Geometrical formulation of physics.

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Abstract

We define a unique geometrical poisson bracket for classical physics and construct geometrical quantum operators.

1 Introduction.

Consider a particle moving in a bundle $\mathcal{E}$ over a Lorentzian spacetime $(\mathcal{M}, g)$ where the fibres are equipped with a metric field and the associated connection preserves the total metric (which is usually a product metric). Regard the wordline as an immersion $\gamma : \mathbb{R} \to \mathcal{E}$ and the momentum as its the push forward of ∂_t with equals

$$\frac{D}{dt} := \nabla_{\frac{d}{dt}\gamma(t)}$$

where ∇ is the bundle connection. Given that we shall only work with functions $f : \mathcal{E} \to \mathbb{R}$, the latter expression can be taken for $(\partial_t)_\star$ as an ordinary vectorfield instead of a general derivative operator. To every curve γ and function f we can attach a function $\gamma_f : \mathbb{R} \to \mathbb{R} : t \to f(\gamma(t))$. We can now define a $C^\infty(\mathbb{R})$ algebra of operators $\mathbb{L}$ on the function space $f : \mathcal{E} \to \mathbb{R}$ mappping them to functions from $\mathbb{R}$ to $\mathbb{R}$. Concretely

$$[(\gamma_f)(g)](t) := f(\gamma(t))g(\gamma(t))$$

and

$$[p_\gamma f](t) := \frac{d}{dt} f(\gamma(t)).$$

We have moreover,

$$\gamma_f(gh) = \gamma_f(g)\gamma_f(h)$$

and

$$[(\partial_t)(\gamma_f g)](t) := [(\partial_t)_\star f](t)g(\gamma(t)) + f(\gamma(t))[(\partial_t)_\star(g)](t).$$

This suggests to extend the definition of the momentum in this way to functions $\mathbb{R} \to \mathbb{R}$. The same comment holds for γ_f. In this vein,

$$[\gamma_g \gamma_f h](t) = g(\gamma(t))f(\gamma(t))h(\gamma(t))$$

and

$$[p_\gamma \gamma_f h](t) := \partial_t(f(\gamma(t))h(\gamma(t)))$$

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as well as

$$[\gamma_f p_\gamma h](t) := f(\gamma(t))\partial_t h(\gamma(t)).$$

Finally,

$$[p_\gamma p_\gamma h](t) = (\partial_t)^2 h(\gamma(t))$$

which induces a real algebra generated by

$$\gamma_g, p_\gamma$$

where γ varies over all immersions. This algebra is represented by means of linear operators on the function algebra

$$\mathcal{B} := C^\infty(\mathbb{R}) \otimes C^\infty(\mathcal{E})$$

which may be given the structure of an Hilbert algebra in the usual L^2 sense by introducing an einbein on the "time line" $\mathbb{R}$. Concretely

$$[\gamma_f, \gamma_h]\,(g) = 0 = [p_\gamma, p_\gamma]\,(g), [p_\gamma, \gamma_f]\,(g) = p_\gamma(f)\gamma_\star(g) = \gamma_{p_\gamma(f)}(g)$$

where $\gamma_\star$ is the pull back defined by the immersion γ. Here, the commutation relations employ the full $\mathcal{B}$ action but are understood to apply on $f, g, h \in C^\infty(\mathcal{E})$ and result in an element of $C^\infty(\mathbb{R})$.

2 Classical dynamics.

Covariant dynamics requires dynamics without potential energy terms; therefore, any force has to be implemented in the momentum what explains the bundle $\mathcal{E}$. Moreover, according to Einstein himself, every force apart from the gravitational onde can be gauged away in some point so that locally and physically every particle is a free one meaning that the correct equation is the geodesic bundle equation. Therefore, the classical Hamiltonian is a constraint and moreover, commuting it with a vector leaves a covector if it were an invariant energy so that

$$[\mathcal{H}(\gamma_f, p_\gamma), p_\gamma]$$

cannot represent $\frac{D}{dt}p_\gamma$ unless we would make an extra metric contraction. Actually, the whole Hamiltonian edifice is kind of meaningless as we shall see now. Indeed, taking $\mathcal{H}(\gamma_f, p_\gamma)$ to be p_γ with equations of motion given by

$$[\frac{D}{dt}\triangle\gamma_f](g) := [p_\gamma, \gamma_f](g) = \gamma_{p_\gamma(f)}(g)$$

and

$$[\frac{D}{dt}\triangle p_\gamma](g) := [p_\gamma, p_\gamma](g) = 0$$

where

$$[\frac{D}{dt}\triangle\zeta](g) = [\frac{D}{dt}, \zeta](g).$$

There is nothing more to say really apart from the constraint $g(p_\gamma, p_\gamma) = m^2$ which is the mass energy relation. This is all what is allowed in classical physics of point particles really and we now proceed to quantum theory. Notice that the dynamical content is completely implied by the commutator algebra which constitutes a total unision between dynamics and kinematics. Physically, this is entirely trivial and completely justified given that the momentum just corresponds to the energy in a rest frame.

3 Quantum theory of a free particle.

Unlike in classical physics, quantum mechanics cannot use an external time given that a particle is not specified anymore by a worldline but by a wave. In a way, it is the complex dual of the classical situation where "worldlines" correspond to functions $\psi : \mathcal{E} \to \mathbb{C}$ which are C^∞. The operators γ_f and p_γ are replaced then by x_f and $i\nabla_V$ where V is a real vectorfield over $\mathcal{E}$ and f is a real valued function over $\mathcal{E}$. Here, $[x_f](g)(x) = f(x)g(x)$ and

$$P(V)(g) := i\nabla_V(g) = iV(g).$$

They obey the algebra

$$[x_f, x_h] = 0, [i\nabla_V, i\nabla_W] = -R(V,W)(\cdot) - \nabla_{[V,W]}$$

and finally

$$[i\nabla_V, x_f] = x_{iV(f)}.$$

The momentum commutation relations have been put in this exotic form because the covariant derivative can work on vectorfields and higher objects too. The i is just there to ensure that the momentum operator is real given that the commutator of two real operators is imaginary. The situation here is very different as one cannot just pick a Hamiltonian linear in the momenta given that one would as thus preselect a nondynamical arrow of time. Hence our only choice is given by

$$H = \sum_{i,j=1}^{n} \eta^{ij} \nabla_{E_i} \nabla_{E_j}$$

where the E_i correspond to loical vielbeins and η^{ij} is the inverse of the standard flat metric. In order for this to work ∇ must be extended to the spin connection to digest local boost transformations. Furthermore, one has

$$H = m^2$$

as constraint. It is clear one has no Heisenberg type dynamics here as the vectorfields really are spacetime vectorfields; hence, the entire theory is encapsulated by the constraint and the geometry of the bundle $\mathcal{E}$. It has been shown by Ashtekar and Magnon that this theory only works out fine in stationary spacetimes with Minkowski as the prime example due to the existence of scalar products on leafs of a foliation for which the latter is preserved in "time". In the next section on free Quantum Field theory we explain why only the Minkowskian theory has to be taken seriously.

4 Free Quantum Field Theory.

There is no miracle here, one solves the free one particle theory of the previous section and proceeds with the construction of the Bosonic or Fermionic Fock space associated to the Hilbert algebra $C^\infty(\mathcal{E})$. This allows one to define creation and annihilation operators as well as a Hermitian representation of the Poincaré Lie-algebra. The energy-momentum vector operator squares to m^2 and this is the theory. In a general curved space-time, I have shown how to glue these different representations together by means of a bi-field formalism represented on a Hilbert bundle; all of this is canonical. The interacting theory has subsequently been worked out as a theory of Feynman diagrams and the reader may find all material on this in my publications.