The Additional Pressure of The White Dwarf Star Caused By The Rest Positive Charges

Ting-Hang Pei

Thpei142857@gmail.com

Abstract The explanation for the pressure against gravity in the white dwarf star is based on the ideally degenerate Fermi electron gas at the temperature of absolute zero. It predicts the upper mass limit of the white dwarf star is 1.44 $M_{\odot}$. However, more conditions have to be considered like temperature and charges. In this research, first we use the grand partition function in statistical mechanics to build the expressions of the electron gas pressure and the particle number depending on temperature and find the electron gas pressure is weakly temperature dependence at 10^7 K. At this temperature, there is about $1.50\text{x}10^{-4}$ of total electrons exceeding the Fermi energy. Because some of this Fermi electron gas are the relativistic electrons, then we consider that those relativistic electrons can escape the gravity resulting in a positively charged star. These rest positive charges produce the strong repulsive force and the pressure to against gravity. Furthermore, due to quantum effect, some electrons can tunnel the potential barrier even their energy is less than the maximal potential. From theoretical calculations, the 75 GeV electrons escaping to infinity can gradually accumulate through 5 billion years resulting in $2.067\text{x}10^{20}$ *C* positively charged star as long as the attraction force is strong enough. This positive-charged effect makes the upper mass limit higher than the previous value to 4.073 $M_{\odot}$ when the radius of the white dwarf star is twice that of Earth.

Keywords: white dwarf star, degenerate Fermi electron gas, pressure, upper mass limit, Coulomb's interaction

I. Introduction

The white dwarf star is thought to be the type of the low to medium mass stars in the final evolution stage and it was named first in 1922 [1]. Its density is usually very high with the mass similar to our sun but the radius as small as Earth. The nucleus of the white dwarf star sops the nuclear-fusion reaction and cools down. It is believed that the pressure of the degenerate Fermi electron gas mainly support the balance with the gravitation so as to the mass upper limit exist [2-7]. The calculation is based on the assumption that all electrons like free particles occupy all energy levels until to Fermi energy as they are at temperature *T* of absolute zero. It is surprising that the ideal Fermi electron gas still works in the high-temperature and high-pressure situation even some electrons have kinetic energy higher than the Fermi energy.

To deal with the general relativity, some appropriate metrics have been found such as the Schwarzschild metric, the Kerr metric, and the Kerr-Newman metric [7-11]. Especially, the rotating and charged star can be described by the Kerr-Newman metric. Some special problems about the Kerr-Newman black hole have been reported [12,13]. From astronomical observations, most stars are rotating and might be easily charged because the relativistically massive particles can escape the gravitational attraction. According to the calculations of statistical mechanics [2-6], the relativistic electrons have more possibility to escape gravity than helium nuclei at the same temperature. Because of this factor, we consider the positive charged star and the Coulomb interaction existing in the rest positive charges, and further calculate the pressure produced by these rest charges. The Coulomb force also an important one against gravity so the upper mass limit of the white dwarf star should be higher.

II. The Degenerate Fermi Electron Gas For The White Dwarf Star

The calculation of the upper mass limit for the white dwarf star adopts the ideally degenerate Fermi electron gas at and considers the relativistic kinetic energy [3,4]. Each energy state permits two electrons occupied because of the electron spin s$= \pm\frac{1}{2}$. Each electron has the rest mass $m_{e,}$ and its relativistic kinetic energy E at momentum p is

$$E_k = m_e c^2 \left\{ \left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{1/2} - 1 \right\}. \tag{1}$$

The Fermi electron gas has total kinetic energy

$$\begin{aligned} E_0 &= 2m_e c^2 \sum_{|\vec{p}|<p_F} \left\{ \left[1 + \left(\frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\} \\ &= \frac{2Vm_e c^2}{h^3} \int_0^{p_F} dp 4\pi p^2 \left\{ \left[1 + \left(\frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\}, \end{aligned} \tag{2}$$

where h is the Planck's constant and Fermi momentum is defined

$$p_F = h \left(\frac{3N}{8\pi V} \right)^{1/3}. \tag{3}$$

The pressure produced by the ideal Fermi electron gas is [4]

$$P_0 = -\frac{\partial E_0}{\partial V} = \frac{8\pi m_e^4 c^5}{h^3} \left[\frac{1}{3} x_F^3 \sqrt{1 + x_F^2} - f(x_F) \right], \tag{4}$$

where

$$E_0 = \frac{8\pi m_e^4 c^5 V}{h^3} \left[f(x_F) - \frac{1}{3} x_F^3 \right], \tag{5}$$

$$f(x_F) = \int_0^{x_F} dx x^2 \left[(1 + x^2)^{1/2} \right], \tag{6}$$

and

$$x_F \equiv \frac{p_F}{m_e c} = \frac{h}{2m_e c}\left(\frac{3N}{8\pi V}\right)^{1/3}, \tag{7}$$

A white dwarf star mainly consists of helium nuclei so the total mass M is

$$M = \left(m_e + m_p + m_n\right)N \approx 2m_p N \approx 2m_n N, \tag{8}$$

where m_p is the mass of a proton and m_n is the mass of a neutron. Then the relationship between the radius R and mass M of the star is

$$\bar{R} = \bar{M}^{2/3}\left[1 - \left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right]^{1/2}, \tag{9}$$

where

$$\bar{M}_0 = \left(\frac{27\pi}{64\delta}\right)^{3/2}\left(\frac{hc}{2\pi G m_n^2}\right)^{3/2}, \tag{10}$$

$$\bar{R} = \left(\frac{2\pi m_e c}{h}\right)R, \tag{11}$$

and

$$\bar{M} = \frac{9\pi}{8}\frac{M}{m_n}. \tag{12}$$

In Eq. (10), δ is a parameter of pure number and G is the gravitational constant. Further calculations [4] the upper mass limit M_0 is given by

$$M_0 \approx 1.44 M_{\odot}. \tag{13}$$

III. The Temperature Effect On The Pressure of The Ideal Fermi Electron Gas

The inner temperature of a star is usually about 10^7 K, and the upper mass limit in Eq. (13) is calculated at T=0 which seems to be unreasonable and necessarily improved. Then we consider the grand partition function for T>>0 in statistical mechanics [4] is

$$q(T, V, z) = \ln Z = \textstyle\sum_k \ln[1 + z \cdot exp(-\beta E_k)], \tag{14}$$

where E_k is the kinetic energy, $\beta=1/k_B T$, $z=\exp(\mu\beta)$, and μ the chemical potential of the Fermi electron gas. The grand partition function can change to the integral from

$$\ln Z = \int_0^{\infty} dE g(E_k) \ln[1 + zexp(-\beta E_k)]. \tag{15}$$

When integrating it by parts, then it gives [4]

$$\ln Z = g\frac{4\pi V}{h^3}\frac{\beta}{3}\int_0^\infty p^3 dp\,\frac{dE_k}{dp}\frac{1}{z^{-1}exp(\beta E_k)+1}, \tag{16}$$

where

$$p^2c^2 = E_k^2 + 2m_e c^2 E_k \tag{17}$$

and g=2s+1 is the degeneracy factor. Substituting Eq. (17) into Eq. (16), it gives

$$\ln Z = g\frac{4\pi V\beta}{3h^3c^3}\int_0^\infty dE_k\,\frac{E_k^3\left[1+\frac{2m_e c^2}{E_k}\right]^{3/2}}{z^{-1}exp(\beta E_k)+1}. \tag{18}$$

Using the Taylor series expansion to the first-order term, then we have

$$\ln Z \approx g\frac{4\pi V}{3h^3c^3\beta^3}\int_0^\infty d(\beta E_k)\,\frac{(\beta E_k)^3\left[1+3\left(\frac{m_e c^2\beta}{\beta E_k}\right)\right]}{z^{-1}exp(\beta E_k)+1}. \tag{19}$$

Similarly, we can obtain the expression of the total number $N(T, V, z)$ is

$$\begin{aligned} N(T,V,z) &= \mathrm{g}\frac{4\pi V}{h^3}\int_0^\infty p^2 dp\,\frac{1}{z^{-1}exp(\beta E_k)+1} \\ &= \mathrm{g}\frac{4\pi V}{h^3c^3}\int_0^\infty dE_k\,\frac{E_k^2\left(1+\frac{2m_e c^2}{E_k}\right)^{1/2}\left(1+\frac{m_e c^2}{E_k}\right)}{z^{-1}exp(\beta E_k)+1}. \end{aligned} \tag{27}$$

A useful function in both above integrals is

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty dx\,\frac{x^{n-1}}{z^{-1}e^x+1}, \tag{21}$$

where $x = \beta E_F$. When z>>1, Eq. (21) approximates [4]

$$f_n(z) \approx \frac{(\ln z)^n}{n!}. \tag{22}$$

For example, the calculation of $\ln Z$ can be written as

$$\ln Z \approx g\frac{4\pi V}{3h^3c^3\beta^3}\left[\Gamma(4)f_4(z)+3\left(\frac{m_e c^2}{k_B T}\right)\Gamma(3)f_3(z)\right], \tag{23}$$

where $\Gamma(n)$ is the gamma function. The chemical potential depends on T and it is

$$\mu = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right]. \tag{24}$$

Further calculations in Eq. (20) give

$$N(T,V,z) \approx \frac{8\pi V (k_B T)^3}{h^3 c^3} \times \left[\Gamma(3) f_3(z) + 2\left(\frac{m_e c^2}{k_B T}\right)\Gamma(2) f_2(z) + \left(\frac{m_e c^2}{k_B T}\right)^2 f_1(z)\right]. \tag{25}$$

The results in Eqs. (23) and (25) can also be applied to neutron when the electron mass is changed to the neutron mass, so we can also use the similar way to discuss the neutron star.

IV. The Improvement of The Upper Mass Limit Considering The Escaping Particles

However, above discussions are based on the neutral star condition that the negative charges balance the positive charges. The relativistic electrons have possibility to escape the gravity of a star much higher than the nuclei, so the star would very be the positively charged star. In the classical statistical mechanics, the Maxwell-Planck velocity distribution tells us the most probable, the mean, and the root mean square root absolute velocities v^*, $\langle v \rangle$, and $\sqrt{\langle v^2 \rangle}$ in the ideal gas are

$$v^* = \sqrt{\frac{2k_B T}{m}}, \tag{26}$$

$$\langle v \rangle = \sqrt{\frac{8k_B T}{m\pi}}, \tag{27}$$

and

$$\sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}}. \tag{28}$$

Although we deal with the indistinguishable quantum particles, in the ultrahigh-temperature case the classical results still are good references. All those three velocities for the Fermi electron gas are very close to c, but they are only $7.3\text{x}10^{-4}c$, $8.2\text{x}10^{-4}c$, and $8.9\text{x}10^{-4}c$ for helium nucleus at the same temperature. Even for the hydrogen nucleus, its average velocity is still much less than electron. According to these, some electrons escape the gravitation of the star and the star is reasonably positive-charged stellar. Especially, the evolution period is usually 5-10 billion years, the accumulation escaping electrons more and more because the fusion reaction provides enough energy to heat electrons. A similar phenomenon is the well-known solar wind raising from the surface of the star and moving outward to the space.

Then considering the total negative and positive charges are -Q and Q+ΔQ. Supposing the rest positive charges ΔQ distribute homogeneously in the star, then the density $\rho_{\Delta \mathrm{Q}}$ of the rest positive charges is

$$\rho_{\Delta \mathrm{Q}} = \frac{\Delta \mathrm{Q}}{\frac{4}{3}\pi R^3}. \tag{29}$$

The self-energy E_{self} of this charged sphere is

$$E_{self} = \frac{3K_e(\Delta \mathrm{Q})^2}{5R} = \frac{3K_e(\Delta \mathrm{Q})^2}{5}\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}. \tag{30}$$

The pressure $P_{\Delta \mathrm{Q}}$ produced by the rest positive charges is

$$P_{\Delta Q} = -\frac{\partial E_{self}}{\partial V} = \frac{K_e(\Delta \mathrm{Q})^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-4/3}. \tag{31}$$

Using Eqs. (5), (11), and (12), then we have

$$P_{\Delta Q} = \frac{K_e(\Delta \mathrm{Q})^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\left(\frac{3M}{8\pi m_n N R^3}\right)^{\frac{4}{3}} = \frac{3K_e(\Delta \mathrm{Q})^2}{5\pi^2 N}\frac{1}{9}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\left(\frac{2\pi m_e c}{h}\right)^4 \frac{\bar{M}^{4/3}}{\bar{R}^4}. \tag{32}$$

This rest-positive-charges pressure has to be also considered into the contribution of the total pressure. When we consider the Fermi electron gas in metal, the rest charges stay in the surface because of the zero electric field inside the perfect metal. However, the solar system is consisting of high-temperature and viscous plasma, and the convection continuously happens. The homogeneous distribution is theoretically reasonable assumption as long as the time is enough to reach this situation.

Next, we combine P_0 with $P_{\Delta \mathrm{Q}}$ as the main pressure P_{main} against the gravitation in the star

$$P_{main} = \frac{2\pi m_e^4 c^5}{3h^3}\left[\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2} + \frac{8K_e(\Delta \mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\frac{\bar{M}^{4/3}}{\bar{R}^4}\right]. \tag{33}$$

Actually, the pressure of the Fermi electron gas has a tiny decrease because of the escaping electrons. It has to add a factor (1-ΔQ/*Ne*) accompanying with P_0 where *e* is the charge of a single electron, so the main pressure becomes

$$P_{main} = \frac{2\pi m_e^4 c^5}{3h^3}\left\{\left(1-\frac{\Delta\mathrm{Q}}{Ne}\right)\left(\frac{\bar{M}^{4/3}}{\bar{R}^4}-\frac{\bar{M}^{2/3}}{\bar{R}^2}\right)+\frac{8K_e(\Delta\mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\frac{\bar{M}^{4/3}}{\bar{R}^4}\right\}. \quad (34)$$

Using these data [20], K_e=8.987x10^9 $N{\cdot}m^2/C^2$, h=6.626x10^{-34} $J{\cdot}m$, c=2.998x10^8 m/s, and N~6x10^{56} for our sun [3], the third coefficient can simplify to

$$\frac{8K_e(\Delta\mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}} \approx 2.341\times10^{-41}(\Delta Q)^2. \quad (35)$$

When ΔQ=2.067x10^{20} *C*, this coefficient is 1.0. This effect is caused by (2.067 x10^{20})/(1.602x10^{-19})=1.29x10^{39} electrons escaping the star. It only occupies about 10^{-17} Fermi electron gas so the ratio (1-ΔQ/*Ne*) approximates 1.0 in Eq. (34). This approximation also satisfies the common case. Similar to Eq. (9), we obtain

$$\bar{R} = \bar{M}^{2/3}\left\{\left[1+\frac{8K_e(\Delta\mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\right]-\left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right\}^{1/2}. \quad (36)$$

Using the conservation of energy between the kinetic energy and the electric potential, we can estimate the maximal number of electrons escaping to infinity. As we know, the Coulomb’s interaction is much larger than the gravitational interaction for two protons or electrons at the same distance, so we only consider the Coulomb’s interaction here. The electric potential at infinity is zero as a reference. Supposing the minimum kinetic energy for escaping the Coulomb’s interaction is E_{min}, then we have

$$(\gamma-1)m_ec^2 = E_{min} \geq \frac{K_e(\Delta\mathrm{Q})_{max}e}{R}. \quad (37)$$

where γ is the relativistic factor for the massive particle with velocity v

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (38)$$

Let $(\Delta\mathrm{Q})_{max}=(\Delta\mathrm{N})_{max}e$, using the Fermi-Dirac distribution gives

$$\frac{E_{min}R}{K_e e^2} \geq (\Delta N)_{max}$$

$$= g\frac{4\pi V}{h^3c^3\beta^3}\int_{E_{min}}^{\infty} d(\beta E_k)\frac{(\beta E_k)^2\left(1+\frac{2m_ec^2}{E_k}\right)^{1/2}\left(1+\frac{m_ec^2}{E_k}\right)}{z^{-1}exp(\beta E_k)+1}$$

$$\approx \mathrm{g}\frac{4\pi V(k_BT)^3}{h^3c^3}\int_{E_{min}}^{\infty} dx\frac{x^2\left[1+2\left(\frac{\beta m_ec^2}{x}\right)+\left(\frac{\beta m_ec^2}{x}\right)^2\right]}{z^{-1}e^x+1}, \qquad (39)$$

where $x=\beta E_k$, $z=exp(\beta\mu)$ and μ is the chemical potential very close to E_F. Then choose the lowest limit is zero, the right-hand side of Eq. (39) is equal to the total electron number N. Using Eq. (20), first we can estimate the ratio of the particle number above E_F. Then we divide the three parts of the integral in Eq. (20) into two integrals for each part, and these integrals are

$$\Gamma(3)f_3(z) = \int_0^{\beta E_F}\frac{x^2}{z^{-1}e^x+1}dx + \int_{\beta E_F}^{\infty}\frac{x^2}{z^{-1}e^x+1}dx, \qquad (40a)$$

$$\Gamma(2)f_2(z) = \int_0^{\beta E_F}\frac{x}{z^{-1}e^x+1}dx + \int_{\beta E_F}^{\infty}\frac{x}{z^{-1}e^x+1}dx, \qquad (40b)$$

and

$$\Gamma(1)f_1(z) = \int_0^{\beta E_F}\frac{1}{z^{-1}e^x+1}dx + \int_{\beta E_F}^{\infty}\frac{1}{z^{-1}e^x+1}dx. \qquad (40c)$$

The second integral of each part can be estimated by changing the variable x to $y = x - \beta E_F$, that is,

$$\int_{\beta E_F}^{\infty}\frac{x^2}{z^{-1}e^x+1}dx = \int_0^{\infty}\frac{(y+\beta E_F)^2}{e^y+1}dy$$

$$\approx (\beta E_F)^2\int_0^{\infty}\left(1+2\frac{y}{\beta E_F}+\left(\frac{y}{\beta E_F}\right)^2\right)e^{-y}dy$$

$$= (\beta E_F)^2\left[\Gamma(1)+2\frac{\Gamma(2)}{\beta E_F}+\frac{\Gamma(3)}{(\beta E_F)^2}\right], \qquad (41a)$$

$$\int_{\beta E_F}^{\infty}\frac{x}{z^{-1}e^x+1}dx = \int_0^{\infty}\frac{y+\beta E_F}{e^y+1}dy$$

$$\approx \beta E_F\int_0^{\infty}\left(1+\frac{y}{\beta E_F}\right)e^{-y}dy$$

$$= \beta E_F\left[\Gamma(1)+\frac{\Gamma(2)}{\beta E_F}\right], \qquad (41b)$$

and

$$\frac{1}{\Gamma(1)}\int_{\beta E_F}^{\infty}\frac{1}{z^{-1}e^{x}+1}dx=\frac{1}{\Gamma(1)}\int_{0}^{\infty}\frac{1}{e^{y}+1}dy\approx\frac{1}{\Gamma(1)}\int_{0}^{\infty}e^{-y}dy=1. \qquad (41c)$$

The ratio of the second integral to the whole integrals in Eqs. (40a)-(40c) is

$$\approx\frac{[(\beta E_F)^2+\beta(2+2\beta m_ec^2)E_F+4+2(\beta m_ec^2)+(\beta m_ec^2)^2]}{\left[\frac{(\beta E_F)^3}{3}+\beta^3(m_ec^2)E_F^2+\beta^3(m_ec^2)^2E_F\right]}$$

$$\approx\frac{3k_BT}{E_F}\approx 1.50\times10^{-4}, \qquad (42)$$

where βE_F~20000.0. It means that there are $1.50\text{x}10^{-4}$ of the total electrons above Fermi energy at T=$1.16\text{x}10^{7}$ K or k_BT~1000 eV. When the total number of electrons is N~$6\text{x}10^{56}$, there are $9.0\text{x}10^{52}$ electrons above Fermi energy in our sun.

Next, we want to obtain the electron number above energy E_{min} in Eq. (39). The integral in Eq. (39) can be divided into two parts. Then we have the

$$\frac{(\Delta N)_{max}}{N}\left[\Gamma(3)f_3(z)+2\left(\frac{m_ec^2}{k_BT}\right)\Gamma(2)f_2(z)+\left(\frac{m_ec^2}{k_BT}\right)^2f_1(z)\right]$$

$$\approx(\beta E_F)^2\int_{\beta(E_{min}-E_F)}^{\infty}\left(1+2\frac{y}{\beta E_F}+\left(\frac{y}{\beta E_F}\right)^2\right)e^{-y}\,dy$$

$$+2\beta^2(m_ec^2)E_F\int_{\beta(E_{min}-E_F)}^{\infty}\left(1+\frac{y}{\beta E_F}\right)e^{-y}dy$$

$$+\beta^2(m_ec^2)^2\int_{\beta(E_{min}-E_F)}^{\infty}e^{-y}dy$$

$$=\beta^2E_F^2\left[\begin{array}{c}\Gamma[1,\beta(E_{min}-E_F)]\left(1+2\left(\frac{m_ec^2}{E_F}\right)+\left(\frac{m_ec^2}{E_F}\right)^2\right)\\ +2\Gamma[2,\beta(E_{min}-E_F)]\left(1+\frac{m_ec^2}{E_F}\right)\left(\frac{k_BT}{E_F}\right)\\ +\Gamma\big(3,\beta(E_{min}-E_F)\big)\left(\frac{k_BT}{E_F}\right)^2\end{array}\right], \qquad (43)$$

where $\Gamma(n,y)$ is the incomplete Gamma function [19]. Substituting Eq. (43) into Eq. (39), it becomes

$$\frac{E_{min}R}{K_ee^2}\frac{\left[\Gamma(3)f_3(z)+2\left(\frac{m_ec^2}{k_BT}\right)\Gamma(2)f_2(z)+\left(\frac{m_ec^2}{k_BT}\right)^2f_1(z)\right]}{N}$$

$$\gtrsim \beta^2 E_F^2 \begin{bmatrix} \Gamma[1,\beta(E_{min}-E_F)]\left(1+2\left(\frac{m_e c^2}{E_F}\right)+\left(\frac{m_e c^2}{E_F}\right)^2\right) \\ +2\Gamma[2,\beta(E_{min}-E_F)]\left(1+\frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +\Gamma\big(3,\beta(E_{min}-E_F)\big)\left(\frac{k_B T}{E_F}\right)^2 \end{bmatrix}. \tag{44}$$

Using Eq. (42) in Eq. (44), it is simplified to

$$\frac{E_{min}R}{K_e e^2} \gtrsim (1.50\times 10^{-4}N) \times \left\{ \frac{\begin{bmatrix} \Gamma[1,\beta(E_{min}-E_F)]\left(1+2\left(\frac{m_e c^2}{E_F}\right)+\left(\frac{m_e c^2}{E_F}\right)^2\right) \\ +2\Gamma[2,\beta(E_{min}-E_F)]\left(1+\frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +\Gamma\big(3,\beta(E_{min}-E_F)\big)\left(\frac{k_B T}{E_F}\right)^2 \end{bmatrix}}{\left[1+\left(\frac{2+2\beta m_e c^2}{\beta E_F}\right)+\frac{4+2(\beta m_e c^2)+(\beta m_e c^2)^2}{(\beta E_F)^2}\right]} \right\}. \tag{45}$$

The incomplete Gamma function has another expression [19]

$$\Gamma(n,y)=(n-1)!\,e^{-y}\sum_{m=0}^{n-1}\frac{y^m}{m!}. \tag{46}$$

Substituting Eq. (46) into Eq. (45), then we obtain

$$\frac{E_{min}R}{K_e e^2} \gtrsim (1.50\times 10^{-4}N)e^{-\beta(E_{min}-E_F)} \times \left\{ \begin{matrix} \left[1+2\left(\frac{m_e c^2}{E_F}\right)+\left(\frac{m_e c^2}{E_F}\right)^2\right] \\ +2\left(1+\frac{E_{min}-E_F}{k_B T}\right)\left(1+\frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +2\left[1+\left(\frac{E_{min}-E_F}{k_B T}\right)+\frac{1}{2}\left(\frac{E_{min}-E_F}{k_B T}\right)^2\right]\left(\frac{k_B T}{E_F}\right)^2 \end{matrix} \right\}. \tag{47}$$

Then we substitute some constants [20] into Eq. (47) to obtain E_{min} and $(\Delta Q)_{max}$. The radius is R=6.96x10^8 m of sun, K_e=8.987x10^9 nt·m^2·C^{-2}, e=1.6022x10^{-19} C,

k_B=1.38066x10^{-23} J·K^{-1}, T=1.16x10^7 K, E_F=20 MeV [3], and z=exp(20000.0). After substitution, it gives

$$\frac{E_{min}(eV)}{2.069 \times 10^{-18}(eV)} \gtrsim (9.0 \times 10^{52})e^{-\beta(E_{min}-E_F)}. \tag{48}$$

According to Eq. (48), the condition of E_{min} is

$$E_{min} \gtrsim E_F + 6.44 \times 10^4\ (eV). \tag{49}$$

It means that the electron with kinetic energy 6.44x10^4 eV more than E_F can escape the Coulomb's attraction to infinity. Furthermore, the maximally positive charges are

$$(\Delta \mathrm{Q})_{max} \lesssim 1.554 \times 10^6\ C. \tag{50}$$

Because we deal with the electron as a quantum particle, quantum mechanics tells us that particles can tunnel the potential even its energy is less than the potential. Using the concept of the tunneling effect in quantum mechanics, we can further estimate the possible number of escaping electrons or the rest positive charges. Then we consider a model that the Fermi electron gas is in the symmetric three-dimensional quantum barrier with the Coulomb potential, gravitational potential, or both at r>R as shown in Fig. 1. The potential is chosen as which one is maximal. The ratio of the attracted force on one electron causing by one Coulomb positive charges to the gravity of the sun on its surface is

$$\left(\frac{K_e(\Delta \mathrm{Q})e}{R^2}\right)\Big/\left(\frac{GM_{sun}m_e}{R^2}\right) \approx 11.90. \tag{51}$$

It means that only few charges the Coulomb's interaction can cause the force on one electron is much larger than the gravity from the sun. This tunneling model is the same as alpha-decay model and the tunneling probability for the nonrelativistic case is [17,18]

$$|T|^2 = e^{-\xi}. \tag{52}$$

According to this model, when the kinetic energy of an electron is over the maximal potential energy at r=R, the electron moves like free particle. Then

$$\xi = 2\left(\frac{8\pi^2 m_e}{h^2}\right)^{1/2} \int_R^b dr \left(\frac{K_e(\Delta \mathrm{Q})e}{r} - E_k\right)^{1/2}, \tag{53}$$

where b is the turning point determined by

$$E_k = \frac{K_e(\Delta Q)e}{b}. \tag{54}$$

From Eq. (53), it gives

$$\xi = 2\left(\frac{8\pi^2 m_e}{h^2}\right)^{1/2} [K_e(\Delta Q)eb]^{1/2}\left[\cos^{-1}\left(\frac{R}{b}\right)^{1/2} - \left(\frac{R}{b} - \frac{R^2}{b^2}\right)^{1/2}\right]. \tag{55}$$

Combing Eqs. (52) and (55) with the Fermi-Dirac distribution, the ratio of the tunneling electrons to the total electrons can be calculated, that is,

$$\frac{\Delta N_{tunnel}}{N} = \frac{\int_{E_{low}}^{E_{up}} dx e^{-\xi(x)} \frac{(x)^2\left[1 + 2\left(\frac{m_e c^2}{k_B T}\right)\frac{1}{x} + \left(\frac{m_e c^2}{k_B T}\right)^2 \frac{1}{x^2}\right]}{z^{-1}exp(\beta E_k) + 1}}{\left[\Gamma(3)f_3(z) + 2\left(\frac{m_e c^2}{k_B T}\right)\Gamma(2)f_2(z) + \left(\frac{m_e c^2}{k_B T}\right)^2 f_1(z)\right]}. \tag{56}$$

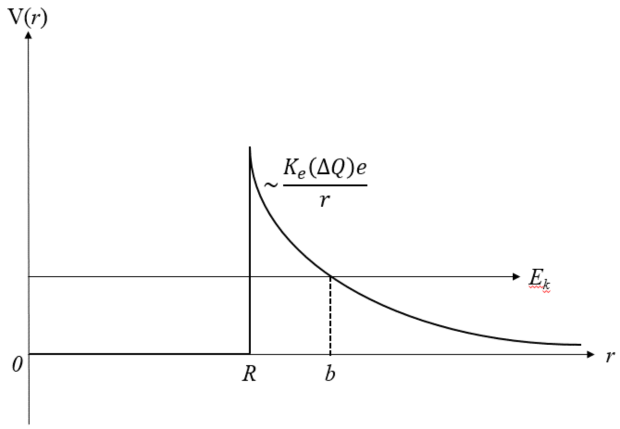

Fig. 1 The potential barrier for the electrons tunneling.

However, as more and more electrons tunnel the potential, the rest positive charges ΔQ also increase. Light passing through the diameter of the sun needs 4.64 seconds and the propagation velocity of light in plasma should be slower than it in free space. As we know, the propagation velocity of the electric force is the same as light. Theoretically speaking, the delay time for the escaping electrons affected by the rest positive charges should be in the order of 10 seconds. The escaping electron has velocity high enough to tunnel the potential and avoid to return back due to the Coulomb's attraction. According to this delay time, we can build an equation describing the relationship between $(\Delta Q)_{max}$ and E_k

$$\frac{\frac{K_e(\Delta Q)_{max}e}{E_k} - R}{c} + 10 = \frac{\frac{K_e(\Delta Q)_{max}e}{E_k}}{v}, \tag{57}$$

where v is the velocity of electron. Using Eq. (38) in Eq. (17), it gives

$$\frac{v}{c} = \frac{(E_k^2 + 2m_e c^2 E_k)^{1/2}}{E_k + m_e c^2} = \frac{\left(1 + \frac{2m_e c^2}{E_k}\right)^{1/2}}{1 + \frac{m_e c^2}{E_k}}$$

$$\approx \frac{1 + \frac{m_e c^2}{E_k} - \frac{1}{2}\left(\frac{m_e c^2}{E_k}\right)^2}{1 + \frac{m_e c^2}{E_k}} \quad (E_k \gg m_e c^2). \tag{58}$$

Substituting Eq. (58) into Eq. (57), then we have

$$\frac{K_e(\Delta Q)_{max}}{E_k\ (eV)} \frac{1}{2}\left(\frac{m_e c^2}{E_k}\right)^2 \approx 10c - R\ . \tag{59}$$

When the maximal charges $(\Delta Q)_{max}$ is 2.067x10^{20} C, Eq. (59) gives

$$E_k = 7.50 \times 10^{10}\ (eV). \tag{60}$$

If the rest positive charges can continuously accumulate without loss, the average accumulation rate is 1.31x10^{2} *C/s* for the period of 5 billion years. It means that there are 8.18x10^{20} electrons per second tunneling the potential barrier to infinity. The total energy of the escaping electrons is at least

$$8.18 \times 10^{20} \times 7.50 \times 10^{10} = 6.136 \times 10^{31}\ (eV). \tag{61}$$

Using the mass-energy equivalence in the special relativity, we can estimate how much mass (Δm) provides this total energy in Eq. (61) during the nuclear-fusion process in the sun. It is

$$\Delta m = \frac{(6.136 \times 10^{31}) \times (1.6022 \times 10^{-19})}{(2.998 \times 10^{8})^2} = 1.094 \times 10^{-4}\ kg. \tag{62}$$

It is only about transferring mass of 0.1 *g* per second for producing 1.31x10^{2} *C* escaping electrons. This lost mass occupies the nuclear-fusion reaction per second very tiny and theoretically speaking, to produce 1.31x10^{2} *C* electrons of which each one has kinetic energy at least 75.0 GeV is possible. As long as the accumulation is continuous and the attraction force is strong enough, after 50 billion years the sun will have rest positive charges 2.067x10^{20} *C*. In this case, Eq. (36) gives the upper limit of $\bar{M}$

$$\bar{M} = 2^{3/2}\bar{M}_0 = 2.8284\bar{M}_0. \tag{63}$$

Then we have the new upper mass limit M_0^{new} as it in Eq. (13)

$$M_0^{new} = 2^{3/2} \times 1.44 M_\odot = 4.073 M_\odot. \quad (64)$$

However, during this period, the star is over positively charged and it provides the positive nuclei like hydrogen ones escaping the Coulomb's and gravitational interactions so the rest positive charges decrease. Even the decrease of the rest positive charges takes place, the tunneling effect continuously happens. There is another possible way to quickly accumulate the rest positive charges. After supernova, it is possible to produce so many high-energy electrons tunneling the potential barrier and the gravitational force is possibly strong enough to keep these positive charges until it becomes a white dwarf star.

V. Conclusion

In summary, the calculation from statistical mechanics shows that the temperature effect is very weak on pressure at 10^7 K. However, the Coulomb interaction should be considered because the relativistic electrons easily escape gravity to infinity. According to our calculations, the maximally positive charges in the star has relationship with the radius. By this condition, we can calculate the pressure produced by the rest positive charges due to the Coulomb force. When this term is significant and comparable with the degenerate Fermi gas pressure, the number of the positive charges is about 10^{20} C. The number of electrons exceeding the Fermi energy is about $9.0\text{x}10^{52}$ and the maximal charges is about $1.55\text{x}10^6$ C for the sun,. However, the electron has the quantum effect and can tunnel the potential barrier even the kinetic energy is less than the maximal potential. The theoretical calculations show that the 75 GeV electron escaping to infinity can gradually accumulate the positive charges to $2.067\text{x}10^{20}$ C after 5 billion years when the escaping rate is $1.31\text{x}10^2$ C per second and it only exhausts 0.1 g mass from the sun. This case results in the upper mass limit of 4.073 $M_\odot$ when the radius is twice as large as Earth. The contribution of the rest charges is significant and makes the upper mass limit over the traditional value of 1.44 $M_\odot$. The calculation results tell us that when we consider the pressure of the white dwarf star, we should not ignore the contribution from the Coulomb's interaction.

Reference:

[1]. J. B. Holberg, "How Degenerate Star Came To Be Known White Dwarf?" *Bulletin of the American Astronomical Society* **37**, 1503 (2005).

[2]. Charles Kittel and Herbert Kroemer, *Thermal Physics* (W. H. Freeman and Company, 2nd ed., San Francisco, 1980), p.196.

[3]. Kerson Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., 2nd ed., 1987), p. 247.

[4]. Walter Greiner, Ludwig Neise, and Horst Stocker, *Thermodynamics And Statistical Mechanics* (Springer, New York, 1995), p. 359.

[5]. J. Honerkamp, *Statistical Physics-An Advanced Approach with Applications* (Springer, 2nd ed., 2002), p.233.

[6]. F. Schwabl, *Statistical Mechanics* (Springer, 2002), p.184.

[7]. Bernard F. Schutz, *A First Course In General Relativity* (Cambridge University Press, Cambridge, 1985).

[8]. Hans C. Ohanian and Remo Ruffini, *Gravitation and Spacetime* (W. W. Norton & Company, 2nd ed., New York, 1994).
[9]. F. De Felice & C. J. S. Clarke, *Relativity On Curved Manifolds* (Cambridge University Press, Cambridge, 1990).
[10]. Richard A. Mould, *Basic Relativity* (Springer, New York, 2002).
[11]. Hans Stephani, *Relativity-An Introduction to Special and General Relativity* (Cambridge, 3rd ed., 2004).
[12]. Ting-Hang Pei, "The Black Hole with A Finite-size Nucleus Based on The Conservation of Energy And Asymptotic Freedom," http://vixra.org/abs/1806.0443.
[13]. Ting-Hang Pei, "The Averagely Radial Speed of Light For The Rotating And Charged Black Hole," https://arxiv.org/abs/1804.03825.
[14]. Harry L. Shipman, "Masses and Radii of The White-Dwarf-Stars. III-Results For 110 Hydrogen-rich And 28 Helium-rich Stars," The Astrophysical Journal **228**, 240 (1979).
[15]. F. S. Levin, *An Introiduction to Quantum Theory* (Cambridge, Cambridge, 2002), p.421.
[16]. Roger G. Newton, *Quantum Physics* (Springer, New York, 2002), p.100.
[17]. Stephen Gasiorowicz, *Quantum Physics* (Joh Wiley & Sons, Inc., 1974), p.89 & p.185.
[18]. Kurt Gottfried and Tung-Mow Yan, *Quantum Mechanics: Fuandamentals* (Springer, 2nd ed., 2003), p.225.
[19]. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, And Products* (Academic Press, 7th ed., 2007).
[20]. Graham Woan, *The Cambridge Handbook of Physics Formulas* (Cambridge, 2000).