A new solution to the linear harmonic oscillator equation

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Abstract

It is well known that amplitude-dependent frequency features only nonlinear dynamical systems. This paper shows that, however, within the framework of the theory of nonlinear differential equations introduced recently by the authors of this work, such a property may also characterize the linear harmonic oscillator equation. In doing so it has been found as another major result that the linear harmonic oscillator is nothing but the nonlocal transformation of equation of the free particle motion under constant forcing function.

Introduction

In mathematical physics and analysis of differential equations, it is well known that isochronicity, that is amplitude-independent frequency property is a fundamental feature of linear differential equations which may exhibit periodic solutions. In this way consider the linear harmonic oscillator equation

$$\ddot{x}(t) - c\,x(t) = 0 \tag{1}$$

where $c < 0$, is an arbitrary parameter, and a dot over a symbol means a derivative with respect to the argument. The linear harmonic oscillator is known as the prototype of second order linear dynamical systems, which has been widely used in a rich variety of physical and engineering applications. Usually the solution of (1) is written in the form

$$x(t) = A_0 \sin\left(\sqrt{-c}\,t + \alpha\right) \tag{2}$$

where A_0 and α are two constants of integration, which may be determined from initial conditions. In other words, A_0 does not dependent on c, and vice versa, due to the isochronicity property of (1). However, the literature shows the existence of nonlinear differential equations which may exhibit such a property. As such a lot of papers has been devoted to criteria for which nonlinear differential equation may exhibit isochonicity [1, 2, 3]. The isochronicity is an unusual behavior of nonlinear differential equation. Therefore nonlinear

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dynamical systems exhibit in general amplitude-dependent frequency. The amplitude-dependent frequency is closely related, for example in mechanics, to hardening or softening property exhibited by nonlinear dynamical mechanical systems [4]. Due to the above, it appears reasonable to ask whether the linear harmonic oscillator equation may exhibit a new solution. To be more precise the question may be formulated as follows: Can the linear harmonic oscillator equation exhibit amplitude-dependent frequency property? To our best knowledge, such a question has never been asked and solved adequately in the literature. Therefore this question may lead to prove the existence of unusual solution that is, unusual behavior to second order linear differential equations and also amplitude-dependent frequency property for the linear harmonic oscillator equation, for the first time. This work assumes such predictions which may be shown within the framework of nonlinear differential equations theory introduced recently by Adjaï et al. [5]. To do so, it is convenient first to briefly review the theory of Adjaï et al. [5] (section 2), to establish and solve using this exactly and explicitly the equation (1), (section 3). Finally, the obtained results are discussed (section 4) and a conclusion is drawn for the work.

2. Review of extended theory of Akande and coworkers [6]

In order to compute exact and explicit general solution to the cubic Duffing equation and to some Painlevé-Gambier equations, Adjaï et al. [5] have been able to carry out an extension to the theory of nonlinear differential equations introduced recently by Akande et al. [6]. The theory developed by Adjaï et al. [5] requires to consider the forced linear harmonic oscillator equation

$$y''(\tau) + a^2 y(\tau) = c \tag{3}$$

where a and c are arbitrary parameters and prime means differentiation with respect to the argument and the generalized Sundman transformation

$$y(\tau) = F(t,x), \quad d\tau = G(t,x)dt, \quad G(t,x)\frac{\partial F(t,x)}{\partial x} \neq 0 \tag{4}$$

where

$$F(t,x) = \int g(x)^l \, dx, \quad G(t,x) = \exp(\gamma\, \varphi(x))$$

such that l and γ are arbitrary constants and, $g(x) \neq 0$ and $\varphi(x)$ are arbitrary functions of x. In this regard the nonlocal transformation of (3) yields

$$\ddot{x}+\left(l\frac{g'(x)}{g(x)}-\gamma\,\varphi'(x)\right)\dot{x}^2+\frac{a^2\exp(2\gamma\,\varphi(x))\int g(x)^l\,dx}{g(x)^l}-\frac{c\exp(2\gamma\,\varphi(x))}{g(x)^l}=0 \qquad (5)$$

For $\varphi(x)=\ln(f(x))$, (5) reduces to

$$\ddot{x}+\left(l\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)}\right)\dot{x}^2+\frac{a^2 f(x)^{2\gamma}\int g(x)^l\,dx}{g(x)^l}-\frac{c\,f(x)^{2\gamma}}{g(x)^l}=0 \qquad (6)$$

which, for $f(x)=x^2$ and $g(x)=x$, becomes

$$\ddot{x}+(l-2\gamma)\frac{\dot{x}^2}{x}+\frac{a^2}{l+1}x^{4\gamma+1}-c\,x^{4\gamma-l}=0 \qquad (7)$$

Now, the objective is to establish equation (1) from (7) in order to perform its exact and explicit general solution.

3. Solution of (1) with amplitude-dependent frequency

The aim in this section is to establish equation (1) in the context of the theory defined by [5] and to compute the solution.

3.1 Derivation of equation (1)

In this perspective consider the following theorem.

***Theorem 1**. Let $a=0$. Then equation (3) reduces to*

$$y''(\tau)=c \qquad (8)$$

which admits the solution

$$y(\tau)=\frac{1}{2}c\,\tau^2+k_1\tau+k_2 \qquad (9)$$

where k_1 and k_2 are constants of integration.

Proof. By substituting $a=0$, into (3), one may immediately obtain (8). By integration, (8) yields

$$y'(\tau)=c\,\tau+k_1 \qquad (10)$$

so that after integration, equation (9) is obtained as result. One may prove also the following theorem

Theorem 2. *Let* $a=0$*. Let also* $l=2\gamma=1$*. Then (7) reduces to the linear harmonic oscillator equation (1) where* $c<0$*.*

Proof. It suffices to set in (7) $l=2\gamma=1$, and $a=0$, to obtain directly equation (1) with the condition that $c<0$. One may now establish the solution of (1) with amplitude-dependent frequency.

3.2 Solution of equation (1)

From the above, one may prove the following theorem

Theorem 3. *Consider equation (9). Then the application of the nonlocal transformation (4) yields the solution of (1) in the form*

$$x(t)=\pm A\cos\left(\sqrt{-c}t+K\sqrt{-c}\right) \tag{11}$$

where

$$A^2=2k_2-\frac{k_1^2}{c} \tag{12}$$

and K *is an arbitrary constant.*

Proof.

According to Theorem 1, the use of the nonlocal transformation (4) leads to compute

$$x(t)=\pm\sqrt{2y(\tau)} \tag{13}$$

such that

$$dt=\pm\frac{d\tau}{\sqrt{2y(\tau)}} \tag{14}$$

Substituting (9) into (14) yields, after integration [7]

$$\tau=\pm\frac{\sqrt{4k_1^2-8ck_2}\sin\left[\sqrt{-c}\left(t+K\right)\right]-k_1}{2c} \tag{15}$$

Substituting now (15) into (13) leads immediately to

$$x(t)=\pm\sqrt{\frac{2k_2c-k_1^2}{c}}\cos\left(\sqrt{-c}\,t+K\sqrt{-c}\right) \tag{16}$$

which becomes (11) when $A=\sqrt{\frac{2k_2c-k_1^2}{c}}$.

Theorem 3 is a new result in the fields of mathematics and mathematical physics, which requires therefore a discussion.

4. Discussion

As can be seen, the frequency $\sqrt{-c}$ depends on the amplitude A, that is

$$-c=\frac{k_1^2}{A^2-2k_2} \tag{17}$$

so that the solution (11) has the behavior of harmonic form but whit amplitude-dependent frequency. One may also distinguish in (11) the presence of three constants of integration. However But these may easily be determined from initial conditions. In this perspective, consider $x(0)=x_0$ and $\dot{x}(0)=v_0$. Then one may obtain

$$x_0^2=A^2\cos^2(K\sqrt{-c}) \tag{18}$$

and

$$v_0^2=-cA^2\sin^2(K\sqrt{-c}) \tag{19}$$

so that adding the equations (18) and (19), leads to

$$A^2=x_0^2-\frac{v_0^2}{c} \tag{20}$$

The comparison of (20) with (12) yields

$$k_1=v_0^2 \tag{21}$$

and

$$k_2=\frac{x_0^2}{2} \tag{22}$$

The integration constant K may be computed from

$$\frac{v_0}{x_0} = -\sqrt{-c}\tan\left(K\sqrt{-c}\right)$$

that is

$$K = -\frac{\sqrt{-c}}{c}\arctan\left(\frac{v_0\sqrt{-c}}{x_0 c}\right) \tag{23}$$

That being so, one may easily observe that the usual solution of (1) that is equation (2), may be deduced from (11) by taking $k_1 = 0$, such that

$$x(t) = x_0 \cos\left(\sqrt{-c}\,t + K\sqrt{-c}\right) \tag{24}$$

where the amplitude $A = x_0$. For $k_2 = 0$, the unusual solution (11) remains amplitude-dependent frequency. The above shows that the amplitude-dependent frequency characterizes not only nonlinear dynamical systems but may feature also linear dynamical systems. It is worth to note that another major finding of this work has been to show that the free particle motion equation with a constant forcing function (8) is closely related to the linear harmonic oscillator equation, for the first time, that is these two equations are mathematically equivalent. In other words the linear harmonic oscillator equation is nothing but a nonlocal transformation of equation of the free particle motion under a constant forcing function, and vice versa. So with that a conclusion may be drawn to this research contribution.

Conclusion

It is well known that amplitude-dependent frequency consists of a fundamental property of nonlinear dynamical systems. Also, very few researchers can suspect that the well known linear harmonic oscillator equation can exhibit solution with amplitude-dependent frequency. However, the present work has successfully shown, for the first time, this behavior for the linear harmonic oscillator equation. This has been possible by the application of the theory of nonlinear differential equations introduced recently by the authors of this paper. In doing so it has been observed that the equation of the free particle motion under constant forcing function, is closely related to the linear harmonic oscillator equation. In other words the linear harmonic oscillator equation is nothing but the nonlocal transformation of the equation of the free particle motion with a constant forcing function.

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