Trigonometric Tutorial: Pythagorean Theorem, Planar Rotations, Chord Lengths, and Trig Function Features

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Abstract

This tutorial takes trigonometry to be the study of the transformation of a two-dimensional vector's Cartesian coordinates when it is rotated about the Cartesian origin of coordinates in its plane. Since the Pythagorean theorem underlies the idea of Cartesian coordinates, the tutorial commences with a plane-geometry recapitulation of that theorem. Three characteristics of the planar rotation transformations of a two-dimensional vector's Cartesian coordinates are pointed out: their linearity, their preservation of the vector's length and their additivity in successive rotation angles. The rotated vector's Cartesian coordinates themselves aren't thus successively additive; they reflect mapping of the rotation angles into the more intricate corresponding changes of the vector's location in two dimensions. The machinery which maps successive rotation angles into the corresponding two-dimensional locations is the "angle-addition formula"; it performs that task via application of the sine and cosine functions to the rotation angles. The last part of the tutorial studies properties of the sine and cosine functions, one of the most fascinating is that they are the imaginary and real parts of the exponential function of imaginary argument.

Review of the Pythagorean theorem in plane geometry

Some plane geometry texts gloss over the Pythagorean theorem *without mentioning its centrality to Cartesian coordinates*, or *emphasizing* that it *follows* from the equality of the ratios of the corresponding side lengths *of three particular similar right triangles*. Because the Pythagorean theorem isn't given the prominence it merits in some plane geometry texts, we *reprise its demonstration* here.

Given a right triangle whose two legs have lengths denoted l_1 and l_2, and whose hypotenuse has length denoted h, we construct the the line segment from its right-angle vertex to its hypotenuse which is perpendicular to the latter. We note that this line segment, whose length we denote p, *divides the right triangle into two right triangles, each of which is similar to the original right triangle because their angles are the same*. The intersection point of this line segment with the hypotenuse divides the hypotenuse into two line segments: we denote as s the length of the hypotenuse line segment which intersects the leg of length l_1; the remaining hypotenuse line segment, whose length of course is $(h-s)$, intersects the leg of length l_2. Because the three right triangles are similar to each other, the following equalities of the ratios of their corresponding side lengths hold,

$$s/l_1 = p/l_2 = l_1/h \ \text{ and } \ p/l_1 = (h-s)/l_2 = l_2/h, \tag{1a}$$

where the last equality turns out to be redundant; we ignore it. Solving the remaining equalities for p yields,

$$p = s\,l_2/l_1 = l_1 l_2/h = (h-s)l_1/l_2, \tag{1b}$$

which can in turn be solved for s and $(h-s)$ in terms of l_1, l_2 and h, with the results,

$$s = (l_1)^2/h \ \text{ and } \ (h-s) = (l_2)^2/h. \tag{1c}$$

Adding the two equalities of Eq. (1c) to eliminate s yields,

$$h = \big((l_1)^2 + (l_2)^2\big)/h \ \ \Rightarrow \ \ h^2 = (l_1)^2 + (l_2)^2. \tag{1d}$$

The final equality of Eq. (1d) is the Pythagorean theorem.

The linear changes of coordinates produced by planar rotation of a vector

A fundamental concept underlying trigonometry is that the new coordinates of a unit-length vector which has been rotated in the $x-y$ plane are a linear transformation of its previous $x-y$ coordinates, with the coefficients of that linear transformation being, aside from certain particular signs, the sine and cosine of the rotation angle θ; those planar rotations naturally leave the the vector's unit length unchanged, which turns out to be a consequence of the identity $\cos^2\theta + \sin^2\theta = 1$ in conjunction with the Pythagorean theorem.

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An $x-y$ planar vector described as (x, y) has, according to the Pythagorean theorem, the length $(x^2+y^2)^{\frac{1}{2}}$ from the origin point $(0, 0)$ to its (x, y) point. Now consider the unit-length vector $(1, 0)$ that points in the x-direction. Its counterclockwise rotation by angle θ is given by,

$$R(\theta)(1, 0) = (\cos\theta,\, \sin\theta), \tag{2a}$$

whose length from the origin point $(0, 0)$ is by the Pythagorean theorem $(\cos^2\theta + \sin^2\theta)^{\frac{1}{2}}$, which, because of the identity $\cos^2\theta + \sin^2\theta = 1$, is equal to unity. Thus $R(\theta)$ indeed leaves the unit length of the vector $(1, 0)$ unchanged.

The fact that $R(\theta)$ is a *linear* transformation allows us to conclude that,

$$R(\theta)(x,\, y) = R(\theta)[x(1, 0) + y(0, 1)] = xR(\theta)(1, 0) + yR(\theta)(0, 1) = x(\cos\theta,\, \sin\theta) + yR(\theta)(0, 1). \tag{2b}$$

We see from Eq. (2b) that to obtain $R(\theta)(x,\, y)$ we must be able to obtain $R(\theta)(0, 1)$. A *partial step* toward obtaining $R(\theta)(0, 1)$ is to note from Eq. (2a) that $R(\pi/2)(1, 0) = (\cos(\pi/2),\, \sin(\pi/2)) = (0, 1)$, so,

$$R(\theta)(0, 1) = R(\theta)R(\pi/2)(1, 0). \tag{2c}$$

With regard to Eq. (2c), *another* important property of $x - y$ *planar* rotations, in *addition* to their linearity, is their *angle-additivity*, namely that,

$$R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1) = R(\theta_1 + \theta_2). \tag{2d}$$

Therefore from Eqs. (2c), (2d) and (2a),

$$R(\theta)(0, 1) = R(\theta + \pi/2)(1, 0) = (\cos(\theta + \pi/2),\, \sin(\theta + \pi/2)) = (-\sin(\theta),\, \cos(\theta)), \tag{2e}$$

where to obtain the last equality we have applied—albeit only to a very limited extent—our knowledge of trigonometric identities. Inserting the Eq. (2e) result into Eq. (2b) yields,

$$R(\theta)(x,\, y) = x(\cos\theta,\, \sin\theta) + y(-\sin(\theta),\, \cos(\theta)) = (x\cos\theta - y\sin\theta,\; x\sin\theta + y\cos\theta). \tag{2f}$$

Eq. (2f) shows that the *length* of the planar-rotated vector $R(\theta)(x,\, y)$ is,

$$|R(\theta)(x,\, y)| = \left((x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2\right)^{\frac{1}{2}} = \left(x^2 + y^2\right)^{\frac{1}{2}} = |(x,\, y)|, \tag{2g}$$

so planar rotation *doesn't change a vector's length*; in particular it leaves a vector of unit length *still having unit length after the rotation*.

If we *specialize the* $(x,\, y)$ *in* Eq. (2f) *to the unit-length vector* $(\cos\theta_1,\, \sin\theta_1) = R(\theta_1)(1, 0)$, where the equality follows from Eq. (2a), we can then reexpress Eq. (2f) as,

$$R(\theta_2)R(\theta_1)(1, 0) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,\; \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2), \tag{2h}$$

and since from Eqs. (2d) and (2a),

$$R(\theta_2)R(\theta_1)(1, 0) = R(\theta_1 + \theta_2)(1, 0) = (\cos(\theta_1 + \theta_2),\, \sin(\theta_1 + \theta_2)), \tag{2i}$$

combining Eq. (2i) with Eq. (2h) produces *the full trigonometric angle-addition result*,

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \;\text{ and }\; \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \tag{2j}$$

The above *derivation* of this *general* result *assumed in* Eq. (2e) *only its very limited special case*,

$$\cos(\theta + \pi/2) = -\sin(\theta) \;\text{ and }\; \sin(\theta + \pi/2) = \cos(\theta). \tag{2k}$$

The special case of the Eq. (2j) trigonometric angle-addition formula where $\theta_1 = +\theta$ and $\theta_2 = -\theta$ yields,

$$\cos(0) = \cos^2\theta + \sin^2\theta \;\text{ and }\; \sin(0) = \sin\theta\cos\theta - \cos\theta\sin\theta = 0. \tag{2l}$$

Eq. (2l) tells us that $\sin(0) = 0$, and in light of the basic trigonometric postulate that $\cos^2\theta + \sin^2\theta = 1$, it also tells us that $\cos(0) = 1$. Basic trigonometry *is encompassed by* $\cos^2\theta + \sin^2\theta = 1$ *and the* Eq. (2j) *angle-addition formula*.

The *derivatives* with respect to θ of $\cos\theta$ and $\sin\theta$ *also* follow from $\cos^2\theta + \sin^2\theta = 1$ and Eq. (2j) *once it is verified that* $\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta) = 1$. In the limit that $|\delta\theta| \to 0$, $|\delta\theta|$ is *the arc length* of a very short arc of the smooth unit-circle curve that starts from $\delta\theta = 0$; that very short arc's arc length is by its nature well-approximated by the length of the straight-line-segment chord which joins the two ends of that arc. Therefore we now obtain the length of the chord of an arc of the unit circle, and verify that the ratio of $|\sin\delta\theta|$ to that chord length approaches unity in the limit that the corresponding arc length $|\delta\theta| \to 0$.

The length of the chord of an arc of the trigonometric unit circle

Consider an arc of the trigonometric unit circle that starts from $\theta = 0$ and has arc length $|\theta|$. We denote the length of the straight-line-segment chord which joins the two ends of that arc as $\mathrm{chl}\,\theta$. Of course that chord length $\mathrm{chl}\,\theta$ is less than the arc length $|\theta|$ of the arc, *but it is greater than* $|\sin\theta|$: the length $\mathrm{chl}\,\theta$ of the straight-line-segment chord which joins the two ends of the unit-circle arc of arc length $|\theta|$ is,

$$\mathrm{chl}\,\theta = \big((\sin\theta)^2 + (1-\cos\theta)^2\big)^{\frac{1}{2}} \geq |\sin\theta| \ \text{ and } \ \mathrm{chl}\,\theta = (2(1-\cos\theta))^{\frac{1}{2}}. \tag{3a}$$

By the nature of the arc length of a smooth curve, that arc length is well approximated by the corresponding chord length when the arc length is sufficiently short, namely,

$$\lim_{\delta\theta\to 0}(\mathrm{chl}\,\delta\theta/|\delta\theta|) = 1. \tag{3b}$$

Inverting the Eq. (3a) result for the chord length $\mathrm{chl}\,\theta$ corresponding to the arc length $|\theta|$ yields,

$$\cos\theta = 1 - \tfrac{1}{2}\mathrm{chl}^2\,\theta \quad \Rightarrow \quad |\sin\theta| = \big(1-\cos^2\theta\big)^{\frac{1}{2}} = \big(\mathrm{chl}^2\,\theta - \tfrac{1}{4}\mathrm{chl}^4\,\theta\big)^{\frac{1}{2}} = \mathrm{chl}\,\theta\big(1 - \tfrac{1}{4}\mathrm{chl}^2\,\theta\big)^{\frac{1}{2}}. \tag{3c}$$

Eq. (3b) in conjunction with the Eq. (3c) result $|\sin\theta| = \mathrm{chl}\,\theta\big(1 - \tfrac{1}{4}\mathrm{chl}^2\,\theta\big)^{\frac{1}{2}}$ implies that,

$$\lim_{\delta\theta\to 0}(|\sin\delta\theta|/|\delta\theta|) = 1, \tag{3d}$$

and since $(|\sin\theta|/|\theta|) = (\sin\theta/\theta)$ when $0 < |\theta| < \pi$, Eq. (3d) implies that,

$$\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta) = 1. \tag{3e}$$

Interesting properties of the sine and cosine functions

With Eq. (3e) in hand, we apply it together with $\cos^2\theta + \sin^2\theta = 1$ and Eq. (2j) to obtain the derivatives of $\cos\theta$ and $\sin\theta$. We write Eq. (2j) in a form conducive to taking the limits that define those derivatives,

$$\cos(\theta+\delta\theta) = \cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta \ \text{ and } \ \sin(\theta+\delta\theta) = \sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta. \tag{4a}$$

Proceeding further in this vein, we note that $d\cos\theta/d\theta = \lim_{\delta\theta\to 0}((\cos(\theta+\delta\theta) - \cos(\theta))/\delta\theta)$ and $d\sin\theta/d\theta = \lim_{\delta\theta\to 0}((\sin(\theta+\delta\theta) - \sin(\theta))/\delta\theta)$. In light of those facts, we work out from Eq. (4a) that,

$$\begin{aligned} d\cos\theta/d\theta &= \lim_{\delta\theta\to 0}((\cos(\theta+\delta\theta) - \cos(\theta))/\delta\theta) = \lim_{\delta\theta\to 0}[\cos\theta((\cos\delta\theta - 1)/\delta\theta) - \sin\theta(\sin\delta\theta/\delta\theta)] \ \text{ and} \\ d\sin\theta/d\theta &= \lim_{\delta\theta\to 0}((\sin(\theta+\delta\theta) - \sin(\theta))/\delta\theta) = \lim_{\delta\theta\to 0}[\sin\theta((\cos\delta\theta - 1)/\delta\theta) + \cos\theta(\sin\delta\theta/\delta\theta)]. \end{aligned} \tag{4b}$$

The two *key limits* on the right sides of Eq. (4b) are (1) $\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta) = 1$ from Eq. (3e), and (2) $\lim_{\delta\theta\to 0}((\cos\delta\theta - 1)/\delta\theta)$. Since $(\cos\delta\theta - 1) = (\cos^2\delta\theta - 1)/(\cos\delta\theta + 1) = -\sin^2\delta\theta/(\cos\delta\theta + 1)$, we obtain,

$$\lim_{\delta\theta\to 0}((\cos\delta\theta - 1)/\delta\theta) = \lim_{\delta\theta\to 0}[-(\sin\delta\theta)(\sin\delta\theta/\delta\theta)/(\cos\delta\theta + 1)] = 0. \tag{4c}$$

The Eq. (4c) and (3e) limit results *permit the right sides of* Eq. (4b) *to be evaluated*; they yield,

$$d\cos\theta/d\theta = -\sin\theta \quad \text{and} \quad d\sin\theta/d\theta = \cos\theta. \tag{4d}$$

A certain linear combination of $\cos\theta$ and $\sin\theta$ very usefully *turns out to be an exponential*. We write,

$$\cos\theta + \beta\sin\theta = \exp(\gamma\theta), \tag{5a}$$

making the two sides of Eq. (5a) *agree at* $\theta = 0$. Differentiating those two sides with respect to θ yields,

$$\beta\cos\theta - \sin\theta = \gamma\exp(\gamma\theta) = \gamma(\cos\theta + \beta\sin\theta), \tag{5b}$$

which implies that $\beta = \gamma$ and $\gamma^2 = -1$, so $\beta = \gamma = \pm i$. Thus,

$$\cos\theta \pm i\sin\theta = \exp(\pm i\theta). \tag{5c}$$

The *two* signs of $\pm i$ in fact *are redundant*; their *effect is already accounted for when* $\theta \to -\theta$.

This *exponential version* of trigonometry *very readily yields the angle-addition formulas:* on one hand,

$$\exp(i\theta_1)\exp(i\theta_2) = \exp(i(\theta_1 + \theta_2)) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \tag{5d}$$

but on the other hand,

$$\begin{gathered}\exp(i\theta_1)\exp(i\theta_2) = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) = \\ (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)),\end{gathered} \tag{5e}$$

and of course it is Eqs. (5d) and (5e) *together* which produce the angle-addition formulas, namely,

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \quad \text{and} \quad \sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2). \tag{5f}$$

As well as the angle-addition formulas, exponential trigonometry *categorically yields* $\cos^2\theta + \sin^2\theta = 1$ since,

$$1 = \exp(0) = \exp\big((i\theta) + (-i\theta)\big) = \exp(i\theta)\exp(-i\theta) = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta. \tag{5g}$$

We proceed to the Taylor expansions of the trigonometric functions; that of $\exp(i\theta)$ *itself* is *elementary*,

$$\begin{gathered}\exp(i\theta) = \textstyle\sum_{k=0}^{\infty}(i)^k(\theta)^k/k! = \sum_{m=0}^{\infty}(i)^{2m}(\theta)^{2m}/(2m)! + \sum_{n=1}^{\infty}(i)^{2n-1}(\theta)^{2n-1}/(2n-1)! = \\ \textstyle\sum_{m=0}^{\infty}(-1)^m(\theta)^{2m}/(2m)! + i\,\sum_{n=1}^{\infty}(-1)^{n-1}(\theta)^{2n-1}/(2n-1)!\,.\end{gathered} \tag{6a}$$

Since $\exp(i\theta) = \cos\theta + i\sin\theta$, the Taylor expansions of $\cos\theta$ and $\sin\theta$ *can now be read off from* Eq. (6a),

$$\cos\theta = \textstyle\sum_{m=0}^{\infty}(-1)^m(\theta)^{2m}/(2m)! \quad \text{and} \quad \sin\theta = \sum_{n=1}^{\infty}(-1)^{n-1}(\theta)^{2n-1}/(2n-1)!\,. \tag{6b}$$