A fully relativistic description of stellar or planetary objects orbiting a very heavy central mass

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Abstract

A fully relativistic numerical program is used to calculate the advance of the peri-helium of Mercury or the deflection of light by the Sun is here used also to discuss the case of S2, a star orbiting a very heavy central mass of the order of $4.3\,10^6$ solar masses.

1 Equations of motion

Given an space-time metric:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{1}$$

it is usual to refer to the equations of motion of free bodies as the the geodesic equations:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \tag{2}$$

but it is important to keep in mind that this is only the case if the space-time trajectories of free particles are parameterized by the proper space-time s or any other affine parameter $s' = k_1 s + k_2$. This is not always the best choice to make.

I shall use here a different parametrization that was introduced by Eisenhart ([1]) when dealing with general linear connections, but can be used

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also when dealing with Riemannian ones. This consists in using instead the equations of motion:

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=a\frac{dx^\alpha}{ds} \tag{3}$$

where a depends on the parameter that is used to describe the solutions.

The purpose of this paper is to look for models of test objects orbiting a central mass using the Schwarzschild solution in isotropic coordinates as an example.

$$ds^2=-\frac{\left(1-\frac{m}{2r}\right)^2}{\left(1+\frac{m}{2r}\right)^2}c^2dt^2+\left(1+\frac{m}{2r}\right)^4(dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)) \tag{4}$$

Considering that the space trajectory of the object lies on the plane $\theta=\pi/2$ the only Christoffel symbols to consider are the following:

$$\Gamma^1_{33}=\frac{r(-2r+m)}{2r+m},\ \Gamma^3_{13}=-\frac{-2r+m}{r(2r+m)}, \tag{5}$$

$$\Gamma^1_{44}=-\frac{64r^4(-2r+m)c^2m}{(2r+m)^7},\ \Gamma^4_{14}=\frac{4m}{(2r+m)(-2r+m)},\Gamma^1_{11}=-\frac{2m}{r2r+m}, \tag{6}$$

My second choice is to choose the coordinate φ as the parameter and therefore describe the motion of the object with the functions $r(\phi)$ and $t(\phi)$. In which case we have:

$$a=-\frac{2(-2r+m)vr}{r(2r+m)} \tag{7}$$

where here and later:

$$vr=\frac{dr}{d\varphi},\ vt=\frac{dt}{d\varphi}, \tag{8}$$

The equations to solve are then the following

$$\frac{dvr}{d\varphi}+\Gamma^1_{33}+\Gamma^1_{44}vt^2+\Gamma^1_{11}vr^2=a\,vr \tag{9}$$

$$\frac{dvt}{d\varphi}+2\Gamma^4_{14}vr\,vt=a\,vt \tag{10}$$

or more explicitly:

$$\frac{dvr}{d\varphi} = \frac{2vr^2m}{r(2r+m)} + \frac{64vt^2r^4(-2r+m))mc^2}{2r+m)^7} - \frac{2(-2r+m)vr^2}{r^2}r(2r+m) - \frac{(-2r+m)r}{2r+m} \quad (11)$$

$$\frac{dvt}{d\varphi} = \frac{8vrm\,vt}{(2r+m)(-2r+m)} - \frac{2(-2r+m)vr\,vt}{r(2r+m)} \quad (12)$$

An important feature of this system of differential equations is its scale invariance. This meaning that the substitutions:

$$r,\ t,\ m \to kr,\ kt,\ km \quad (13)$$

leaves the system invariant.

And also the fact that elliptic-like orbits where the eccentricity is by definition:

$$\epsilon = \frac{ra - rp}{ra + rp} \quad (14)$$

can have either positive or negative values concomitantly with the value of:

$$\mu = \frac{vp - va}{va + vp} \quad (15)$$

where va is the linear velocity at ra and vp is the linear velocity at rp, the extreme values of the distance of a point of the trajectory to the location of the central mass.

The quasi newtonian limit of the preceding system of differential equations can be defined as the result of a formal expansion with respect to m to order 2, followed with an asymptotic expansion with respect to r to order 2. The result is:

$$Fr = \frac{2vr^2}{r} + r\left(\frac{vr^2 - vt^2c^2}{r^2} - 2\right)m \quad (16)$$

$$Ft = \frac{2vrvt}{r}\left(1 - \frac{2m}{r}\right) \quad (17)$$

Noteworthy is the fact that the third law of Kepler, namely $dA/d\phi = 0$ with $A = 1/2r^2d\phi/dt$ is strictly speaking not satisfied since:

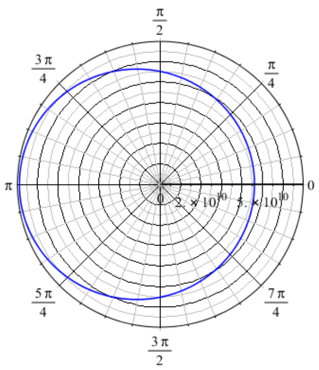
$$\frac{dA}{d\varphi} = \frac{2vr\,m}{vt} \quad (18)$$

2 The planet Mercury

In this case we know the mass m of the Sun, the length of the perihelion rp and the maximal orbital velocity $vmax$:

$$m = 1484.851528, \;\; rp = 46\,10^9, \;\; vmax = 58.98\,10^3 \tag{19}$$

With these data the numerical integration proceeds smoothly, wherefrom we can obtain known Mercury's data including its sidereal orbit period: 86.64 days (NASA's value is 87.969). The polar plot of the trajectory and the plot of the linear velocity around it are:



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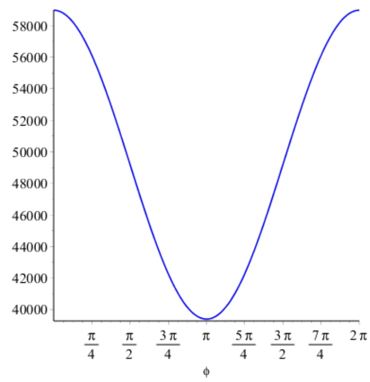

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The maximum discrepancy with the third law of Kepler, is:

$$\frac{DA(0) - DA(\pi)}{DA(0) + DA(\pi)} \approx 10^{-8}, \;\; with \; DA = \frac{dA}{d\varphi} \tag{20}$$

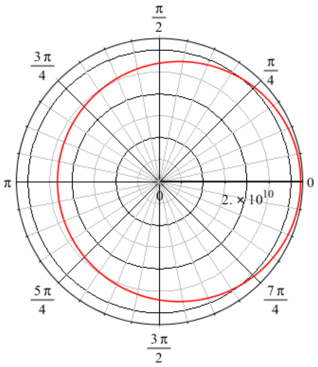
The relativistic advance of the peri-helium per century comes out as 43.46 arcseconds per century derived from the first value of $\phi = 2\pi + \delta\phi$ such that $vr = 0$. A very simple interpolation program is useful.

Let us consider a planet that differs from Mercury in the sense that his perihelion is 0.7 times the perihelion of Mercury. This would suffice to lead to an orbit with negative eccentricity whose polar plot trajectory and linear velocity around it would be:

This should lead us to ask: Is it by chance that all our planets have orbits with positive eccentricity, or should we restrict with a new principle the generality of the system of differential equations we started with?

3 The Star S2

S2 is a star circling Sagittarius A^* with a mass estimated to $4.3\,10^6$, solar masses, along an ellipse with eccentricity 0.87 and a period of 15.2 years.



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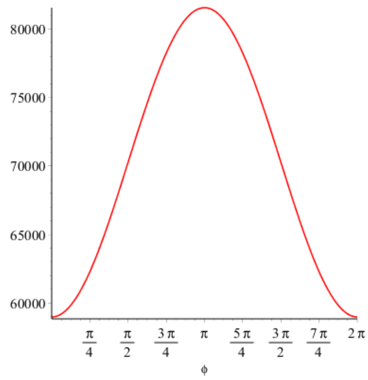

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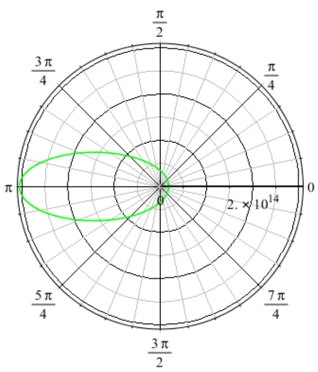
Also known is its peri-center length of $120\,au$.

Mass and peri-center length are not sufficient initial conditions to derive a unique model integrating the system of differential equations (11)-(12). The value of vt0 or equivalently the value of the linear velocity $rp/vt0$ must be also known.

A trial and error method leads easily to the value $vt0 = rp(38.6/c)$ and to the observed value of the period: 15.2 years. This result has been celebrated ([3]) as a confirmation of General relativity. This is incorrect because the same result holds if instead of integrating the system of equations (11)-(12), we integrate its firs its quasi Newtonian approximation.

The precession of the peri-center of S2 can now be calculated easily by solving the equation $r(\pi + \delta) = rp$ or $vr(\pi + \delta) = 0$. The result is $\delta\varphi = 0.3393$ arcseconds per revolution

The polar plot of the trajectory and the plot of the linear velocity around it are:



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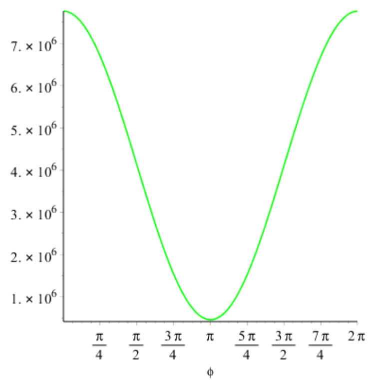

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4 The deflection of light by the Sun or by Sagittarius A^*

In this case the system of differential equations has to be integrated with the inial conditions:

$$r(0) = 6.95\,10^8, \;\; vr(0) = 0, \;\; vt(0) = \frac{1}{4}\frac{(2r0+m)^3}{r0(2r0-m)c} \tag{21}$$

($vt(0)$ follows from $vr(0) = 0$ and $ds^2 = 0$) and the integration has to proceed until the tangent to the trajectory:

$$\delta\varphi_\infty \equiv \frac{dy}{dx} = \frac{r\sin\varphi - vr\cos\varphi}{r\cos\varphi + vr\sin\varphi} \tag{22}$$

reaches a stationary value. Twice this value:

$$\delta\varphi_\infty = 1.76\, arcseconds \tag{23}$$

is the value of of the angular deviation of a light ray skimming the surface of the Sun. and:

$$\delta\varphi_\infty = 705\, arcseconds \tag{24}$$

is the corresponding deviation for a light-ray approaching Sagittarius A^* to a distance of $50\, au$.

5 Appendix: Radial motion

Of course the coordinate φ can not be used to describe the dynamics of a particle moving in a radial direction $\varphi = const.$. In this case is appropriate to use the coordinate t as parameter. Starting again with equations (27) we get:

$$a = \frac{2mvr}{r(r+2m)}, \text{with } vr = \frac{dr}{dt} \tag{25}$$

and:

$$\frac{dvr}{dt} = \frac{2mvr^2}{r(2r+m)} + \frac{64r^4(-2r+m)c^2m}{(2r+m)^7} + \frac{8mvr^2}{(2r+m)(-2r+m)} \tag{26}$$

And solving the partial differential equation:

$$\frac{\partial f(r,vr)}{\partial r}vr + \frac{\partial f(r,vr)}{\partial vr}\frac{dvr}{dt} = 0 \tag{27}$$

we get the Energy function:

$$E = \frac{1}{32}\frac{(2r+m)^8 m^2 vr^2}{(-2r+m)^4 r^4} - \frac{4mr}{(-2r+m)^2} \tag{28}$$

where an appropriate arbitrary multiplicative constant has been chosen to obtain the correct non relativistic limit.

References

[1] L. P. Eisenhart, *Non Riemannian geometry* , American Mathematical Society (1929)

[2] Nature, **419**, October 2002

[3] Astronomy and Astrophysics, **615** July 2018