq. (3o) into Eq. (3j) *yields for the sum of the masses of the two stars*,

$$(m_1 + m_2) = M = G^{-1}\big((L/m)^2/\rho_0\big) = G^{-1}(2\pi/T)^2\big(\rho_0/\big(1 - \beta^2\big)\big)^3 = G^{-1}(2\pi/T)^2R_0^3. \tag{3p}$$

In view of Eq. (3p) a simple observational link to R_0 would be welcome. One observational link to R_0 and r_0 arises from the fact that the orbital perigee and apogee distances, *the smallest and greatest distances between the two stars*, are respectively $R_0 \mp (R_0^2 - r_0^2)^{\frac{1}{2}}$, so R_0 *is the arithmetic mean of the smallest and greatest distances between the two stars*, whereas r_0 *is the geometric mean of those smallest and greatest distances*.

It is worth mentioning that the parameters ρ_0 and β of the elliptical locus $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ are related to r_0 and R_0 by $\rho_0 = r_0^2/R_0$, $\beta = (1 - (r_0/R_0)^2)^{\frac{1}{2}}$ and $(1 - \beta^2) = (r_0/R_0)^2$; *the inverse relations* $r_0 = \rho_0/(1 - \beta^2)^{\frac{1}{2}}$ *and* $R_0 = \rho_0/(1 - \beta^2)$ *which apply when* $\beta^2 < 1$ *were noted in* Eq. (3n).

Below Eq. (3l) we mentioned that $(L/m) = 2|dA/dt|$ can simply be read off from orbital data, which is doubtless *the best option*. *Alternatively*, if the period T of the orbit has been observed, we have from Eq. (3o) that $(L/m) = (2\pi/T)\, r_0 R_0$. Below Eq. (3p) we mentioned that the basic elliptical-locus parameters R_0 and r_0 are respectively the arithmetic and geometric means of the smallest and greatest distances between the two stars. *Alternatively*, just as a circle is in principle determined by three or more of the points which lie on it, an elliptical locus is in principle determined by five or more of the points which lie on it, *the best option*.

We have also mentioned that the Eq. (3h) representation $u(\theta) = (1 - \beta\cos(\theta + \delta))/\rho_0$ of the orbit locus *is as opaque* as it is *simple*. To *remedy* its opacity *we convert it to rectangular coordinates*. Since $u = 1/r$, $\cos(\theta + \delta) = \cos\theta\cos\delta - \sin\theta\sin\delta$, $x = r\cos\theta$ and $y = r\sin\theta$, after multiplying the relation $u = 1/r = (1 - \beta(\cos\theta\cos\delta - \sin\theta\sin\delta))/\rho_0$ by $(r\rho_0)$, it is readily expressed in terms of y and x as,

$$\left(y^2 + x^2\right)^{\frac{1}{2}} = \rho_0 + \beta(x\cos\delta - y\sin\delta). \tag{4a}$$

In Eq. (4a) we set x to $(x'\cos\delta + y'\sin\delta)$ and y to $(y'\cos\delta - x'\sin\delta)$—and then discard the primes—*the rotated coordinates explicitly display the mirror invariance of the locus*, which is now *manifest when* $y \to -y$,

$$\left(y^2 + x^2\right)^{\frac{1}{2}} = \rho_0 + \beta x. \tag{4b}$$

We square both sides of Eq. (4b) and regroup the resulting terms *to reveal that this locus is the conic section*,

$$y^2 + \left(1 - \beta^2\right)\Big(x - \left(\beta\rho_0/\left(1 - \beta^2\right)\right)\Big)^2 = \rho_0^2/\left(1 - \beta^2\right). \tag{4c}$$

The Eq. (4c) locus is a circle when $\beta = 0$, an ellipse when $0 < \beta^2 < 1$, a parabola when $\beta^2 = 1$ and a hyperbola when $\beta^2 > 1$. *Thus this locus only applies to a binary-star system if* $\beta^2 < 1$. *In that case* Eq. (4c) *yields the semi-minor axis length* $r_0 = \rho_0/(1 - \beta^2)^{\frac{1}{2}} > 0$ *and the semi-major axis length* $R_0 = \rho_0/(1 - \beta^2) \geq r_0$; these axis lengths were specifically mentioned in Eq. (3n). Since $\rho_0 = r_0^2/R_0$, $\beta = (1 - (r_0/R_0)^2)^{\frac{1}{2}} = (R_0^2 - r_0^2)^{\frac{1}{2}}/R_0$ and $(1 - \beta^2) = (r_0/R_0)^2$, expressing the Eq. (4c) elliptical locus in terms of r_0 and R_0 produces,

$$\begin{gathered} y^2 + (r_0/R_0)^2\Big(x - \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big)^2 = r_0^2, \text{ which enforces the bounds } -r_0 \leq y \leq r_0 \text{ and} \\ -R_0 \leq \Big(x - \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big) \leq R_0, \text{ so } \Big(-R_0 + \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big) \leq x \leq \Big(R_0 + \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big). \end{gathered} \tag{5a}$$

The square of the distance from the origin (0, 0) to this elliptical locus is, for x in its Eq. (5a) domain,

$$x^2 + y^2 = x^2 + r_0^2 - (r_0/R_0)^2\Big(x - \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big)^2 = \left(1 - (r_0/R_0)^2\right)\Big(x + \left(r_0^2/R_0\right)\left(1 - (r_0/R_0)^2\right)^{-\frac{1}{2}}\Big)^2, \tag{5b}$$

which is, except in the $R_0 = r_0$ case of a circle, *strictly increasing over the entire* Eq. (5a) *x-domain because,*

$$\begin{gathered} d(x^2 + y^2)/dx = 2\left(1 - (r_0/R_0)^2\right)\Big(x + \left(r_0^2/R_0\right)\left(1 - (r_0/R_0)^2\right)^{-\frac{1}{2}}\Big) = \\ 2\left(1 - (r_0/R_0)^2\right)\Big[\Big(x - \Big(-R_0 + \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big)\Big) + \Big(\Big(R_0 - \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big)\left(1 - (r_0/R_0)^2\right)^{-\frac{1}{2}}\Big)\Big]. \end{gathered} \tag{5c}$$

The perigee therefore occurs at $x = -R_0 + (R_0^2 - r_0^2)^{\frac{1}{2}}$, $y = 0$, and the apogee at $x = R_0 + (R_0^2 - r_0^2)^{\frac{1}{2}}$, $y = 0$, so the perigee and apogee *distances* are respectively $R_0 \mp (R_0^2 - r_0^2)^{\frac{1}{2}}$, as is mentioned below Eq. (3p).

The *area* A of this Eq. (5a) elliptical locus is,

$$\begin{gathered} A = 2r_0 \int_{-R_0 + \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}}^{R_0 + \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}} \Big(1 - \Big(\Big(x - \left(R_0^2 - r_0^2\right)^{\frac{1}{2}}\Big)/R_0\Big)^2\Big)^{\frac{1}{2}} dx = 2r_0R_0 \int_{-1}^{1} \left(1 - u^2\right)^{\frac{1}{2}} du = \\ 2r_0R_0 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = r_0R_0 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)d\theta = r_0R_0[\pi + \tfrac{1}{2}(\sin\pi - \sin(-\pi))] = \pi\, r_0R_0. \end{gathered} \tag{5d}$$

Thus Eq. (3n) properly accords with the Eq. (5d) area $A = \pi\, r_0R_0$ of the Eq. (5a) elliptical locus.