Vortex Gradient Formula

We expound the gradient of Vorticity tensor formula in general coordinates as treated in relativistic mechanics .

The formula of the gradient of the Vorticity tensor is derived in general coordinates as treated in classical and relativistic continuum mechanics and as groundwork of Tailherer's theory [2].
The basic equations are those of vortex kinematics encountered in lagrangian description of continua [1] relating the angular velocity tensor to the deformation velocity $K_{\alpha\beta} = ½\partial_\tau g_{\alpha\beta}$ (as remarked in [2] identified with the second fundamental tensor relative to V_4): let us start by considering all the points-event of the space spanned by the particles of a continuum as parameterized with their co-ordinates representing the position vector OP with respect to an arbitrary origin O , and a local frame referred to a local basis of vectors $\boldsymbol{e}_\alpha = \partial OP/\partial x^\alpha$ whose the metric tensor $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ and the countervariant frame $g^{\alpha\beta}$ associated with, such that $g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha{}^\beta$. We consider the lagrangian metric $g_{\alpha\beta}(x^\alpha/\tau)$ as function of the trajectory line's variables x^α and time τ , and so $\boldsymbol{e}_\alpha$. Since our reasoning might be done in the 4-dimensional cronotope too (so τ is referred to as the proper time), it follows that if the relations hold in each tern subspace, as we shall see they do, they will keep holding in the whole 4-dimensional space for the same equation that we shall get. So, let us choose without loss of generality the tern referring to the space indexes h=1,2,3. Let us consider now the gradient of the space components of the velocity which will be of the type:

$$\partial_h \boldsymbol{v} = q_{hk} \boldsymbol{e}^k \qquad (\partial_h = \partial / \partial x^h \quad h,k = 1,2,3) \tag{A.1}$$

The matrix q_{hk} can always be split up in a symmetrical part and a skew-symmetric one

$$q_{hk} = \partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k = K_{hk} + \omega_{hk} \tag{A.2}$$

with symmetrical part

$$K_{hk} = 1/2(\partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k + \partial_k \boldsymbol{v} \cdot \boldsymbol{e}_h) = K_{kh} \tag{A.3}$$

and skew-symmetric

$$\omega_{hk} = 1/2(\partial_h \boldsymbol{v} \cdot \boldsymbol{e}_k - \partial_k \boldsymbol{v} \cdot \boldsymbol{e}_h) = -\omega_{kh} \tag{A.4}$$

Since

$$\partial_\tau \boldsymbol{e}_h = \frac{\partial^2 OP}{\partial \tau \, \partial x^h} = \frac{\partial^2 OP}{\partial x^h \, \partial \tau} = \partial_h \boldsymbol{v} \tag{A.5}$$

equ.(A.3) will be written as:

$$K_{hk} = 1/2(\partial_\tau \boldsymbol{e}_h \cdot \boldsymbol{e}_k + \partial_\tau \boldsymbol{e}_k \cdot \boldsymbol{e}_h) = 1/2\partial_\tau (\boldsymbol{e}_h \cdot \boldsymbol{e}_k) = 1/2\partial_\tau g_{hk} \tag{A.6}$$

which can be referred to the second fundamental tensor as already outlined in [2] where it was denoted as deformation velocity of the metric. For what concerns ω_{hk} by taking (A.5) into account let us introduce the vector

$$\boldsymbol{\omega} = ½\, \boldsymbol{e}^h \times \partial_\tau \boldsymbol{e}_h = ½\, \boldsymbol{e}^h \times \partial_h \boldsymbol{v} \qquad \text{with} \quad \times \quad \text{exterior product} \tag{A.7}$$

Then from the (A.2) and (A.3) we have successively:

$$\boldsymbol{\omega} = ½\, \boldsymbol{e}^h \times (K_{hk} + \omega_{hk})\, \boldsymbol{e}^k = ½\, (K_{hk}\, \boldsymbol{e}^h \times \boldsymbol{e}^k) + ½\, (\omega_{hk}\, \boldsymbol{e}^h \times \boldsymbol{e}^k) \tag{A.8}$$

which in account of the symmetry of K_{hk} and the skew-symmetry of $\boldsymbol{e}^{h}\times \boldsymbol{e}^{k}$ becomes:

$$\boldsymbol{\omega} = ½\,(\omega_{hk}\,\boldsymbol{e}^{h}\times \boldsymbol{e}^{k}) \tag{A.9}$$

$\boldsymbol{\omega}$ will be named angular velocity and characterised by the coefficients ω_{hk}.
Moreover, if we multiply (A.7) by $\boldsymbol{e}_h$ through the exterior product, taking into account that $\boldsymbol{e}^{h}\cdot \boldsymbol{e}_k=\delta^{h}{}_{k}$, we get:
$\boldsymbol{\omega}\times \boldsymbol{e}_h = ½\,(\omega_{lk}\,\boldsymbol{e}^{l}\times \boldsymbol{e}^{k})\times \boldsymbol{e}_h = ½\,\omega_{lk}(\boldsymbol{e}^{l}\cdot \boldsymbol{e}_h\,\boldsymbol{e}^{k} - \boldsymbol{e}^{k}\cdot \boldsymbol{e}_h\;\boldsymbol{e}^{l}) = ½\,\omega_{lk}(\delta^{l}{}_{h}\,\boldsymbol{e}^{k} - \delta^{k}{}_{h}\,\boldsymbol{e}^{l})$ and therefore the relation:

$$\boldsymbol{\omega}\times \boldsymbol{e}_h = \omega_{hk}\,\boldsymbol{e}^{k} \qquad (h,k=1,2,3) \tag{A.10}$$

Let us now make some recalls. By differentiating the vectors $\boldsymbol{e}_h\,(x^{\mu}/\tau)$ of the local base with respect to the proper time we get the gradient of the space components of 4-velocity as from (A.5). In deriving them with respect to x^{μ} we get for definition the Christoffel symbols as well-known in differential geometry:

$$\partial_j \boldsymbol{e}_h = \Gamma^{k}{}_{jh}\,\boldsymbol{e}_k \qquad\qquad \partial_j\, \boldsymbol{e}^{h} = -\,\Gamma^{h}{}_{jk}\,\boldsymbol{e}^{k} \tag{A.11}$$

Let us recall the links between the Christoffel symbols of the first and second kind:

$$\Gamma^{k}{}_{jh} = g^{kr}\,\Gamma_{jh,r} \qquad\qquad \Gamma_{jh,r} = g_{rk}\,\Gamma^{k}{}_{jh} \tag{A.12}$$

From (A.11) it turns out that

$$\partial_j \boldsymbol{v} = \partial_j (v_h\,\boldsymbol{e}^{h}) = (\partial_j v_h - \Gamma^{k}{}_{jh}\,v_k)\,\boldsymbol{e}^{h} = (\nabla_j v_h)\,\boldsymbol{e}^{h} \tag{A.13}$$

leading via (A.4) to the expression:

$$\omega_{hk} = 1/2\,(\nabla_h v_k - \nabla_k v_h) = 1/2\,(\partial_h v_k - \partial_k v_h) \tag{A.14}$$

and by taking advantage of the symmetry of Christoffel symbols with respect to inferior indexes.
Analogously we get for (A.3):

$$K_{hk} = 1/2\,(\nabla_h v_k + \nabla_k v_h) \tag{A.15}$$

as usual for the deformation tensor.
The gradient of the velocity expressed in terms of deformation and angular velocity follows as from (A.1), (A.2) and (A.10):

$$\partial_h \boldsymbol{v} = \underline{K}_h + \boldsymbol{\omega}\times \boldsymbol{e}_h \tag{A.16}$$

with $\boldsymbol{K}_h$ following as from (A.6):

$$\boldsymbol{K}_h = K_{hk}\,\boldsymbol{e}^{k} = ½\,\partial_\tau K_{hk}\,\boldsymbol{e}^{k} \tag{A.17}$$

From (A.16) we can infer $\boldsymbol{\omega}$ to depend on $\boldsymbol{K}$, that is to say, on the deformation velocity as will be seen better next. To see that let us derive both the members of (A.9) with respect to x^{j}. We get:

$$\partial_j \boldsymbol{\omega} = ½\,\partial_j \boldsymbol{e}^{h}\times \partial_\tau \boldsymbol{e}_h + ½\,\boldsymbol{e}^{h}\times \partial_\tau(\partial_j \boldsymbol{e}_h) \tag{A.18}$$

as well as on using (A.11):

$$\partial_j \boldsymbol{\omega} = -\,½\,\Gamma^{h}{}_{jk}\,\boldsymbol{e}^{k}\times \partial_\tau\,\boldsymbol{e}_h + ½\,\boldsymbol{e}^{h}\times \Gamma^{k}{}_{jh}\,\partial_\tau\,\boldsymbol{e}_k + ½\,\partial_\tau\,(\Gamma^{k}{}_{jh})\,\boldsymbol{e}^{h}\times\,\boldsymbol{e}_k \tag{A.19}$$

Since the first two terms vanish as it is understood by changing the indexes h and k, it turns out:

$$\partial_j \boldsymbol{\omega} = ½\, \partial_\tau (\Gamma^{\,k}_{\,j\,h})\, \boldsymbol{e}^{\,h} \times \boldsymbol{e}_{\,k} \tag{A.20}$$

On the other hand, since $\Gamma_{j\,h\,,\,r} = ½\,(\partial_j g_{hr} + \partial_h g_{rj} - \partial_r g_{jh})$ and taking into account (A.6) we have:

$$\partial_\tau \Gamma_{j\,h\,,\,r} = \partial_j K_{\,h\,r} + \partial_h K_{\,r\,j} - \partial_r K_{\,j\,h} = \nabla_j K_{\,h\,r} + \nabla_h K_{\,r\,j} - \nabla_r K_{\,j\,h} + 2\,\Gamma^{\,k}_{\,j\,h} K_{\,k\,r}$$

where we used the definition of covariant derivative:

$$\nabla_j K_{\,hr} = \partial_j K_{\,hr} - \Gamma^{\,k}_{\,j\,h} K_{\,kr} - \Gamma^{\,k}_{\,j\,r} K_{\,h\,k} \tag{A.21}$$

Making use of the triple tensor $q_{\,j\,h\,,\,r} = \nabla_j K_{\,hr} + \nabla_h K_{\,rj} - \nabla_r K_{\,jh}$ we obtain the following expression of the time derivative of Christoffel symbols of first kind:

$$\partial_\tau \Gamma_{j\,h\,,\,r} = q_{\,j\,h\,,\,r} + \Gamma^{\,k}_{\,j\,h}\, \partial_\tau g_{k\,r} \tag{A.22}$$

Moreover, by differentiating $(A.12)_2$ with respect to proper time we get for the precedent relation:

$$\partial_\tau g_{r\,k}\, \Gamma^{\,k}_{\,j\,h} + g_{r\,k}\, \partial_\tau \Gamma^{\,k}_{\,j\,h} = q_{\,j\,h\,,\,r} + \Gamma^{\,k}_{\,j\,h}\, \partial_\tau g_{k\,r} \qquad \text{i.e.}$$

$$\partial_\tau \Gamma^{\,k}_{\,j\,h} = g^{kr}\, q_{\,j\,h\,,\,r} = q^{\,k}_{\,j\,h} \tag{A.23}$$

which is plainly a tensor. Hence equ.(A.20) becomes:

$$\partial_j \boldsymbol{\omega} = ½\, q_{\,jh\,\,,k}\, \boldsymbol{e}^{\,h} \times \boldsymbol{e}^{\,k} \tag{A.24}$$

or because of (A.21) and the skew-symmetry of the exterior product:

$$\partial_j \boldsymbol{\omega} = \nabla_h K_{kj}\;\; \boldsymbol{e}^{\,h} \times \boldsymbol{e}^{\,k} \tag{A.25}$$

Then, by differentiating (A.9) we obtain:

$\partial_j\, \boldsymbol{\omega} = ½\;\partial_j \omega_{\,hk}\,(\,\boldsymbol{e}^{\,h} \times \boldsymbol{e}^{\,k}\,) + ½\;\omega_{\,hk}\,\partial_j\,(\,\boldsymbol{e}^{\,h} \times \boldsymbol{e}^{\,k}\,)$ and taking $(A.11)_2$ into account and the definition of covariant derivative for $\omega_{\,hk}$ we finally arrive to the differential expressions:

$$\nabla_j\, \omega_{\,hk} = \nabla_h K_{k\,j} - \nabla_k K_{h\,j} \qquad (\,j,h,k = 1,2,3\,) \tag{A.26}$$

Extending (A.26) to the 4-dimensional cronotope (also making $K_{\mu\nu}$ and $\omega_{\mu\nu}$ dimensionally as a $[\text{length}]^{-1}$ by re-defining them dividing by the light speed c) and entering the Tailherer's ansatz: $C_{\mu\nu} = S\omega_{\mu\nu}$, $C_{\mu\nu} = R_{\mu\nu\rho\sigma} \in^{\rho\sigma}$, with $\in^{\alpha\beta}$ any constant skew-symmetric tensor, we have a second gravitational equation:

$$\nabla_\sigma C_{\mu\nu} = S\,(\,\nabla_\mu K_{\nu\sigma} - \nabla_\nu K_{\mu\sigma}\,) \qquad (\,\mu,\nu,\sigma = 1,2,3,4\,)$$

with S = (2.5±1.2)E-19 m^{-1} [3] . By choosing $\in^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ Lorentz invariance is yet preserved, however general one is broken as discussed in [4], just regarding the gravitational wave phenomenon as symmetry breaking of general relativity .

References

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[4] Antonelli, S. 2018 Appraisal of a new gravitational constant. *International Journal of Physics*, 3, 4, 139-149.