

Observations of structure of a possible unification algebra

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Abstract. A C-loop algebra, designated \mathbb{U} is assembled as the product: $M_4(C) \otimes \mathbb{T}$. When $M_4(C)$ is assigned to represent $Cl_{1,3}(R) \otimes \mathbb{C}$ and the principle of spatial equivalence is invoked, a sub-algebra designated \mathbb{W} is found to have features that suggest it could provide an underlying basis for the standard model of fundamental particles. \mathbb{U} is of the same order as $Cl_{0,10}(R)$, but has a “natural” partition into $Cl_{1,3}(R) \otimes \mathbb{C} \otimes \mathbb{W}$, suggesting that its use in string/M theories in the place of $Cl_{0,10}(R)$ may generate a description of reality.

1. Introduction

This paper describes algebraic structures using labels for unit elements which highlight features related to spatial equivalence, as set out in section 2.

Sections 3 to 6 document the algebraic structures. They have been investigated by using the multiplication tables for unit elements combined with random coefficients to generate random products, which are then used to check the properties of the algebras, testing for distributivity, associativity, flexibility, alternativity and power associativity. The Loops package[1] for GAP4[2] has also been used.

In sections 3 to 6, observations and postulates are presented relating the algebraic structures to physics. The author has limited understanding of subjects such as torsion, manifolds and fiber bundles, so the postulates are speculative, but they demonstrate the potential for the algebraic structures to provide a basis for the unification of general relativity with quantum mechanics.

Sections 7 to 10 present details relating to the assembly of the algebraic structures and their properties.

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2. Notation for algebras \mathbb{T} , \mathbb{M} , \mathbb{U} , \mathbb{W} and \mathbb{D}

Labels for algebras and their unit elements are based on patterns related to spatial equivalence when a sub-algebra is used to represent the space-time Clifford algebra.

Unit elements for $\mathbb{M} \cong M_4(C)$ are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
S	L	M	N	V	D	E	F	iU	iX	iY	iZ	iT	iP	iQ	iR
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
iS	iL	iM	iN	iV	iD	iE	iF	U	X	Y	Z	T	P	Q	R

Unit elements for the trigtintaduonion algebra, \mathbb{T} , are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
σ_o	σ_l	σ_j	σ_κ	λ_o	λ_l	λ_j	λ_κ	μ_o	μ_l	μ_j	μ_κ	ν_o	ν_l	ν_j	ν_κ
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
α_o	α_l	α_j	α_κ	β_o	β_l	β_j	β_κ	γ_o	γ_l	γ_j	γ_κ	δ_o	δ_l	δ_j	δ_κ

Cayley tables for these algebras are shown in sections 8.1 and 9.3. They have been arranged so that, if the signs of products are ignored, they form the same latin square. This is referred to as “alignment” of the algebras. Note that the subscripts l, j, κ identify orientation with respect to iX, iY, iZ for the alignment.

An algebra labeled \mathbb{U} is generated as the tensor product $\mathbb{T} \otimes \mathbb{M}$. For \mathbb{U} , all unit elements of its sub-algebra, \mathbb{T} , commute with all unit elements of its sub-algebra, \mathbb{M} . Unit elements of \mathbb{U} are labeled using combinations of the labels assigned to \mathbb{T} and \mathbb{M} , such as $\nu_\kappa iR$. The labls are listed in Section 10.

A further sub-algebra of \mathbb{U} , labeled \mathbb{W} , is identified which has a Cayley table, shown in section 3.4.2, which is aligned with those of \mathbb{T} and \mathbb{M} . It is a “resonant” subalgebra of \mathbb{U} , where “resonance” is defined as meeting a requirement for spatial equivalence for all sub-algebras when three unit elements of \mathbb{M} are used to represent unit spatial vector elements for a Clifford algebra.

Unit elements for \mathbb{W} are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$\sigma_o S$	$\sigma_l L$	$\sigma_j M$	$\sigma_\kappa N$	$\lambda_o V$	$\lambda_l D$	$\lambda_j E$	$\lambda_\kappa F$	$\mu_o iU$	$\mu_l iX$	$\mu_j iY$	$\mu_\kappa iZ$	$\nu_o iT$	$\nu_l iP$	$\nu_j iQ$	$\nu_\kappa iR$
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
$\alpha_o iS$	$\alpha_l iL$	$\alpha_j iM$	$\alpha_\kappa iN$	$\beta_o iV$	$\beta_l iD$	$\beta_j iE$	$\beta_\kappa iF$	$\gamma_o U$	$\gamma_l X$	$\gamma_j Y$	$\gamma_\kappa Z$	$\delta_o T$	$\delta_l P$	$\delta_j Q$	$\delta_\kappa R$

A label, \mathbb{D} is used for algebras isomorphic to the algebra of real 4×4 diagonal matrices.

3. Algebraic structures

3.1. Structure of $\mathbb{M} \cong M_4(C)$

The structure of \mathbb{M} is well known.

3.1.1. Embedded group and automorphism group. \mathbb{M} contains an embedded group, the Dirac matrix group, of order 64, generated by its 32 basis elements. The automorphism group for the Dirac matrix group, as determined using the Loops package for GAP4, has 42 conjugacy classes, and its structure description is:

$$((C2 \times C2 \times C2 \times C2) : A6) : (C2 \times C2)$$

3.1.2. Sub-groups of order 32. The Dirac matrix group has 31 sub-groups of order 32, which can be sorted into 3 classes, as shown in table 10 in section 7.

3.1.3. Sub-groups of order 16. The Dirac matrix group has 155 sub-groups of order 16. There are five types of these sub-groups which differ in the signature of their unit components or in their commutation properties. Once unit matrices for \mathbb{M} are assigned to represent complexified unit multivector elements for a space-time Clifford algebra, further distinctions are found which identify sub-groups based on their relationship to space-like and time-like vectors. These sub-groups have been labeled as shown in table 2 in section 7.

3.1.4. Sub-groups of order 8. The Dirac matrix group has 155 sub-groups of order 8. There are 80 non-abelian sub-groups. There are 60 abelian sub-groups which exclude the unit imaginary, and 15 abelian sub-groups which include the unit imaginary. Once unit matrices for \mathbb{M} are assigned to represent complexified unit multivector elements for a space-time Clifford algebra, further distinctions are found which identify sub-groups based on their relationship to space-like and time-like vectors. These sub-groups are shown in tables 4-6 in section 7.

3.1.5. Graded algebras. \mathbb{M} can be used to represent multivectors for graded polar vector algebras such as the Clifford algebras $Cl_4(C)$, $Cl_{0,5}(R)$, $Cl_{2,3}(R)$, $Cl_{1,4}(R)$.

3.2. Structure of \mathbb{T}

The structure of the trigintaduonion algebra, \mathbb{T} , has been described by Cawagas et al[3], but that description does not detail the differing ways in which lower order subalgebras participate in sedenion-type subalgebras. These details are shown in tables 3-9 in section 7.

3.2.1. Embedded loop and automorphism group. \mathbb{T} contains an embedded loop T_L of order 64 generated by its 32 basis elements. The automorphism group for T_L , as determined using the Loops package[1] for GAP4[2], has 42 conjugacy classes, and its structure description is:

$$C2 \times C2 \times ((C2 \times C2 \times C2) . PSL(3,2))$$

3.2.2. Sub-loops of order 32. T_L has 31 sedenion-type subloops of \mathbb{T} of order 32, falling into four isomorphism classes, which Cawagas et al. designated $S_L, S_L^\alpha, S_L^\beta, S_L^\gamma$, as shown in table 10 in section 7.

3.2.3. Sub-loops of order 16. T_L has 155 octonion-type sub-loops of order 16, falling into two isomorphism classes: octonion loops which Cawagas et al. designated O_L and quasi-octonion loops which they designated \tilde{O}_L . These octonion-type sub-loops of T_L participate in sedenion-type subloops of T_L in a variety of ways, as shown in table 3 in section 7.

3.2.4. Sub-loops of order 8. T_L has 155 quaternionic subloops of order 8, falling into one isomorphism class which Cawagas et al. designated Q_8 . These quaternionic subloops of T_L participate in sedenion-type subloops of T_L in a variety of ways, as shown in tables 4-6 in section 7.

3.2.5. Graded algebras. It is postulated that \mathbb{T} can be used to represent graded multivectors for axial vector algebras, which can be associated with graded multivectors for polar vector algebras represented by Clifford algebras such as $Cl_4(C), Cl_{0,5}(R), Cl_{2,3}(R), Cl_{1,4}(R)$.

3.3. Structure of $\mathbb{U} \cong \mathbb{M} \otimes \mathbb{T}$

3.3.1. Embedded loop and automorphism group. \mathbb{U} contains an embedded loop U_L of order 2048 generated by its 1024 basis elements. The automorphism group for U_L , as determined using the Loops package for GAP4, is:
 $C2xC2x(((C2xC2xC2xC2):A6):(C2xC2))x((C2xC2xC2).PSL(3,2))$.

3.3.2. Associative subalgebras of \mathbb{U} . \mathbb{M} is associative. \mathbb{T} is di-associative, pairs of its imaginary unit elements generate sub-algebras isomorphic to \mathbb{H} . As noted by Cawagas et al, there are 155 of these sub-algebras. So, for \mathbb{U} , there are 155 associative sub-algebras isomorphic to $\mathbb{H} \otimes \mathbb{M}$.

3.3.3. Aligned sub-algebras of \mathbb{U} . The Cayley tables for \mathbb{M} and \mathbb{T} , as presented in tables 12 and 13 in sections 8 and 9, have been arranged so that, if the signs of products are ignored, they form the same latin square. Referring to this as “alignment”, there are many possible ways of aligning unit elements of the two algebras. For sub-algebras of \mathbb{M} and \mathbb{T} , paired combinations of unit elements from this alignment generate “aligned” sub-algebras of \mathbb{U} .

3.3.4. Resonant subalgebras of \mathbb{U} . For the alignment of table 12 with 13, unit elements of aligned sub-algebras are listed in tables 4-10 in section 7. It can be seen that all \mathbb{M} subgroups that are related by spatial rotation are aligned with \mathbb{T} subloops which have related patterns of participation in sedenion type subloops. This property is designated as a “resonance”. A sub-algebra generated using paired products for a resonant alignment, such as \mathbb{W} , is designated a “resonant” sub-algebra.

3.3.5. Partition of \mathbb{U} . For \mathbb{U} , all unit elements from \mathbb{W} together with the unit imaginary element from \mathbb{M} all commute with all unit elements from a subalgebra of \mathbb{M} isomorphic to $Cl_{1,3}(R)$. If \mathbb{U} is used as a basis for a total space for a manifold with fiber bundles, this suggests the choice of a $Cl_{1,3}(R)$ base manifold. For a complexified unit element of \mathbb{W} such as $\mu_\iota X + \mu_\iota iX$, the partial derivative with respect to a coordinate, x , associated with the unit vector, would be a complex function of the associated unit element, μ_ι , of \mathbb{T} . This suggests association of $\mathbb{C} \otimes \mathbb{W}$ with fiber bundles, as covariant derivatives of solder forms for tangent vectors define torsion on tangent frame bundles[4].

3.4. Structure of \mathbb{W} and $\mathbb{C} \otimes \mathbb{W}$

Each unit element of \mathbb{W} is a product of a unit element of \mathbb{M} with a unit element of \mathbb{T} . As all imaginary unit elements of \mathbb{T} square to -1 and anti-commute, unit elements of \mathbb{W} other than the identity have opposite signature and opposite commutation properties to the corresponding unit elements of \mathbb{M} . This relates the Lie bracket of products for \mathbb{W} to the Jordan brace of products for \mathbb{M} and vice-versa, a form of supersymmetry.

3.4.1. Embedded loop and automorphism group. \mathbb{W} contains an embedded loop W_L of order 64 generated by its 32 basis elements. The automorphism group for W_L , as determined using the Loops package for GAP4, has 40 conjugacy classes, and its structure description is:

$(C2) \times (C2) \times (C2) \times (S4)$.

The automorphism group of its complexification has 80 conjugacy classes and its structure description is:

$(C2) \times (C2) \times (C2) \times (C2) \times (S4)$.

3.4.2. Cayley table for \mathbb{W} basis elements. The Cayley table for \mathbb{W} basis elements (excluding negative elements) is shown in table 1.

TABLE 1. Cayley table for \mathbb{W} basis elements

Table with 31 columns labeled 0 to 31 and 31 rows labeled with basis elements like e, s, t, u, v, w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z. Each cell contains a product of two basis elements, e.g., 0 1 is e s, 1 2 is s t, etc.

3.4.3. Sub-algebras of \mathbb{W} and $\mathbb{C} \otimes \mathbb{W}$. The Cayley tables of \mathbb{M} , \mathbb{T} and \mathbb{W} have been configured in a resonant alignment, their sub-algebras are also aligned, and are configured for spatial equivalence when \mathbb{M} is used to represent $Cl_{1,3}(R) \otimes \mathbb{C}$. The sub-algebras of \mathbb{W} are in one-to-one correspondence with sub-algebras of \mathbb{M} and \mathbb{T} .

The 155 sub-algebras of \mathbb{W} with four unit elements, all of which are associative, are listed in tables 4-6 in section 7. 80 of them relate to non-abelian sub-algebras of \mathbb{M} and 75 to abelian sub-algebras of \mathbb{M} . The sub-algebras of \mathbb{W} with eight unit elements, none of which are associative, are listed in tables 7-9 in section 7.

Analysis of the 155 sub-algebras of \mathbb{W} with 8 unit elements reveals that 15 of them are isomorphic to the split octonions. The other 140 are not power associative. Its 15 sub-algebras isomorphic to the split octonions generate, when complexified, 15 sub-algebras of $\mathbb{C} \otimes \mathbb{W}$ isomorphic to $\mathbb{C} \otimes \mathbb{O}$. Kohl Furey has postulated that minimal left ideals of a $Cl_6(C)$ algebra extracted from $\mathbb{C} \otimes \mathbb{O}$ correspond to one family of fundamental particles[5][6], and refers to others who have advocated the existence of a connection between non-associative algebras and particle theory[7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25]. $\mathbb{U} \cong [\mathbb{C} \otimes \mathbb{W} \otimes Cl_{1,3}(R)]$ has, as sub-algebras, the algebras on which all of these approaches are based, suggesting that it has the potential to provide a basis for the standard model of fundamental particles.

It is postulated that these 15 $\mathbb{C} \otimes \mathbb{O}$ sub-algebras, correspond to three spatial orientations for five families of particles - three families of standard model fermions and two families of dark matter particles. These 15 sub-algebras have, as sub-algebras, all the sub-algebras generated by complexification of the 75 four element subalgebras of \mathbb{W} related to abelian sub-algebras of \mathbb{M} .

The other 80 sub-algebras of \mathbb{W} with four unit elements, when complexified, generate sub-algebras isomorphic to $\mathbb{C} \otimes \mathbb{D}$, the algebra of 4×4 complex diagonal matrices. These are commuting and associative. It is postulated that they are associated with vector bosons.

It is postulated that the sub-algebra of \mathbb{W} with unit elements $[\sigma_o S, \sigma_o i S, \alpha_o S, \alpha_o i S]$, a complex doublet, is associated with the Higgs mechanism[26].

3.4.4. Grading distinctions. For sub-algebras of $\mathbb{C} \otimes \mathbb{W}$ with fewer than 32 unit elements, there are distinctions between otherwise isomorphic sub-algebras arising from the patterns of participation of quaternionic and octonion-type sub-loops of T_L in $S_L, S_L^\alpha, S_L^\beta, S_L^\gamma$ sub-loops of T_L . Once \mathbb{M} is assigned to represent $Cl_{1,3}(R) \otimes \mathbb{C}$, further distinctions arise between otherwise isomorphic sub-algebras of $\mathbb{C} \otimes \mathbb{W}$. It is postulated that these distinctions generate the complexity of the standard model.

3.4.5. Matrix representation of $\mathbb{C} \otimes \mathbb{W}$. The notation used to generate \mathbb{T} features dis-association operators, symbols that define the products of unit elements of \mathbb{T} using a modified form of the usual Moufang loop construction for octonions, as setout in table 14 in section 9.2.

A general element of $\mathbb{C} \otimes \mathbb{W}$ can be represented using a complex 4×4 matrix with entries with dis-association operators and using coefficients $a_{-}, b_{-i}, c_{-}, d_{-i}$ with subscripts which are lower case versions of the associated unit element from \mathbb{M} .

$$\left(\begin{array}{cccc} +a_s+b_s i+c_s \sigma_o+d_s i \beta_o & +a_u+b_u i+c_u \lambda_o+d_u i \beta_o & +a_m+b_m i+c_m \sigma_j+d_m i \alpha_j & +a_e+b_e i+c_e \lambda_j+d_e i \beta_j \\ +a_q+b_q i+c_q \delta_j+d_q i \nu_j & +a_y+b_y i+c_y \gamma_j+d_y i \mu_j & +a_t+b_t i+c_t \delta_o+d_t i \nu_o & +a_u+b_u i+c_u \gamma_o+d_u i \mu_o \\ +a_d+b_d i+c_d \lambda_i+d_d i \beta_i & +a_f+b_f i+c_f \lambda_\kappa+d_f i \beta_\kappa & +a_n+b_n i+c_n \sigma_\kappa+d_n i \alpha_\kappa & +a_n+b_n i+c_n \sigma_\kappa+d_n i \alpha_\kappa \\ +a_z+b_z i+c_z \gamma_\kappa+d_z i \mu_\kappa & +a_r+b_r i+c_r \delta_\kappa+d_r i \nu_\kappa & +a_x+b_x i+c_x \gamma_l+d_x i \mu_l & +a_p+b_p i+c_p \delta_l+d_p i \nu_l \\ \\ -a_v-b_v i-c_v \lambda_o-d_v i \beta_o & +a_s+b_s i+c_s \sigma_o+d_s i \alpha_o & -a_e-b_e i-c_e \lambda_j-d_e i \beta_j & +a_m+b_m i+c_m \sigma_j+d_m i \alpha_j \\ +a_y+b_y i+c_y \gamma_j+d_y i \mu_j & -a_q-b_q i-c_q \delta_j-d_q i \nu_j & +a_u+b_u i+c_u \gamma_o+d_u i \mu_o & -a_t-b_t i-c_t \delta_o-d_t i \nu_o \\ -a_l-b_l i-c_l \sigma_l-d_l i \alpha_l & +a_d+b_d i+c_d \lambda_i+d_d i \beta_i & -a_n-b_n i-c_n \sigma_\kappa-d_n i \alpha_\kappa & +a_f+b_f i+c_f \lambda_\kappa+d_f i \beta_\kappa \\ +a_r+b_r i+c_r \delta_\kappa+d_r i \nu_\kappa & -a_z-b_z i-c_z \gamma_\kappa-d_z i \mu_\kappa & +a_p+b_p i+c_p \delta_l+d_p i \nu_l & -a_x-b_x i-c_x \gamma_l-d_x i \mu_l \\ \\ -a_m-b_m i-c_m \sigma_j-d_m i \alpha_j & -a_e-b_e i-c_e \lambda_j-d_e i \beta_j & +a_s+b_s i+c_s \sigma_o+d_s i \alpha_o & +a_v+b_v i+c_v \lambda_o+d_v i \beta_o \\ -a_t-b_t i-c_t \delta_o-d_t i \nu_o & -a_u-b_u i-c_u \gamma_o-d_u i \mu_o & +a_q+b_q i+c_q \delta_j+d_q i \nu_j & +a_y+b_y i+c_y \gamma_j+d_y i \mu_j \\ +a_f+b_f i+c_f \lambda_\kappa+d_f i \beta_\kappa & +a_n+b_n i+c_n \sigma_\kappa+d_n i \alpha_\kappa & -a_d-b_d i-c_d \lambda_i-d_d i \beta_i & -a_l-b_l i-c_l \sigma_l-d_l i \alpha_l \\ +a_x+b_x i+c_x \gamma_l+d_x i \mu_l & +a_p+b_p i+c_p \delta_l+d_p i \nu_l & -a_z-b_z i-c_z \gamma_\kappa-d_z i \mu_\kappa & -a_r-b_r i-c_r \delta_\kappa-d_r i \nu_\kappa \\ \\ +a_e+b_e i+c_e \lambda_j+d_e i \beta_j & -a_m-b_m i-c_m \sigma_j-d_m i \alpha_j & -a_v-b_v i-c_v \lambda_o-d_v i \beta_o & +a_s+b_s i+c_s \sigma_o+d_s i \alpha_o \\ -a_u-b_u i-c_u \gamma_o-d_u i \mu_o & +a_t+b_t i+c_t \delta_o+d_t i \nu_o & +a_y+b_y i+c_y \gamma_j+d_y i \mu_j & -a_q-b_q i-c_q \delta_j-d_q i \nu_j \\ -a_n-b_n i-c_n \sigma_\kappa-d_n i \alpha_\kappa & +a_f+b_f i+c_f \lambda_\kappa+d_f i \beta_\kappa & +a_l+b_l i+c_l \sigma_l+d_l i \alpha_l & -a_d-b_d i-c_d \lambda_i-d_d i \beta_i \\ +a_x+b_x i+c_x \gamma_l+d_x i \mu_l & +a_p+b_p i+c_p \delta_l+d_p i \nu_l & -a_z-b_z i-c_z \gamma_\kappa-d_z i \mu_\kappa & -a_r-b_r i-c_r \delta_\kappa-d_r i \nu_\kappa \end{array} \right)$$

4. Numbers of physical dimensions

The quest for a theory of everything has generated models with various physical dimensionalities - five dimensions for the original Kaluza-Klein theory[27], ten for string theories[28], eleven for M-theory[29]. For nature to be embedded in dimensionalities such as four, five, ten or eleven, whether real or complex, appears arbitrary.

We observe three spatial dimensions and one temporal dimension, which sets a minimum for the number of physical dimensions. The original Kaluza-Klein theory added a further physical dimension and extended general relativity unifying it with classical electromagnetism, suggesting the existence of five physical dimensions.

The Big Bang comprises a transition from a singularity to a manifold expanding through time. The singularity can be regarded as a one dimensional amplitude for zero dimensionality. If that was an unstable configuration, so that a transition to finite amplitudes for more than one dimension was favoured, this raises the question - why a transition to three, four, five, ten or eleven dimensions?

If physical n-space is regarded as composed of an assembly of distorted n-spherical quanta, and if the degree of distortion required is related to the densest possible packing ratio, those densest possible packing ratios vary with dimensionality (n). Suppose that a singularity corresponds to stacking those quanta in an n-needle, one “above” the other. That stack also has a packing ratio.

The packing ratios for assemblies of spheres for continua of different dimensionalities are:

The area of a 2 dimensional circle is $\pi R^2 = 3.142R^2$.
 The volume of a 3 dimensional 3-ball is $4\pi/3.R^3 = 4.189R^3$.
 The volume of a 4 dimensional 4-ball is $\pi^2/2.R^4 = 4.935R^4$.
 The volume of a 5 dimensional 5-ball is $8\pi^2/15.R^5 = 5.264R^5$.
 The volume of a 6 dimensional 6-ball is $\pi^3/6.R^6 = 5.168R^6$.
 The volume of a 7 dimensional 7-ball is $16\pi^3/105.R^7 = 4.725R^7$.
 The volume of an 8 dimensional 8-ball is $\pi^4/24.R^8 = 4.059R^8$.
 The volume of a 9 dimensional 9-ball is $32\pi^4/945.R^9 = 3.299R^9$.
 The volume of a 10 dimensional 10-ball is $\pi^5/120.R^8 = 2.550R^{10}$.

For an assembly of n-balls stacked into an n-needle of unit quantum radius, the packing fractions are:

For a stack of 2 dimensional unit circles on edge: $(\pi R^2)/(2R \times 2R) \rightarrow 78.6\%$
 For a 3-needle of unit 3-balls: $(4\pi^3/3.R^3)/(\pi R^2 \times 2R) \rightarrow 66.6\%$
 For a 4-needle of unit 4-balls: $(\pi^2/2.R^4)/(4\pi^3/3.R^3 \times 2R) \rightarrow 58.9\%$
 For a 5-needle of unit 5-balls: $(8\pi^2/15.R^5)/((\pi^2/2.R^4) \times 2R) \rightarrow 53.3\%$
 For a 6-needle of unit 6-balls: $(\pi^3/6.R^6)/(8\pi^2/15.R^5 \times 2R) \rightarrow 49.1\%$
 For a 7-needle of unit 7-balls: $(16\pi^3/105.R^7)/(\pi^3/6.R^6 \times 2R) \rightarrow 45.7\%$
 For an 8-needle of unit 8-balls: $(\pi^4/24.R^8)/(16\pi^3/105.R^7 \times 2R) \rightarrow 45.9\%$
 For a 9-needle of unit 9-balls: $(32\pi^4/945.R^9)/(\pi^4/24.R^8 \times 2R) \rightarrow 34.9\%$
 For a 10-needle of unit 10-balls: $(\pi^5/120.R^{10})/(32\pi^4/945.R^9 \times 2R) \rightarrow 38.6\%$

For euclidean n-space the densest packing fractions, as listed by Cohn and Elkies[30], are:

For 2 dimensional 2-balls (circles) of equal radius: 91%
 For 3 dimensional 3-balls of equal radius: 74%
 For 4 dimensional 4-balls of equal radius: in the range 61.7 to 64.8%
 For 5 dimensional 5-balls of equal radius: in the range 46.5 to 52.5%
 For 6 dimensional 6-balls of equal radius: in the range 37.3 to 41.8%
 For 7 dimensional 7-balls of equal radius: in the range: 29.5 to 32.8%
 For 8 dimensional 8-balls of equal radius: 25.4%
 For 9 dimensional 9-balls of equal radius: in the range: 14.6 to 19.5%
 For 10 dimensional 10-balls of equal radius: in the range: 10.0 to 14.9%

For n -balls arranged in a n -disc, that is an euclidean n -space extended in $n - 1$ dimensions and limited to unit quantum diameter in the n th dimension, the densest packing fractions are product of the densest packing fractions for the euclidean $(n - 1)$ space and the packing fraction for the n -needle of the same dimensionality. For instance, for three dimensions spheres would be packed into the disc in the densest packing of parallel cylinders in a plane.

For euclidean space the densest packing fractions for n -discs are:

For 2 dimensional 2-balls (circles) of equal radius: 71%

For 3 dimensional 3-balls of equal radius: 49%

For 4 dimensional 4-balls of equal radius: in the range 36 to 38%

For 5 dimensional 5-balls of equal radius: in the range 25 to 28%

For 6 dimensional 6-balls of equal radius: in the range 18 to 21%

For 7 dimensional 7-balls of equal radius: in the range: 13 to 15%

For 8 dimensional 8-balls of equal radius: 11.7%

For 9 dimensional 9-balls of equal radius: in the range: 5 to 7%

For 10 dimensional 10-balls of equal radius: in the range: 4 to 6%

In dimensionalities lower than 5 an euclidean n -space has a higher densest possible packing fraction than that of an n -needle and that of an n -disc. This suggests the hypothesis that, for a four dimensional manifold, a singularity would be unstable, tending to expand into a 4 sphere. If that expansion were to overshoot, becoming disc like, there would be a tendency for it to contract again.

However, space is not necessarily euclidean. Packing fractions for hyperbolic space are higher than for euclidean space, but are difficult to calculate. For 3 dimensions and 4 dimensions, densest packing fractions have been calculated as:

For 3 dimensional 3-balls of equal radius: 85.3%

For 4 dimensional 4-balls of equal radius: 71.6%

Compared to euclidean n -space, these figures are 15% and 13% higher respectively. Extrapolating to higher dimensions, for dimensionalities higher than 5, euclidean n -space would still have a lower packing fraction than an n -needle. However, in 5 dimensions, it is possible that for hyperbolic 5-space there could be a denser packing than for a 5-needle.

This analysis suggests that expansion of a singularity into four dimensions for euclidean space or five dimensions for hyperbolic space could be favoured.

5. Types of dimensions

The original Kaluza-Klein theory[27] accounted for electro-magnetism by introducing an additional dimension. This dimension is not observed, suggesting that it would differ from the observed spatial dimensions. Time is observed, but also differs from spatial dimensions. \mathbb{M} is isomorphic to the Clifford algebra $Cl_{0,5}(R)$ and to $Cl_{1,3}(R) \otimes \mathbb{C}$. $Cl_{0,5}(R)$ is generated using five polar vector unit elements with negative signature. It is postulated that four dimensions corresponds to the dimensions of space and imaginary time and the fifth dimension to the extra dimension for 5D Kaluza-Klein theory, and that conventional time is emergent. This suggests the concept of reality as a 3-dimensional wavefront distorted into a fourth dimension propagating in a fifth dimension.

String and M-theories[28][29] also postulate additional dimensions. \mathbb{U} is of the same order as $Cl_{0,10}(R)$, but has a “natural” partition into $Cl_{1,3}(R) \otimes \mathbb{C} \otimes \mathbb{W}$, suggesting that its use in string/M theories in the place of $Cl_{0,10}(R)$ may generate a description of reality. However, as \mathbb{U} may constitute a representation of a combination of a manifold with 5 polar vectors and 5 axial vectors, this suggests that reality may be embedded in five-dimensional space with torsion, rather than 10-dimensional space. General relativity[31] is usually formulated using the assumption that affine connection has a vanishing torsion tensor, but non-vanishing torsion has been proposed for Einstein-Cartan-Sciama-Kibble and other theories. In an overview[32], Tejinder Singh comments:

“Thus on the one hand we have the torsion-dominated limit, which are the Dirac equations, and on the other hand we have the gravity dominated limit, which are the Einstein equations. In the former case, gravity is absent (Minkowski space-time) and matter behaviour is quantum. In the latter case matter behaviour is classical, and gravity dominates over torsion. Thus we may conclude that there must be a more general underlying theory in which the torsion-free part and the torsion part of the spin-connection are both present, and to which GTR and quantum theory are both approximations.”

The number of degrees of freedom for torsion for a given dimensionality are limited. For a manifold of dimension $d = 5$ with a maximally symmetric submanifold of dimension $n = 4$, there are up to $1 + 4 + 6 = 11$ allowed torsion components which are in general functions of the fifth coordinate[33]. The Einstein field equations (for four dimensions) have 10 degrees of freedom, four of which are unphysical. This suggests a correlation between the $4 + 6$ allowed torsion components and the degrees of freedom for physical space-time with imaginary time substituted for time to make the four dimensions symmetric.

6. The Higgs mechanism

A grand unification algebra should provide a basis for the Higgs mechanism[26]. The mexican hat potential is unusual. This section describes algebraic elements that could account for that potential.

The Higgs mechanism acts on a complex doublet and involves scalar fields. For \mathbb{U} a scalar subalgebra can be assembled as the product:

$$[\sigma_o S, \sigma_o i S, \alpha_o S, \alpha_o i S] \otimes [\sigma_o S, \sigma_o T, \sigma_o V, \sigma_o U] \otimes [\sigma_o S, \lambda_o S, \mu_o S, \nu_o S].$$

$[\sigma_o S, \sigma_o T, \sigma_o V, \sigma_o U] \otimes [\sigma_o S, \lambda_o S, \mu_o S, \nu_o S]$ is isomorphic to $\mathbb{H} \otimes \mathbb{H}$ and to $M_4(R)$. Its unit elements can be assigned unit matrices from table 1 as follows:

$$[\sigma_o S] = [S], [\sigma_o T, \sigma_o V, \sigma_o U] = [TVU], [\lambda_o S, \mu_o S, \nu_o S] = [LMN]$$

$$[\lambda_o T, \mu_o T, \nu_o T] = [PQR], [\lambda_o V, \mu_o V, \nu_o V] = [DEF], [\lambda_o U, \mu_o U, \nu_o U] = [XYZ]$$

Subalgebras of $M_4(R)$ for which the scalar component is associated with a mexican hat potential can be found by considering unitary abelian subgroups of $M_4(R)$. Unitary abelian subgroups of $M_4(R)$ can be represented by diagonal 4×4 matrices.

$$\begin{bmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{bmatrix}$$

where $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$, allowing it to be rewritten:

$$\begin{bmatrix} e^{ia} & 0 & 0 & 0 \\ 0 & e^{ib} & 0 & 0 \\ 0 & 0 & e^{ic} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of two elements of this type with parameters a, b, c and a', b', c' has parameters $a + a', b + b', c + c'$. A subgroup of the Heisenberg group $H(5)$ shares this property:

$$\begin{bmatrix} 1 & a & b & c + ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has determinant = 1, and the commuting products of the form:

$$\begin{bmatrix} 1 & a + a' & b + b' & c + c' + (a + a') \times (b + b') \\ 0 & 1 & 0 & b + b' \\ 0 & 0 & 1 & a + a' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix can be written in terms of unit elements of $M_4(R)$ as:

$$[S] + a/2[V + Y] + b/2[M + F] + (c + ab)/4[E + U + N + P].$$

There are other combinations of unit elements of $M_4(C)$ with similar properties. To find these combinations, it is useful to arrange the matrices in an array with anticommuting basis matrices and the identity in each row/column, forming a 6×6 array:

$$\begin{bmatrix} S & V & T & X & Y & Z \\ V & S & U & P & Q & R \\ T & U & S & D & E & F \\ X & P & D & S & N & M \\ Y & Q & E & N & S & L \\ Z & R & F & M & L & S \end{bmatrix}$$

Interchanging rows and matching columns preserves commutation/anticommutation relationships and group properties with respect to position in the array. For example, rows and columns 1 and 2 can be interchanged to make the array:

$$\begin{bmatrix} S & V & U & P & Q & R \\ V & S & T & X & Y & Z \\ U & T & S & D & E & F \\ P & X & D & S & N & M \\ Q & Y & E & N & S & L \\ R & Z & F & M & L & S \end{bmatrix}$$

Inspecting this array to assign unit matrices for an equivalent H5 subgroup group, they would be:

$$[S] + a/2[V + Q] + b/2[M + F] + (c + ab)/4[E + N + T + X]$$

This combination has the same properties. Interchanging rows and columns 1 and 2 has not changed the signatures of the matrices allocated to each position.

If a further interchange is made that does affect the signatures, e.g interchanging rows and columns 1 and 4, to generate:

$$\begin{bmatrix} S & Q & U & P & V & R \\ Q & S & E & N & Y & L \\ U & E & S & D & T & F \\ P & N & D & S & X & M \\ V & Y & E & N & S & L \\ R & L & F & M & Z & S \end{bmatrix}$$

For the combination:

$$[S] + a/2[Y + Q] + b/2[M + F] + (c + ab)/4[P + U + T + X]$$

The determinant is no longer 1. To make this combination generate a unitary matrix, a factor has to be applied to $[S]$. That factor is $\sqrt{(\pm 1 \pm 2(a/2)^2)}$, provided that the factor is real and not imaginary.

For the resulting matrix, there are four plus/minus permutations, for which the possible matrices for $[S]$ are:

$$\begin{bmatrix} \sqrt{(1+a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(1+a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(1+a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(1+a^2/2)} \end{bmatrix}$$

Which always has real entries, and determinant = $1 + a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(-1-a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(-1-a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(-1-a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(-1-a^2/2)} \end{bmatrix}$$

Which never has real entries, and determinant = $1 + a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(1-a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(1-a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(1-a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(1-a^2/2)} \end{bmatrix}$$

Which has real entries for $a^2/2 \leq 1$, and determinant = $1 - a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(-1+a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(-1+a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(-1+a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(-1+a^2/2)} \end{bmatrix}$$

Which has real entries for $a^2/2 \geq 1$, and determinant = $1 - a^2 + a^4/4$

The function $f(a) = 1 - a^2 + a^4/4$ has the form of a mexican hat potential.

For the assignment of unit elements of \mathbb{U} to matrices:

$$[\sigma_o S] = [S], [\sigma_o T, \sigma_o V, \sigma_o U] = [TVU], [\lambda_o S, \mu_o S, \nu_o S] = [LMN]$$

$$[\lambda_o T, \mu_o T, \nu_o T] = [PQR], [\lambda_o V, \mu_o V, \nu_o V] = [DEF], [\lambda_o U, \mu_o U, \nu_o U] = [XYZ]$$

The group represented by a plus/minus choice for:

$$\sqrt{(\pm 1 \pm a^2/2)}[S] + a/2[Y + Q] + b/2[M + F] + (c + ab)/4[P + U + T + X]$$

is isomorphic to that for the same plus/minus choice for:

$$\sqrt{(\pm 1 \pm a^2/2)}[\sigma_o S] + a/2[\mu_o U + \mu_o T] + b/2[\mu_o S + \nu_o V]$$

$$+ (c + ab)/4[\lambda_o T + \sigma_o U + \sigma_o T + \lambda_o U]$$

which can be rearranged into:

$$\sqrt{(\pm 1 \pm a^2/2)}[\sigma_o S] + (U + T) \times [a/2\mu_o + (c + ab)/4\sigma + (c + ab)/4\lambda] + b/2[\mu_o S + \nu_o V]$$

for which $[TVU]$ symmetry is broken.

7. Subalgebra tables

TABLE 2. Classification of sub-algebras with 8 unit elements of $\mathbb{W} \cong M_4(C)$ with respect to unit elements of a $Cl_{1,3}$ multivector

Type 1 subgroups having $[+ - - + + -]$ signature							
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements
1a	SLMNDEFV	1a	SLiMiNDiEiFV	1a	SiLMinDiEiFV	1a	SiLiMnDiEiFV
1b	SLMNpQRT	1b	SLiMinPiQiRT	1b	SiLMinPiQiRT	1b	SiLiMnPiQiRT
1c	SLMNXYZU	1c	SLiMinXiYiZU	1c	SiLMinXiYiZU	1c	SiLiMnXiYiZU
1d	SVTUDPXL	1d	SViTiuDiPiXL	1d	SiVTiuDiPiXL	1d	SiViTuDiPiXL
1d	SVTUEQYM	1d	SViTiuEiQiYM	1d	SiVTiuEiQiYM	1d	SiViTuEiQiYM
1d	SVTUFrZn	1d	SViTiuFiriZn	1d	SiVTiuFiriZn	1d	SiViTuFiriZn
1e	SLiEiFiuRQiX	1e	SLEFiUiriQiX	1e	SiLEFURiQiX	1e	SiLEiFUiRQiX
1e	SMFiDiUPRiY	1e	SMFDiUiPiRiY	1e	SiMFDUPiRiY	1e	SiMFiDUiPRiY
1e	SNiDiEiUQPz	1e	SNDEiUiQiPz	1e	SiNiDEUQiPz	1e	SiNDiEUiQPz
1f	SLiQriViZiYiD	1f	SLQRiViZiYiD	1f	SiLiQRVZiYiD	1f	SiLQrViZiYiD
1f	SMiRiPViXZiE	1f	SMRPViXiZiE	1f	SiMiRPVXiZiE	1f	SiMRiPViXZiE
1f	SNiPiQiVYXiF	1f	SNPQiViYXiF	1f	SiNiPQVYXiF	1f	SiNPiQiVYXiF
1g	SLiYziTfEiP	1g	SLYZiTiFiEiP	1g	SiLiYZTfEiP	1g	SiLYzTfEiP
1g	SMZiXTDFiQ	1g	SMZXiTiDFiQ	1g	SiMzXTDFiQ	1g	SiMziXTDFiQ
1g	SNXiYiTEDiR	1g	SNXYTiEiDiR	1g	SiNXiYTeiDiR	1g	SiNXiYTeiDiR
Type 2 subgroups having $[+ - - - - +]$ signature							
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements
2a	SLMNiDiEiFiv						
2b	SLMNiPiQiRiT						
2c	SLMNXiYiZiU						
2d	SVTUiDiPiXiL						
2d	SVTUiEiQiYiM						
2d	SVTUiFiriZin						
2e	SLiEiFUiRiQX						
Type 3 subgroups having $[+ - + + - + +]$ signature							
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements
3a	SLiMiNDiEiFiv	3a	SLiMiNDiEiFiv	3a	SiLiMnDeiFiv		
3b	SLiMinPiQiRiT	3b	SLiMinPiQiRiT	3b	SiLMinPiQiRiT		
3c	SLiMinXiYiZiU	3c	SLiMinXiYiZiU	3c	SiLMinXiYiZiU		
3d	SViTiuDPXiL	3d	SViTiuDPXiL	3d	SiViTuDPXiL		
3d	SViTiuEQYiM	3d	SViTiuEQYiM	3d	SiViTUEQiYiM		
3d	SViTiuFRZin	3d	SViTiuFRZin	3d	SiViTuFRZin		
3e	SLEFURQX	3e	SLiEiFUiRiQX	3e	SiLEiFUiRiQX		
3e	SMFDUPRY	3e	SMiFDiUiPRiY	3e	SiMFDiUiPRiY		
3e	SNDEUQPZ	3e	SiNiDEiUiQPz	3e	SiNDiEiUQiPz		
3f	SLQRVZYD	3f	SiLiQRiViZiYD	3f	SiLQrViZiYD		
3f	SMRPVXZE	3f	SiMiRPViXiZE	3f	SiMRiPViXiZE		
3f	SNPQVYXF	3f	SiNiPQiViYXF	3f	SiNPiQiVYXiF		
3g	SLYZTEFP	3g	SiLYziTfEiFP	3g	SiLYzTfEiFP		
3g	SMZXTFDQ	3g	SiMzXiTfDiQ	3g	SiMzXTfDiQ		
3g	SNXYTDER	3g	SiNXiYTiDER	3g	SiNXiYTeiDER		
Type 4 subgroups having $[+ + + - - -]$ signature							
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements
4a	SiVdiLviDLiS	4a	SiVeImViEMiS	4a	SiVFiNViFNiS		
4b	SiTPiLtiPLiS	4b	SiTQiMtQiMiS	4b	SiTRiNtiRNiS		
4c	SXiUiLiXULiS	4c	SYiUiMiYUMiS	4c	SZiUiNiZUNiS		
4d	SXERiXiEiRiS	4d	SYPFiYiPiFiS	4d	SZDQiZiDiQiS		
4d	SXQFiXiQiFiS	4d	SYDRiYiDiRiS	4d	SZPEiZiPiEiS		
Type 5 subgroups having $[+ - - + + -]$ signature							
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements
5a	SLMNLiMiNiS						
5b	SLiEiFiLEFiS	5b	SMiFiDiMFDiS	5b	SNiDiEiNDEiS		
5c	SLiQriRiLQRiS	5c	SMiRiPiMRPiS	5c	SNiPiQiNFPQiS		
5d	SLiYziLiLYziS	5d	SMiZiXiMZXiS	5d	SNiXiYiNXyiS		
5e	SVTUiViTiUiS						
5f	SViPiXiVpXiS	5f	SViQiYiVqYiS	5f	SViRiZiVRZiS		
5g	STXiDiTXDiS	5g	STiYiEiTYEiS	5g	STiZiFiTZFiS		
5h	SUIDiPiUDPiS	5h	SUiEiQiUEQiS	5h	SUiFiriUFRiS		

TABLE 10. Unit elements for aligned M, T and W sub-algebras with 16 unit elements

M sub-group		T sub-loop		W sub-loop
Ref.	Unit elements	Ref.	Unit elements	Unit elements
E ₀	SLMNXYZ VDEFTPQR	S ₀	$\sigma_0 \sigma_i \sigma_j \sigma_k \gamma_0 \gamma_i \gamma_j \gamma_k$ $\lambda_0 \lambda_i \lambda_j \lambda_k \delta_0 \delta_i \delta_j \delta_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \gamma_0 U \gamma_i X \gamma_j Y \gamma_k Z$ $\lambda_0 V \lambda_i D \lambda_j E \lambda_k F \delta_0 T \delta_i P \delta_j Q \delta_k R$
E ₁	SLMNiViDiEiF iUiXiYiZTPQR	S ₁	$\sigma_0 \sigma_i \sigma_j \sigma_k \beta_0 \beta_i \beta_j \beta_k$ $\mu_0 \mu_i \mu_j \mu_k \delta_0 \delta_i \delta_j \delta_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \beta_0 iV \beta_i iD \beta_j iE \beta_k iF$ $\mu_0 iU \mu_i iX \mu_j iY \mu_k iZ \delta_0 T \delta_i P \delta_j Q \delta_k R$
E ₂	SLUXiViDiTiP iMiNiYiZEFQR	S ₂	$\sigma_0 \sigma_i \gamma_0 \gamma_i \beta_0 \beta_i \nu_0 \nu_i$ $\alpha_j \alpha_k \mu_j \mu_k \lambda_j \lambda_k \delta_j \delta_k$	$\sigma_0 S \sigma_i L \gamma_0 U \gamma_i X \beta_0 iV \beta_i iD \nu_0 iT \nu_i iP$ $\alpha_j iM \alpha_k iN \mu_j iY \mu_k iZ \lambda_j E \lambda_k F \delta_j Q \delta_k R$
E ₃	SMUYiViEiTiQ iLiNiXiZDFPR	S ₃	$\sigma_0 \sigma_j \gamma_0 \gamma_j \beta_0 \beta_j \nu_0 \nu_j$ $\alpha_i \alpha_k \mu_i \mu_k \lambda_i \lambda_k \delta_i \delta_k$	$\sigma_0 S \sigma_j M \gamma_0 U \gamma_j Y \beta_0 iV \beta_j iE \nu_0 iT \nu_j iQ$ $\alpha_i iL \alpha_k iN \mu_i iX \mu_k iZ \lambda_i D \lambda_k F \delta_i P \delta_k R$
E ₄	SNUZiViFiTiR iMiXiYiZDEPQ	S ₄	$\sigma_0 \sigma_k \gamma_0 \gamma_k \beta_0 \beta_k \nu_0 \nu_k$ $\alpha_i \alpha_j \mu_i \mu_j \lambda_i \lambda_j \delta_i \delta_j$	$\sigma_0 S \sigma_k N \gamma_0 U \gamma_k Z \beta_0 iV \beta_k iF \nu_0 iT \nu_k iR$ $\alpha_i iL \alpha_j iM \mu_i iX \mu_j iY \lambda_i D \lambda_j E \delta_i P \delta_j Q$
E ₅	SLYZiViDiQiR iMiNiUiXEFTP	S ₅	$\sigma_0 \sigma_i \gamma_j \gamma_k \beta_0 \beta_i \nu_j \nu_k$ $\alpha_j \alpha_k \mu_0 \mu_i \lambda_j \lambda_k \delta_0 \delta_i$	$\sigma_0 S \sigma_i L \gamma_j Y \gamma_k Z \beta_0 iV \beta_i iD \nu_j iQ \nu_k iR$ $\alpha_j iM \alpha_k iN \mu_0 iU \mu_i iX \lambda_j E \lambda_k F \delta_0 T \delta_i P$
E ₆	SMXZiViDiFiP iLiNiXiZVFTQ	S ₆	$\sigma_0 \sigma_j \gamma_i \gamma_k \beta_0 \beta_j \nu_i \nu_k$ $\alpha_i \alpha_k \mu_0 \mu_j \lambda_i \lambda_k \delta_0 \delta_j$	$\sigma_0 S \sigma_j M \gamma_i X \gamma_k Z \beta_0 iV \beta_j iE \nu_i iP \nu_k iR$ $\alpha_i iL \alpha_k iN \mu_0 iU \mu_j iY \lambda_i D \lambda_k F \delta_0 T \delta_j Q$
E ₇	SNXYiViFiPiQ iLiMiUiZDDETR	S ₇	$\sigma_0 \sigma_k \gamma_i \gamma_j \beta_0 \beta_k \nu_i \nu_j$ $\alpha_i \alpha_j \mu_0 \mu_k \lambda_i \lambda_j \delta_0 \delta_k$	$\sigma_0 S \sigma_k N \gamma_i X \gamma_j Y \beta_0 iV \beta_k iF \nu_i iP \nu_j iQ$ $\alpha_i iL \alpha_j iM \mu_0 iU \mu_k iZ \lambda_i D \lambda_j E \delta_0 T \delta_k R$
E ₈	SLMNiTiPiQiR iUiXiYiZVDEF	S ₈	$\sigma_0 \sigma_i \sigma_j \sigma_k \nu_0 \nu_i \nu_j \nu_k$ $\mu_0 \mu_i \mu_j \mu_k \lambda_0 \lambda_i \lambda_j \lambda_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \nu_0 iT \nu_i iP \nu_j iQ \nu_k iR$ $\mu_0 iU \mu_i iX \mu_j iY \mu_k iZ \lambda_0 V \lambda_i D \lambda_j E \lambda_k F$
E ₉	SLUXiViDiFiQiR iMiNiYiZVDTQ	S ₉	$\sigma_0 \sigma_i \gamma_0 \gamma_i \beta_j \beta_k \nu_j \nu_k$ $\alpha_j \alpha_k \mu_j \mu_k \lambda_0 \lambda_i \delta_0 \delta_i$	$\sigma_0 S \sigma_i L \gamma_0 U \gamma_i X \beta_j iE \beta_k iF \nu_j iQ \nu_k iR$ $\alpha_j iM \alpha_k iN \mu_j iY \mu_k iZ \lambda_0 V \lambda_i D \delta_0 T \delta_i P$
E ₁₀	SMUYiViDiFiPiR iLiNiXiZVETQ	S ₁₀	$\sigma_0 \sigma_j \gamma_0 \gamma_j \beta_i \beta_k \nu_i \nu_k$ $\alpha_i \alpha_k \mu_i \mu_k \lambda_0 \lambda_j \delta_0 \delta_j$	$\sigma_0 S \sigma_j M \gamma_0 U \gamma_j Y \beta_i iD \beta_k iF \nu_i iP \nu_k iR$ $\alpha_i iL \alpha_k iN \mu_i iX \mu_k iZ \lambda_0 V \lambda_j E \delta_0 T \delta_j Q$
E ₁₁	SNUZiViDiFiPiQ iLiMiUiYVFTTR	S ₁₁	$\sigma_0 \sigma_k \gamma_0 \gamma_k \beta_i \beta_j \nu_i \nu_j$ $\alpha_i \alpha_j \mu_i \mu_j \lambda_0 \lambda_k \delta_0 \delta_k$	$\sigma_0 S \sigma_k N \gamma_0 U \gamma_k Z \beta_i iD \beta_j iE \nu_i iP \nu_j iQ$ $\alpha_i iL \alpha_j iM \mu_i iX \mu_j iY \lambda_0 V \lambda_k F \delta_0 T \delta_k R$
E ₁₂	SLYZiViDiFiTiP iMiNiUiXVDQR	S ₁₂	$\sigma_0 \sigma_i \gamma_j \gamma_k \beta_j \beta_k \nu_0 \nu_i$ $\alpha_j \alpha_k \mu_0 \mu_i \lambda_0 \lambda_i \delta_j \delta_k$	$\sigma_0 S \sigma_i L \gamma_j Y \gamma_k Z \beta_j iE \beta_k iF \nu_0 iT \nu_i iP$ $\alpha_j iM \alpha_k iN \mu_0 iU \mu_i iX \lambda_0 V \lambda_i D \delta_j Q \delta_k R$
E ₁₃	SMXZiViDiFiTiQ iLiNiUiYVVEPR	S ₁₃	$\sigma_0 \sigma_j \gamma_i \gamma_k \beta_i \beta_k \nu_0 \nu_j$ $\alpha_i \alpha_k \mu_0 \mu_j \lambda_0 \lambda_j \delta_i \delta_k$	$\sigma_0 S \sigma_j M \gamma_i X \gamma_k Z \beta_i iD \beta_k iF \nu_0 iT \nu_j iQ$ $\alpha_i iL \alpha_k iN \mu_0 iU \mu_j iY \lambda_0 V \lambda_j E \delta_i P \delta_k R$
E ₁₄	SNXYiViDiEiPiQ iLiMiUiZVFPQ	S ₁₄	$\sigma_0 \sigma_k \gamma_i \gamma_j \beta_i \beta_j \nu_0 \nu_k$ $\alpha_i \alpha_j \mu_0 \mu_k \lambda_0 \lambda_k \delta_i \delta_j$	$\sigma_0 S \sigma_k N \gamma_i X \gamma_j Y \beta_i iD \beta_j iE \nu_0 iT \nu_k iR$ $\alpha_i iL \alpha_j iM \mu_0 iU \mu_k iZ \lambda_0 V \lambda_k F \delta_i P \delta_j Q$
E ₁₅	SLMNXYZ iViDiEiFiTiPiQiR	S ₁₅	$\sigma_0 \sigma_i \sigma_j \sigma_k \gamma_0 \gamma_i \gamma_j \gamma_k$ $\beta_0 \beta_i \beta_j \beta_k \nu_0 \nu_i \nu_j \nu_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \gamma_0 U \gamma_i X \gamma_j Y \gamma_k Z$ $\beta_0 iV \beta_i iD \beta_j iE \beta_k iF \nu_0 iT \nu_i iP \nu_j iQ \nu_k iR$
E ₁₆	SLMNXYZ iSiLiMiNiUiXiYiZ	S ₁₆	$\sigma_0 \sigma_i \sigma_j \sigma_k \gamma_0 \gamma_i \gamma_j \gamma_k$ $\alpha_0 \alpha_i \alpha_j \alpha_k \mu_0 \mu_i \mu_j \mu_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \gamma_0 U \gamma_i X \gamma_j Y \gamma_k Z$ $\alpha_0 iS \alpha_i iL \alpha_j iM \alpha_k iN \mu_0 iU \mu_i iX \mu_j iY \mu_k iZ$
E ₁₇	SLMNiViDiEiFi iSiLiMiNDEF	S ₁₇	$\sigma_0 \sigma_i \sigma_j \sigma_k \beta_0 \beta_i \beta_j \beta_k$ $\alpha_0 \alpha_i \alpha_j \alpha_k \lambda_0 \lambda_i \lambda_j \lambda_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \beta_0 iV \beta_i iD \beta_j iE \beta_k iF$ $\alpha_0 iS \alpha_i iL \alpha_j iM \alpha_k iN \lambda_0 V \lambda_i D \lambda_j E \lambda_k F$
E ₁₈	SLMNiTiPiQiR iSiLiMiNTPQR	S ₁₈	$\sigma_0 \sigma_i \sigma_j \sigma_k \nu_0 \nu_i \nu_j \nu_k$ $\alpha_0 \alpha_i \alpha_j \alpha_k \delta_0 \delta_i \delta_j \delta_k$	$\sigma_0 S \sigma_i L \sigma_j M \sigma_k N \nu_0 iT \nu_i iP \nu_j iQ \nu_k iR$ $\alpha_0 iS \alpha_i iL \alpha_j iM \alpha_k iN \delta_0 T \delta_i P \delta_j Q \delta_k R$
E ₁₉	SLUXiViDiTiP iSiLiUiXVDTQ	S ₁₉	$\sigma_0 \sigma_i \gamma_0 \gamma_i \beta_0 \beta_i \nu_0 \nu_i$ $\alpha_0 \alpha_i \mu_0 \mu_i \lambda_0 \lambda_i \delta_0 \delta_i$	$\sigma_0 S \sigma_i L \gamma_0 U \gamma_i X \beta_0 iV \beta_i iD \nu_0 iT \nu_i iP$ $\alpha_0 iS \alpha_i iL \mu_0 iU \mu_i iX \lambda_0 V \lambda_i D \delta_0 T \delta_i P$
E ₂₀	SMUYiViDiEiTiQ iSiMiUiYVETQ	S ₂₀	$\sigma_0 \sigma_j \gamma_0 \gamma_j \beta_0 \beta_j \nu_0 \nu_j$ $\alpha_0 \alpha_j \mu_0 \mu_j \lambda_0 \lambda_j \delta_0 \delta_j$	$\sigma_0 S \sigma_j M \gamma_0 U \gamma_j Y \beta_0 iV \beta_j iE \nu_0 iT \nu_j iQ$ $\alpha_0 iS \alpha_j iM \mu_0 iU \mu_j iY \lambda_0 V \lambda_j E \delta_0 T \delta_j Q$
E ₂₁	SNUZiViDiFiTiR iSiNiUiZVFTTR	S ₂₁	$\sigma_0 \sigma_k \gamma_0 \gamma_k \beta_0 \beta_k \nu_0 \nu_k$ $\alpha_0 \alpha_k \mu_0 \mu_k \lambda_0 \lambda_k \delta_0 \delta_k$	$\sigma_0 S \sigma_k N \gamma_0 U \gamma_k Z \beta_0 iV \beta_k iF \nu_0 iT \nu_k iR$ $\alpha_0 iS \alpha_k iN \mu_0 iU \mu_k iZ \lambda_0 V \lambda_k F \delta_0 T \delta_k R$
E ₂₂	SLYZiViDiQiR iSiLiYiZVDQR	S ₂₂	$\sigma_0 \sigma_i \gamma_j \gamma_k \beta_0 \beta_i \nu_j \nu_k$ $\alpha_0 \alpha_i \mu_j \mu_k \lambda_0 \lambda_i \delta_j \delta_k$	$\sigma_0 S \sigma_i L \gamma_j Y \gamma_k Z \beta_0 iV \beta_i iD \nu_j iQ \nu_k iR$ $\alpha_0 iS \alpha_i iL \mu_j iY \mu_k iZ \lambda_0 V \lambda_i D \delta_j Q \delta_k R$
E ₂₃	SMXZiViDiFiPiR iSiMiXiZVVEPR	S ₂₃	$\sigma_0 \sigma_j \gamma_i \gamma_k \beta_0 \beta_j \nu_i \nu_k$ $\alpha_0 \alpha_j \mu_i \mu_k \lambda_0 \lambda_j \delta_i \delta_k$	$\sigma_0 S \sigma_j M \gamma_i X \gamma_k Z \beta_0 iV \beta_j iE \nu_i iP \nu_k iR$ $\alpha_0 iS \alpha_j iM \mu_i iX \mu_k iZ \lambda_0 V \lambda_j E \delta_i P \delta_k R$
E ₂₄	SNXYiViFiPiQ iSiNiXiYVFPQ	S ₂₄	$\sigma_0 \sigma_k \gamma_i \gamma_j \beta_0 \beta_k \nu_i \nu_j$ $\alpha_0 \alpha_k \mu_i \mu_j \lambda_0 \lambda_k \delta_i \delta_j$	$\sigma_0 S \sigma_k N \gamma_i X \gamma_j Y \beta_0 iV \beta_k iF \nu_i iP \nu_j iQ$ $\alpha_0 iS \alpha_k iN \mu_i iX \mu_j iY \lambda_0 V \lambda_k F \delta_i P \delta_j Q$
E ₂₅	SLUXiViDiFiQiR iSiLiUiXEFQR	S ₂₅	$\sigma_0 \sigma_i \gamma_0 \gamma_i \beta_j \beta_k \nu_j \nu_k$ $\alpha_0 \alpha_i \mu_0 \mu_i \lambda_j \lambda_k \delta_j \delta_k$	$\sigma_0 S \sigma_i L \gamma_0 U \gamma_i X \beta_j iE \beta_k iF \nu_j iQ \nu_k iR$ $\alpha_0 iS \alpha_i iL \mu_0 iU \mu_i iX \lambda_j E \lambda_k F \delta_j Q \delta_k R$
E ₂₆	SMUYiViDiFiPiR iSiMiUiYDFPR	S ₂₆	$\sigma_0 \sigma_j \gamma_0 \gamma_j \beta_i \beta_k \nu_i \nu_k$ $\alpha_0 \alpha_j \mu_0 \mu_j \lambda_i \lambda_k \delta_i \delta_k$	$\sigma_0 S \sigma_j M \gamma_0 U \gamma_j Y \beta_i iD \beta_k iF \nu_i iP \nu_k iR$ $\alpha_0 iS \alpha_j iM \mu_0 iU \mu_j iY \lambda_i D \lambda_k F \delta_i P \delta_k R$
E ₂₇	SNUZiViDiEiPiQ iSiNiUiZDDEPQ	S ₂₇	$\sigma_0 \sigma_k \gamma_0 \gamma_k \beta_i \beta_j \nu_i \nu_j$ $\alpha_0 \alpha_k \mu_0 \mu_k \lambda_i \lambda_j \delta_i \delta_j$	$\sigma_0 S \sigma_k N \gamma_0 U \gamma_k Z \beta_i iD \beta_j iE \nu_i iP \nu_j iQ$ $\alpha_0 iS \alpha_k iN \mu_0 iU \mu_k iZ \lambda_i D \lambda_j E \delta_i P \delta_j Q$
E ₂₈	SLYZiViDiFiTiP iSiLiYiZEFTP	S ₂₈	$\sigma_0 \sigma_i \gamma_j \gamma_k \beta_j \beta_k \nu_0 \nu_i$ $\alpha_0 \alpha_i \mu_j \mu_k \lambda_j \lambda_k \delta_0 \delta_i$	$\sigma_0 S \sigma_i L \gamma_j Y \gamma_k Z \beta_j iE \beta_k iF \nu_0 iT \nu_i iP$ $\alpha_0 iS \alpha_i iL \mu_j iY \mu_k iZ \lambda_j E \lambda_k F \delta_0 T \delta_i P$
E ₂₉	SMXZiViDiFiTiQ iSiMiXiZDFTQ	S ₂₉	$\sigma_0 \sigma_j \gamma_i \gamma_k \beta_i \beta_k \nu_0 \nu_j$ $\alpha_0 \alpha_j \mu_i \mu_k \lambda_i \lambda_k \delta_0 \delta_j$	$\sigma_0 S \sigma_j M \gamma_i X \gamma_k Z \beta_i iD \beta_k iF \nu_0 iT \nu_j iQ$ $\alpha_0 iS \alpha_j iM \mu_i iX \mu_k iZ \lambda_i D \lambda_k F \delta_0 T \delta_j Q$
E ₃₀	SNXYiViDiEiTiR iSiNiXiYDETR	S ₃₀	$\sigma_0 \sigma_k \gamma_i \gamma_j \beta_i \beta_j \nu_0 \nu_k$ $\alpha_0 \alpha_k \mu_i \mu_j \lambda_i \lambda_j \delta_0 \delta_k$	$\sigma_0 S \sigma_k N \gamma_i X \gamma_j Y \beta_i iD \beta_j iE \nu_0 iT \nu_k iR$ $\alpha_0 iS \alpha_k iN \mu_i iX \mu_j iY \lambda_i D \lambda_j E \delta_0 T \delta_k R$

8. Basis of the notation used for unit elements of $\mathbb{M} \cong M_4(C)$, and its Cayley table

Capital roman letters are used to label real matrix unit elements of $M_4(R)$, as shown in table 11. These labels are combined with i to represent imaginary counterparts. Their Cayley table is shown in table 12.

TABLE 11. Notation used to label 4×4 unit matrices

$$\begin{array}{l}
 S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\
 V = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
 iU = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \quad iX = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad iY = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \quad iZ = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \\
 iT = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \quad iP = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \quad iQ = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \quad iR = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} \\
 iS = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \quad iL = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad iM = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad iN = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \\
 iV = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \quad iD = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \quad iE = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \quad iF = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \\
 U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}
 \end{array}$$

Note that some matrix labels have been changed from those used in a previous paper [34].

9. Basis of notation used for unit elements of \mathbb{T} and its Cayley table

Conventional notation for unit elements of \mathbb{T} would use the numbers 0 to 31, standing alone or subscripted as e_0 to e_{31} . In this paper a different form of notation is used based on a modified Moufang loop construction for \mathbb{T} .

9.1. Moufang Loop construction for octonions

For Moufang loop construction of octonions, based on quaternion pairs, a dis-association operator, ω , is assigned to the second pair, and a product rule which generates octonions can be defined:

For quaternion pair $(p, \omega p') \times$ quaternion pair $(q, \omega q')$:

$$p.q = (pq)$$

$$p.\omega q' = \omega(p^{-1}q')$$

$$\omega p'.q = \omega(qp')$$

$$\omega p'.\omega q' = -(q'p'^{-1})$$

9.2. Modified Moufang Loop construction for trigtaduonions

The Moufang loop construction for octonions uses one dis-association operator. An identity operator, σ , and a set of seven dis-association operators, $\lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$, can be applied to quaternions to define trigtaduonions, where products for two quaternions: p and q , are constructed in accordance with table 14.

TABLE 14. Multiplication procedures for non-associative components

	σq	λq	μq	νq	αq	βq	γq	δq
σp	$+\sigma pq$	$+\lambda qp$	$+\mu qp$	$+\nu qp^{-1}$	$+\alpha qp$	$+\beta qp^{-1}$	$+\gamma qp^{-1}$	$+\delta qp$
λp	$+\lambda pq^{-1}$	$-\sigma q^{-1}p$	$+\nu pq$	$-\mu p^{-1}q$	$+\beta pq$	$-\alpha p^{-1}q$	$-\delta pq$	$+\gamma p^{-1}q$
μp	$+\mu pq^{-1}$	$-\nu qp$	$-\sigma q^{-1}p$	$+\lambda qp^{-1}$	$+\gamma pq$	$+\delta qp$	$-\alpha p^{-1}q$	$-\beta qp^{-1}$
νp	$+\nu pq$	$+\mu q^{-1}p$	$-\lambda pq^{-1}$	$-\sigma p^{-1}q$	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$-\alpha q^{-1}p$
αp	$+\alpha pq^{-1}$	$-\beta qp$	$-\gamma qp$	$-\delta qp^{-1}$	$-\sigma q^{-1}p$	$+\lambda qp^{-1}$	$+\mu qp^{-1}$	$+\nu qp$
βp	$+\beta pq$	$+\alpha q^{-1}p$	$-\delta pq$	$+\gamma p^{-1}q$	$-\lambda pq^{-1}$	$-\sigma p^{-1}q$	$-\nu pq$	$+\mu p^{-1}q$
γp	$+\gamma pq$	$+\delta qp$	$+\alpha q^{-1}p$	$-\beta qp^{-1}$	$-\mu pq^{-1}$	$+\nu qp$	$-\sigma p^{-1}q$	$-\lambda qp^{-1}$
δp	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$+\alpha p^{-1}q$	$-\nu pq$	$-\mu q^{-1}p$	$+\lambda pq^{-1}$	$-\sigma q^{-1}p$

Unit trigtaduonions have been labeled using: $\sigma, \lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$, combined with subscripts: o, i, j, κ . Each label denotes a unit quaternion and a dis-association operator defining a multiplication procedure. Their Cayley table is shown in table 13.

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