The Significant Improvement of The Upper Mass Limit of The White Dwarf Star

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Abstract Traditional explanation for the white dwarf star is based on the ideally degenerate Fermi electron gas to produce the pressure against gravity. This theory predicts the upper mass limit of the white dwarf star is 1.44 times as large as our sun although the Fermi electron gas is treated at the temperature of absolute zero. In this research, first we estimate the temperature effect and find the correction only about 1/100 at 10^7 K and about 1/10 at 10^8 K. Because some parts of this Fermi electron gas are the relativistic electrons, then we consider that some electrons can escape the gravity resulting in a positive charged star. These rest positive charges produce strong repulsive force and the pressure to against gravity. This positive-charged effect makes the upper mass limit higher than the previous value. One demonstration shows that when the rest positive charges is $1.406\text{x}10^4$ *C* and the radius of the white dwarf star is the same as Earth, its upper mass limit can be 1.98 times as large as our sun.

Keywords: white dwarf star, degenerate Fermi electron gas, pressure, upper mass limit, Coulomb's interaction

I. Introduction

The white dwarf star has been investigated many years and it was named first in 1922 [1]. It is thought to be the type of the low to medium mass stars in the final evolution stage. The white dwarf star usually has very high density with the mass similar to our sun but the volume small like Earth. The reported largest mass seems to be the one found in 2007 which is 1.33 times as large as our sun [2]. The early theory to explain its mass upper limit is based on the ideally degenerate Fermi electron gas [3-7]. The calculation adopts all electrons like free particles occupying all energy levels until to Fermi energy as they are at zero temperature. It is surprising that even in the high-temperature and high-pressure situation, the ideal Fermi gas still works. It makes the curiosity to discuss the temperature effect by statistical mechanics.

Since Einstein proposed General Relativity in 1915, some propitiate metrics have been found such as the Schwarzschild metric, the Kerr metric, and the Kerr-Newman metric [8-11]. Especially, the Kerr-Newman metric describes the rotating and charged star. Some detail problems about the Kerr-Newman black hole have been discussed [12,13]. As we know, most stars are rotating and they might be also easily charged because the relativistically massive particles escaping the gravity. According to statistical mechanics, the relativistic electrons have more possibility to escape gravity than helium nuclei at the same high temperature. Because of this factor, we consider the positive charged star and consider the Coulomb interaction existing in the rest positive charges, and further calculate the pressure produced by these rest charges. The Coulomb force also an important one against gravity so the upper mass limit of the white dwarf star should be higher.

II. The Degenerate Fermi Electron Gas For The White Dwarf Star

First, we review the calculation of the upper mass limit for the dwarf star. It adopt the ideally degenerate Fermi electron gas and considers the relativistic kinetic energy in the calculation [3,4]. Because the electron has spin s$= \pm\frac{1}{2}$, each energy state permits two electrons occupied. Each electron has the rest mass m_e, and its relativistic kinetic energy E at momentum p is

$$E = m_e c^2 \left\{ \left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{1/2} \right\}. \tag{1}$$

The Fermi electron gas with the total number N and total volume V has total kinetic energy

$$\begin{aligned} E_0 &= 2m_e c^2 \sum_{|\vec{p}|<p_F} \left\{ \left[1 + \left(\frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\} \\ &= \frac{2V m_e c^2}{h^3} \int_0^{p_F} dp 4\pi p^2 \left\{ \left[1 + \left(\frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\}, \end{aligned} \tag{2}$$

where h is the Planck's constant and p_F is Fermi momentum defined as

$$p_F = h \left(\frac{3N}{8\pi V} \right)^{1/3}. \tag{3}$$

Considering the mass m_p of a proton and the mass m_n of a neutron, the total mass M of a white dwarf star mainly consisting of helium nuclei is

$$M = \left(m_e + m_p + m_n \right) N \approx 2m_p N \approx 2m_n N. \tag{4}$$

If we define the parameter

$$x_F \equiv \frac{p_F}{m_e c} = \frac{h}{2m_e c} \left(\frac{3N}{8\pi V} \right)^{1/3}, \tag{5}$$

then Eq. (1) becomes

$$E_0 = \frac{8\pi m_e^4 c^5 V}{h^3} f(x_F), \tag{6}$$

where

$$f(x_F) = \int_0^{x_F} dx x^2 \left[(1 + x^2)^{1/2} - 1 \right]. \tag{7}$$

The pressure produced by the ideal Fermi electron gas is [4]

$$P_0 = -\frac{\partial E_0}{\partial V} = \frac{8\pi m_e^4 c^5}{h^3}\left[\frac{1}{3}x_F^3\sqrt{1+x_F^2} - f(x_F)\right]. \qquad (8)$$

It is almost 1000 times larger than the pressure of the helium nuclei [4]. Further discussions give the relationship between the radius R and mass M of the star for the relativistically high-density Fermi electron gas

$$\bar{R} = \bar{M}^{2/3}\left[1-\left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right]^{1/2}, \qquad (9)$$

where

$$\bar{R} = \left(\frac{2\pi m_e c}{h}\right)R, \qquad (10)$$

$$\bar{M} = \frac{9\pi}{8}\frac{M}{m_n}, \qquad (11)$$

and

$$\bar{M}_0 = \left(\frac{27\pi}{64\delta}\right)^{3/2}\left(\frac{hc}{2\pi\xi m_n^2}\right)^{3/2}. \qquad (12)$$

In Eq. (18), ξ is the gravitational constant and δ is a parameter of pure number. Some considerations [4] give the upper mass limit M_0 in unit of the mass M_{sun} of our sun

$$M_0 \approx 1.44 M_{sun}, \qquad (13)$$

which is also the upper limit for appearance of the white dwarf star.

III. The Temperature Effect On The Pressure of The Ideal Fermi Electron Gas

The central temperature of a star is usually about 10^7 K, and the upper mass limit in Eq. (13) is calculated at T=0 which seems to be improved. Then we consider the case for T>>0, and the grand partition function in statistical mechanics [4] is

$$q(T,V,z) = \ln Z = \sum_k \ln[1 + z \cdot exp(-\beta E_k)], \qquad (14)$$

where $\beta=1/k_B T$ and $z=\exp(\mu\beta)$ with k μ the chemical potential of the Fermi electron gas. Since the energy eigenstates are treated as arbitrarily close to each other in a very large volume, the grand partition function becomes

$$\ln Z = \int_0^\infty dE g(E) \ln[1 + zexp(-\beta E)]. \qquad (15)$$

When it is integrated by parts, then we have

$$\ln Z = g\frac{4\pi V}{h^3}\frac{\beta}{3}\int_0^\infty p^3 dp\frac{dE}{dp}\frac{1}{z^{-1}exp(\beta E)+1}, \tag{16}$$

where g=2s+1 is the degeneracy factor and

$$p^2 = \frac{E^2}{c^2} + 2m_e E. \tag{17}$$

Substituting Eq. (17) into Eq. (16), it gives

$$\ln Z = g\frac{4\pi V\beta(2m_e)^{3/2}}{3h^3}\int_0^\infty dE\frac{E^{3/2}\left[1+\frac{E}{2m_e c^2}\right]^{3/2}}{z^{-1}exp(\beta E)+1}. \tag{18}$$

Using the Taylor series expansion to the first-order term, then we have

$$\ln Z \approx g\frac{4\pi V(2m_e)^{3/2}}{3h^3\beta^{1/2}}\int_0^\infty d(\beta E)\frac{(\beta E)^{3/2}\left[1+\frac{3}{2}\left(\frac{1}{2m_e c^2\beta}\right)(\beta E)\right]}{z^{-1}exp(\beta E)+1}. \tag{19}$$

It can be written as

$$\ln Z \approx g\frac{4\pi V(2m_e)^{3/2}}{3h^3\beta^{1/2}}\left[\Gamma\left(\frac{5}{2}\right)f_{5/2}(z)+\frac{3}{2}\left(\frac{k_B T}{2m_e c^2}\right)\Gamma\left(\frac{7}{2}\right)f_{7/2}(z)\right], \tag{20}$$

where we define

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty d(\beta E)\frac{(\beta E)^{n-1}}{z^{-1}e^{(\beta E)}+1}. \tag{21}$$

The corresponding Fermi energy E_F is roughly 20 MeV [3] and 1/(2$m_e c^2\beta$)~1/1000. The chemical potential μ~E_F so z=$\beta\mu$~20000. When z>>1, Eq. (21) approximates [4]

$$f_n(z) \approx \frac{(\ln z)^n}{n!}, \tag{22}$$

so the ratio of the second term to the first term is

$$\frac{3}{2}\left(\frac{k_B T}{2m_e c^2}\right)\frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\frac{f_{7/2}}{f_{5/2}} \approx \frac{3}{2}\left(\frac{1}{1000}\right)\frac{5}{2}\cdot\frac{\ln z}{(7/2)} \approx \frac{1}{100}. \tag{23}$$

According to the relationship ln$Z$=pV/k_BT, the result in Eq. (23) means that the temperature effect only increase 1/100 in pressure produced by the high-density Fermi electron gas when T raises from 0 K to 10^7 K. When the temperature rises to 10^8 K, the ratio is 1/10 and the second term becomes important.

IV. The Improvement of The Upper Mass Limit Considering The Escaping Electrons

However, above discussions are based on the neutral star condition that the negative charges balance the positive charges. The relativistic electrons have possibility to escape the gravity of a star much higher than the ions, so the star would very be the positive charged star. In the classical statistical mechanics, the Maxwell-Planck velocity distribution tells us the most probable, the mean, and the root mean square root absolute velocities v^*, $\langle v\rangle$, and $\sqrt{\langle v^2\rangle}$ in the ideal gas are

$$v^* = \sqrt{\frac{2k_BT}{m}}, \tag{24}$$

$$\langle v\rangle = \sqrt{\frac{8k_BT}{m\pi}}, \tag{25}$$

and

$$\sqrt{\langle v^2\rangle} = \sqrt{\frac{3k_BT}{m}}. \tag{26}$$

Although we deal with the indistinguishable quantum particles, in the ultrahigh-temperature case the classical results still give a good approximation. All those three velocities for the Fermi electron gas are very close to c, but they are only 7.3x$10^{-4}c$, 8.2x$10^{-4}c$, and 8.9x$10^{-4}c$ for helium nucleus at the same temperature. According to these, many electrons escape the gravitation of the star and the star is reasonably positive charged stellar. A similar phenomenon is the well-known solar wind raising from the surface of the star and moving outward to the space.

Then considering the total negative and positive charges are -Q and Q+ΔQ. Supposing the rest positive charges ΔQ distribute homogeneously in the star, then the density $\rho_{\Delta Q}$ of the rest positive charges is

$$\rho_{\Delta Q} = \frac{\Delta Q}{\frac{4}{3}\pi R^3}. \tag{27}$$

The self-energy E_{self} of this charged sphere is

$$E_{self} = \frac{3K(\Delta Q)^2}{5R} = \frac{3K(\Delta Q)^2}{5}\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}. \quad (28)$$

The pressure $P_{\Delta Q}$ produced by the rest positive charges is

$$P_{\Delta Q} = -\frac{\partial E_{self}}{\partial V} = \frac{K(\Delta Q)^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-4/3}. \quad (29)$$

Using Eqs. (2), (10), and (11), then we have

$$P_{\Delta Q} = \frac{K(\Delta Q)^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\left(\frac{3M}{8\pi m_n N R^3}\right)^{\frac{4}{3}} = \frac{3K(\Delta Q)^2}{5\pi^2 N}\frac{1}{9}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\left(\frac{2\pi m_e c}{h}\right)^4 \frac{\bar{M}^{4/3}}{\bar{R}^4}. \quad (30)$$

This rest-positive-charges pressure also has to be considered into the contribution of the total pressure.

Next, we combine P_0 with $P_{\Delta Q}$ as the main pressure P_{main} against the gravitation in the star

$$P_{main} = \frac{2\pi m_e^4 c^5}{3h^3}\left[\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2} + \frac{8K(\Delta Q)^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\frac{\bar{M}^{4/3}}{\bar{R}^4}\right]. \quad (31)$$

Actually, the pressure of the Fermi electron gas has a tiny decrease because of the escaping electrons. It has to add a factor $(1-\Delta Q/Ne)$ accompanying with P_0 where e is the charge of a single electron, so the main pressure becomes

$$P_{main} = \frac{2\pi m_e^4 c^5}{3h^3}\left\{\left(1-\frac{\Delta Q}{Ne}\right)\left(\frac{\bar{M}^{4/3}}{\bar{R}^4} - \frac{\bar{M}^{2/3}}{\bar{R}^2}\right) + \frac{8K(\Delta Q)^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\frac{\bar{M}^{4/3}}{\bar{R}^4}\right\}. \quad (32)$$

Using these data, K=8.987x10^{9} $N{\cdot}m^2/C^2$, h=6.626x10^{-34} $J{\cdot}m$, c=2.998x10^{8} m/s, and N~10^{33} for our sun [3], the third coefficient can simplify to

$$\frac{8K(\Delta Q)^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}} \approx 1.187\times 10^{-9}(\Delta Q)^2. \quad (33)$$

When ΔQ=2.903x10^{4} C, this coefficient is 1.0. This effect is caused by (2.903 x10^{4})/(1.602x10^{-19})=1.81x10^{23} electrons escaping the star. It only occupies about 10^{-10} to 10^{-9} Fermi electron gas so the ratio $(1-\Delta Q/Ne)$ approximates 1.0 in Eq. (32). Similar to Eq. (9), we obtain

$$\bar{R} = \bar{M}^{2/3}\left\{\left[1 + \frac{8K(\Delta Q)^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\right] - \left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right\}^{1/2}. \qquad (34)$$

Using the conservation of energy between the kinetic energy and the electric potential, we can estimate the maximal number of electrons against the gravity escaping to infinite. As we know, the Coulomb's interaction is much larger than the gravitational interaction, so we only consider the Coulomb's interaction here. The electric potential at infinity is zero as a reference. The Fermi energy as the kinetic energy of the escaping electrons is used to calculate the maximally positive charges $(\Delta Q)_{max}$ of the star, that is,

$$(\gamma - 1)m_e c^2 \approx 20MeV = \frac{K(\Delta Q)_{max}e}{R}$$

$$= \frac{(8.987 \times 10^9)(\Delta Q)_{max}(1.602 \times 10^{-19})}{R}, \qquad (35)$$

where γ is the relativistic factor for the massive particle with velocity v

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (36)$$

It further gives the ratio between $(\Delta Q)_{max}$ and R

$$\frac{(\Delta Q)_{max}}{R} \approx \frac{1}{450}. \qquad (37)$$

It gives the upper charged limit for a star related to the radius R. The increase of $(\Delta Q)_{max}$ results in the increase of R. When the kinetic energy of the escaping electrons increases twice, the condition in Eq. (37) also increases doubly. If the maximal charge $(\Delta Q)_{max}$ equal to 2.903×10^4 C, the radius of the star is about 13060 km, which is twice larger than Earth. In this case, Eq. (34) gives the upper limit of $\bar{M}$

$$\bar{M} = 2^{3/2}\bar{M}_0 = 2.8284\bar{M}_0. \qquad (38)$$

Then we have the new upper mass limit M_0^{new} as it in Eq. (13)

$$M_0^{new} = 2^{3/2} \times 1.44M_{sun} = 4.073M_{sun}. \qquad (39)$$

If we consider the radius of the white dwarf star matching the astronomical investigation roughly equal to Earth [14], then

$$(\Delta Q)_{max} \approx \frac{6328}{450} = 1.406 \times 10^4\ C. \qquad (40)$$

This result gives

$$\bar{M} = 1.235^{3/2}\bar{M}_0 = 1.3719\bar{M}_0. \quad (41)$$

Then we have the new upper mass limit M_0^{new} is

$$M_0^{new} = 1.235^{3/2} \times 1.44 M_{sun} = 1.98 M_{sun}. \quad (42)$$

It means that the upper mass limit can be higher than the old one in Eq. (13) when the star is positively charged.

V. Conclusion

In summary, the calculation from statistical mechanics shows that the temperature effect is very small at 10^7 K and the ideally degenerate Fermi electron gas is still good enough. However, the Coulomb interaction should be considered because the relativistic electrons easily escape gravity to infinite. According to our calculations, the maximally positive charges in the star has relationship with the radius. By this condition, we can calculate the pressure produced by the rest charges due to the Coulomb force. This term is significant and comparable with the degenerate Fermi gas pressure. In our demonstration, when the maximally positive charge is 1.406×10^4 *C* and the radius is the same as Earth, the upper mass limit of the white dwarf is 1.98 times as large as our sun which is over the traditionally upper mass limit 1.44 M_{sun}. The contribution of the rest charges is significant and even larger than the degenerate Fermi electron gas. The calculation results tell us that when we consider the pressure inside the white dwarf star, we should not ignore the contribution of the Coulomb's interaction because some relativistic electrons can easily escape the gravity.

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