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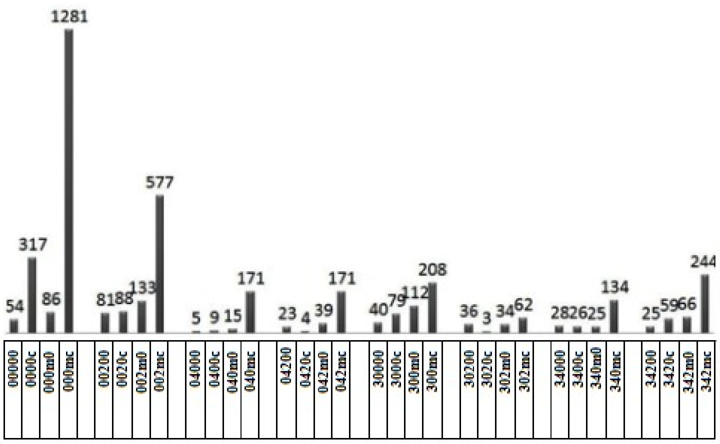
TITLE
5 bits, 32 crystal classes

ABSTRACT
Starting from the 32 crystal classes, we find a complete classification scheme of the same with only 5 bits, and at least in part the meaning of the various bits.
There is no inverse demonstration, i.e. only 5 bits must generate all 32 crystalline classes in nature. However, the proposed classification seems to invoke a logical process of formation of the various classes, doing this way:
the matter first aggregates without any symmetry, then it adopts various rotation symmetries (no symmetry, binary, ternary, etc., simple or composite) and then adds to each symmetry of rotation the additional symmetries c (center), m (planes) or c + m together

None	**c**	**m**	**m,c**
2z	**2z, c**	**2z, m**	**2z, m, c**
4z	**4z,c**	**4z,m**	**4z,m,c**
4z,2y	**4z,2y,c**	**4z,2y, m**	**4z,2y, m,c**
3z	**3z,c**	**3z,m**	**3z,m,c**
3z,2z	**3z,2z,c**	**3z,2z,m**	**3z,2z,m,c**
3z,2y	**3z,2y,c**	**3z,2y, m**	**3z,2y,m,c**
3111, 2z,2y	**3111, 2z,2y,c**	**3111, 2z,2y,m**	**3111, 2z,2y,m,c**



PREMISE

This work is based on some subsequent insights, some of which are illustrated in previous articles, see for example. [1].
But the two main ideas I consider to have been decisive are
1-The demonstration that the gyroidal class 432 does not require the 3(111) symmetry generator, but A4 generators are enough [2]. So I moved it to the tetragonal system;
2-The intuition that crystal classes can be described by groups of 4 classes, the first of which has only rotational symmetries and in the others systematically, in this order, the symmetries c, m and mc get added.
The latter has never been published, I speak here and is the subject of this work
Below you will find details.

ANALYSIS

I came to a demonstration of how the 32 crystal classes can be classified with 5 bits, which I named 3, 4, 2, m, c from the most to the less significant bit.
Specifically below:
A - shows how the 5-bit classification is born and why it is born;
B - shows the meaning of each bit case by case.
However, we can anticipate that:
- in any case the bits 3 and 2 signify the presence of axes of rotational symmetry A3 and A2;
- bit m always means the presence of a reflection plane;
- the bit c is always linked to a symmetry with respect to the center but it is so done that the presence in the crystal of an inversion center involves the presence of the bit c, but not the opposite;
- bit 4 means the presence of A4 rotational symmetry but in some cases it is more problematic to define.
That said, let's move on.
Recall from [1], [2] the classification of the first 8 triclinic + monoclinic + orthorhombic classes.

Hermann Mauguin	**5 bit symbols**	**Generators**
1	**00000**	
1_	**0000c**	**c**
m	**000m0**	**m**
2/m	**000mc**	**m,c**
2	**00200**	**2**
222	**0020c**	**2, c**
mm2	**002m0**	**2, m**
mmm	**002mc**	**2, m, c**

Fig. 1 the 8 triclinic + monoclinic + orthorhombic classes classified as 3 bits

Note the bit sequence 00, 01, 10 and 11, here referred to as the letters m and c, and then 00, 0c, m0, mc. It is obvious in a classification of 32 classes with 5 bits that there is a sequence 00, 01, 10, 11. It simply corresponds to the presence of a binary numbering.
But if it is assumed that bits have a meaning, then this sequence must correspond to a sequence of actions, property, or something.

In a 5-bit classification of the crystal classes what is the physical or 'operational' significance of the particular sequence 00, 0c, m0, mc?
That is:
is there a meaning?
Which?
Note that the sequence 00 0c m0 mc repeats itself, of course, with the 2
200
20c
2m0
2mc
it will be repeated with 4,
4000
400c
40m0
40mc
3 and so on.
What does it mean? That is, is there a meaning?
There is no doubt of a tendentious question, since the sequence, which, as I pointed out, is intrinsic in binary numbering, in fact it repeats itself, and therefore necessarily the question arises.
Fortunately, the answer is (or at least seems) obvious:
somehow there are a number of actions or operations that repeat. These are symmetry operations, 00, 0c, m0, mc. Operation m indicates a symmetry over a plane, and c a symmetry with respect to a center. They repeat at each group of four classes. The first class of each group of four classes, the one where no m or c symmetry is introduced (bits 00), is a class that obviously, as a consequence of the above, has only rotational symmetries.
So the scheme that governs the 5-bit classification of the 32 crystalline classes seems to be (is) this:
1-there are groups of four classes in each of which the sequence of symmetry operations 00, 0c, m0, mc repeats;
2-the first class in each successive sequence 00, 0c, m0, mc is a class of only rotational symmetries (axes only).
Let's imagine step by step a "birth" process of the various classes.
Consider a stack of points, or a composition of objects or dots, or a disposition of objects in space, whatever way it is said.
I take a point: it's where I put it.
I do not do anything.
This operation corresponds to the sequence 00000 or 00. No rotation symmetry, indeed no symmetry at all.
Now I try a more complicated, or more 'rich', or more aesthetic arrangement, whatever way it is said:
at one point I associate another symmetric (compared to a center of symmetry).
This operation corresponds to the sequence 0000c or 0c.
Now I try this arrangement:
at one point I associate another symmetrical with respect to a plane of symmetry.
Or you can say it in a different but equivalent way:
I create a complex object so as to possess the property of symmetry with respect to a plane m.
This operation corresponds to the 000m0 or m0 sequence.
Finally
I create a formed object so that it has both the symmetry property of a center c and the symmetry property of a plane m.
This operation corresponds to the 000mc or mc sequence.

Then, in the order, the operations are:
00000
0000c
000m0
000mc.
In decimal, these operations are enumerated 0, 1, 2, 3.
Let's say that at this point I have exhausted the elementary symmetry operations that can be found in an object. Or, let's say that I have run out the elementary symmetry operations that can be found in an object excluding rotations.
I can therefore try more complicated objects that also have rotational symmetries.
Try the symmetry A2, 180 degree rotation.
I create a formed object so as to have the rotational symmetry A2, axis 2, indicating with 2.
Only this symmetry.
This operation corresponds to the 00200 sequence.
Since I know that there are other symmetries in the object that are symmetry 'c' and symmetry 'm', I repeat the sequence of operations that I had already used.
So in the order:
I create a formed object so that both property 2 and symmetry property c have it.
This operation corresponds to the sequence 0020c;
then I create a formed object so that both the property 2 and the symmetry property m possess.
This operation corresponds to the sequence 002m0;
Finally I create a formed object so that both property 2 and both properties m and c are present.
This operation corresponds to the sequence 002mc.
Then, in the order, the operations are:
00200
0020c
002m0
002mc.
In decimal, these operations are enumerated 4, 5, 6, 7.

Having exhausted all the possible symmetries with symmetry A2, proceed to the same operations / compositions with symmetry A4.
I create a formed object so as to have the symmetry A4 that I indicate with 4.
Only this symmetry.
This operation corresponds to the sequence 04000.
Since I know that there are other symmetries in an object that are symmetry 'c' and symmetry 'm', I repeat the sequence of operations that I had already used.
Then, in the order, the operations are:
04000
0400c
040m0
040mc.
In decimal, 8, 9, 10, 11.

Before abandoning the symmetry A4, I can try to add the symmetry A2 to it. Which way?
It does not make sense to add an axis 2 parallel to axis 4, because this symmetry already exists. But I can add an axis 2 orthogonal to axis 4.

Then I create a formed object so that it has both property 4, and property 2 perpendicular to it.

This operation corresponds to the sequence 04200.
At this point I repeat the sequence of operations I had already used.
Then, in the order, the operations are:
04200
0420c
042m0
042mc.
In decimal, 12, 13, 14, 15.
So I've exhausted all possible operations with symmetry 4.
Now I build a set of points with the A3 symmetry.
This operation corresponds to the sequence 30000.
Then on them I execute the series of operations 00, 0c, m0, mc (16, 17, 18, 19) and I'm not going to repeat it.
Exhausted these, before leaving symmetry A3, I can try to add to it the symmetry A2.
How?

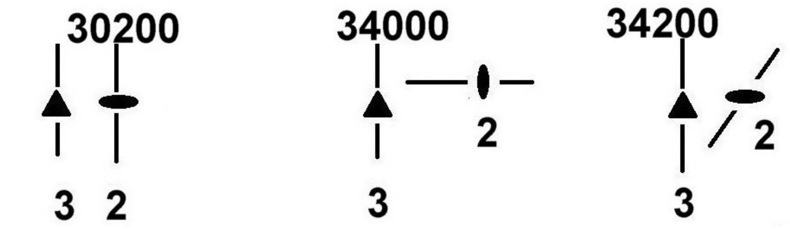


Fig. 2 the combinations that are possible between an axis 3 and an axis 2

This time it makes sense to add an axis 2 parallel to the axis 3, Fig. 2. It becomes clear that it is enough to repeat exactly the symmetry operations already performed with the symmetry A4, repeating them with a vertical rotation A6. You get the same symmetry operations but with an axis 6 instead of an axis 4.
Then I create a formed object so that it has both property 3, and property 2 parallel to it. This operation corresponds to the sequence 30200, and then from this I add the sequence of operations 00, 0c, m0, mc (in decimal 20, 21, 22, 23).

Not enough.
There is another possibility: I can add an axis 2 orthogonal to the axis 3.
Then I create a formed object, a crystal class having rotational symmetries only, so that it has both property 3, and property 2 perpendicular to it. The name of this crystal class is 32.
Two more words are worth (*). This class includes, in addition to the axis 3, a binary axis perpendicular to it: consequently, two other binary axes still occur, always in the horizontal plane, disposed at 120 ° with respect to each other. This can be seen in the simplest way, considering that the ternary axis, by virtue of its symmetry, has to repeat the binary axis three times.
This operation corresponds to the sequence 34000. The bit 4 into the sequence is quite ambiguous. In a broadest sense it could be said that “4” corresponds to “more than one axis 2”.

(*) “Questa classe comprende, oltre all’asse 3, un asse binario ad esso perpendicolare: di conseguenza, derivano ancora altri due assi binari, sempre nel piano orizzontale, disposti a 120° l’uno rispetto all’altro. Questo si può vedere, nel modo più semplice, considerando che l’asse ternario, per effetto della sua simmetria, deve fare ripetere tre volte l’asse binario”. Quoted from Gramaccioli, [3].

At this point I repeat the sequence of operations I had already used.
Then, in the order, the operations are:
34000
3400c
340m0
340mc.
(in decimal 24, 25, 26, 27).

Break time.
Until now, it has occurred that the first class in each subsequent sequence 00, 0c, m0, mc must be (is) a class of only axes. 28 of the 32 crystal classes were accommodated. There are 4 cubic classes remaining. Are they also in this scheme?
Indeed it is, and depends on the peculiarities of axis 3.
Axis 3 has with axis 2 three possibilities (Fig. 2): axis 2 orthogonal to 3, axis 2 parallel to 3, oblique.
(Why? So are the crystals).
The last case corresponds to the cubic system. Indeed, the cubic case is extremely interesting (see Appendix 1).

CHECKS

I write in a table, in the first column, the 32 classes ordered in groups of four, following the logic described above.
In the second column I write the corresponding bit sequence. These sequences are best I've managed to find to reproduce the 32 crystalline classes.
I write in the third column for each class the "generators" (*), following the simplest hypothesis i.e. that there is a direct link between bits and generators. The generators are taken from the Bilbao Server [4].

The nomenclature is that of the Bilbao Server, some pretty intuitive examples: mx means an orthogonal plane to the x axis, 2y means an axis 2 according to the y axis, m111 means an orthogonal plane at the direction 111 and so on. I've already introduced some simplification, for example 3(001) I wrote it as 3z.
The resulting table is that of the next page.
It can be noted that, compared to the previous version in Italian, v1, I exchanged the 6mm and 6_m2 classes. I can say for several reasons, but the most obvious reason is as follows: the group starting with the 32 class has a characteristic that distinguishes it (**) a horizontal axis 2, and a vertical axis 3z. The 6mm class does not have, among its symmetries, any horizontal axis 2. So it could not stay there, instead the 6_m2 class can do it.
For the rest, the graph of Figure 6 below virtually does not change because the two species numbers are similar, 34 and 25.

(*) Generators = the minimum symmetries that are needed and sufficient to generate all the symmetries of that class (generators and symmetries generally are expressed in the form of a matrix, but you may think of using Geometric Algebra notations, or other).
(**) More specifically, all four group's classes share:
- an axis 3z:
- three horizontal axes 2 (120), 2 (210), 2 (1-10) at 120 degrees to each other.
But just consider a horizontal axis 2, the rest are generated by the ternary axis.

Hermann Mauguin	5 bit symbols	Generators
1	00000	
1_	0000c	c
m	000m0	my
2/m	000mc	2y,c
2	00200	2y
222	0020c	2z, 2y
mm2	002m0	2z, my
mmm	002mc	2z, 2y, c
4	04000	4z
4_	0400c	c.4z
4mm	040m0	4z, my
4/m	040mc	4z, c
422	04200	2y, 4z
432	0420c	2z, 2y, 3111,2110
4_2m	042m0	4_, 2y
4/mmm	042mc	2y, 4z, c
3	30000	3z
3_	3000c	3z, c
3m	300m0	3z, m110
3_2/m	300mc	3z, 2(1-10), c
6	30200	3z, 2z
6_	3020c	mz, 3z
6mm	302m0	3z, 2z, m110
6/m	302mc	3, 2z, c
32	34000	3z, 2(1-10
622	3400c	3z, 2z, 2(110)
6_m2	340mc	3z, mz, m110
6/mmm	340mc	3z, 2z, m(110),c
23	34200	2z, 2y, 3111
m3	3420c	2z, 2y, 3111,c
4_3m	342m0	2z, 2y, 3111,m(1-10)
m3m	342mc	2z, 2y, 3111,2110,c

Fig. 3 organization of 32 crystalline classes according to a 5-bit classification. Generators from Bilbao Server [4].

Generators written this way sometimes are difficult to interpret for the present purposes.
For example, m110 can be simply written as a horizontal plane perpendicular to the z axis. Or more generally it can be indicated as a plane m. This generates a sort of confusion for which sometimes differently written generator groups actually are the same.
Moreover, it must be considered that each set of generators constitutes a base that can be changed to make it more suitable for the purpose. For example, a plane + center equals an axis 2 orthogonal to the plane, so for a class with symmetry center, it is possible to adopt as a generator instead of a plane m an axis 2 perpendicular to it.

So before proceeding, since I will use the symbols 3, 4, 2, m, c, to avoid confusion, it is good to define exactly what is meant by 3, 4, 2, m, c.
3: always means an A3 rotation symmetry, that is, a ternary axis, without specifying the direction;
2: always means an A2 rotation symmetry, without specifying the direction, horizontal or oblique vertical;
So too m.
m: always means a plane of symmetry, without specifying the position;
c: always means a property related to a symmetry with respect to a center. In almost all cases it coincides with the 'center' property, but in some cases this is not the case. In particular, these are the 'improper axes' 4 and 6, and classes 222, 432 and 622;
4: bit 4 is for me at the moment the most ambiguous. It can mean the intervention of an A4 symmetry, either because it is present physically or because it can mathematically generate other symmetries such as the axis 3 (111) passing through the vertices of a cube. Its meaning in the last classes 34000 etc. remains ambiguous.
Summing up, at the moment, to be cautious, I do not name the bit as 'symmetries' but 'operations'.

With all the necessary replacements, the table becomes the one on the following page:

Hermann Mauguin	5 bit symbols	Operations 3,4 2,m, c
1	00000	
1_	0000c	c
m	000m0	m
2/m	000mc	m,c
2	00200	2
222	0020c	2, c
mm2	002m0	2, m
mmm	002mc	2, m, c
4	04000	4
4_	0400c	4,c
4mm	040m0	4,m
4/m	040mc	4,m,c
422	04200	4,2
432	0420c	4,2,c
4_2m	042m0	4,2, m
4/mmm	042mc	4,2, m,c
3	30000	3
3_	3000c	3,c
3m	300m0	3,m
3_2/m	300mc	3,m,c
6	30200	3,2
6_	3020c	3,2,c
6mm	302m0	3,2,m
6/m	302mc	3,2,m,c
32	34000	3,2
622	3400c	3,2,c
6_m2	340mc	3,2,m
6/mmm	340mc	3,2,m,c
23	34200	3,2,2
m3	3420c	3,2,2,c
4_3m	342m0	3,2,2,m
m3m	342mc	3,2,2,m,c

Fig. 4 as in the previous figure 3, but with the generators written and / or interpreted differently

The classification highlights the repeat of the theme 00, 0c, m0, mc. By comparison of Figures 3 and 4 the meaning of the bits is derived.

Worth to say two words (about the classification shown in figg.3 and 4).

1- Not so easy to got it. It took to me a long time to arrive;
2- This is the best matching I've found between the crystal classes (left) and the bit sequences (right);
3- The goal requires an accurate analysis even if "a posteriori" the result may appear quite obvious.

More details are in the APPENDIX 4. Here is just a summary.
The crystalline classes are found in literature grouped according to Coxeter, Hermann Mauguin, or according to the works of Hestenes, Hitzer [5], [6]. Here the classification is made this way:
all the classes are organized into eight groups, where the first class of the group has only symmetry of rotation. It can be seen that only a number of possible symmetries are allowed, some simple as example 2, or combinations such as the symmetries 4+ 2. They are in total eight and generate eight groups of classes, adding case by case the operations c, m, and c + m together.
The rotational symmetries that generate the classes are summarized in the Figure 5a.

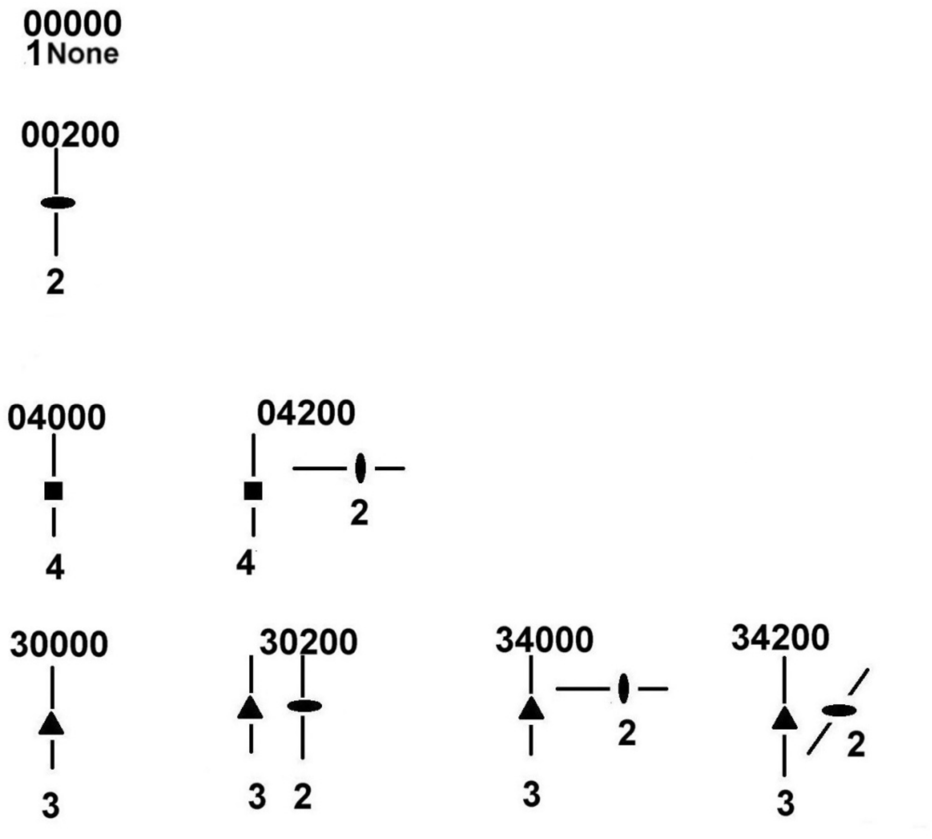


Fig. 5a the eight classes with only rotational symmetries

In Fig. 5b these eight are highlighted in yellow
Fig. 5b also shows the true axis that are involved. By a long analysis by [4] you may check that these symmetries here can be assumed as to generators.
Or better:
by means of a long and patient analysis it can be verified that:
a - or these symmetries coincide with class generators
or
b - these are the symmetries that need to intervene to generate the class.

None	**c**	**m**	**m,c**
2z	**2z, c**	**2z, m**	**2z, m, c**
4z	**4z,c**	**4z,m**	**4z,m,c**
4z,2y	**4z,2y,c**	**4z,2y, m**	**4z,2y, m,c**
3z	**3z,c**	**3z,m**	**3z,m,c**
3z,2z	**3z,2z,c**	**3z,2z,m**	**3z,2z,m,c**
3z,2y	**3z,2y,c**	**3z,2y, m**	**3z,2y,m,c**
3111, 2z,2y	**3111, 2z,2y,c**	**3111, 2z,2y,m**	**3111, 2z,2y,m,c**

Fig. 5b the eight classes of rotational symmetries highlighted in yellow

It is interesting to observe the values (Figure 6) of the number of species contained in each class. The number of species for each class is resumed by [7].
I represent it on a graph together with the 5 bit classification.

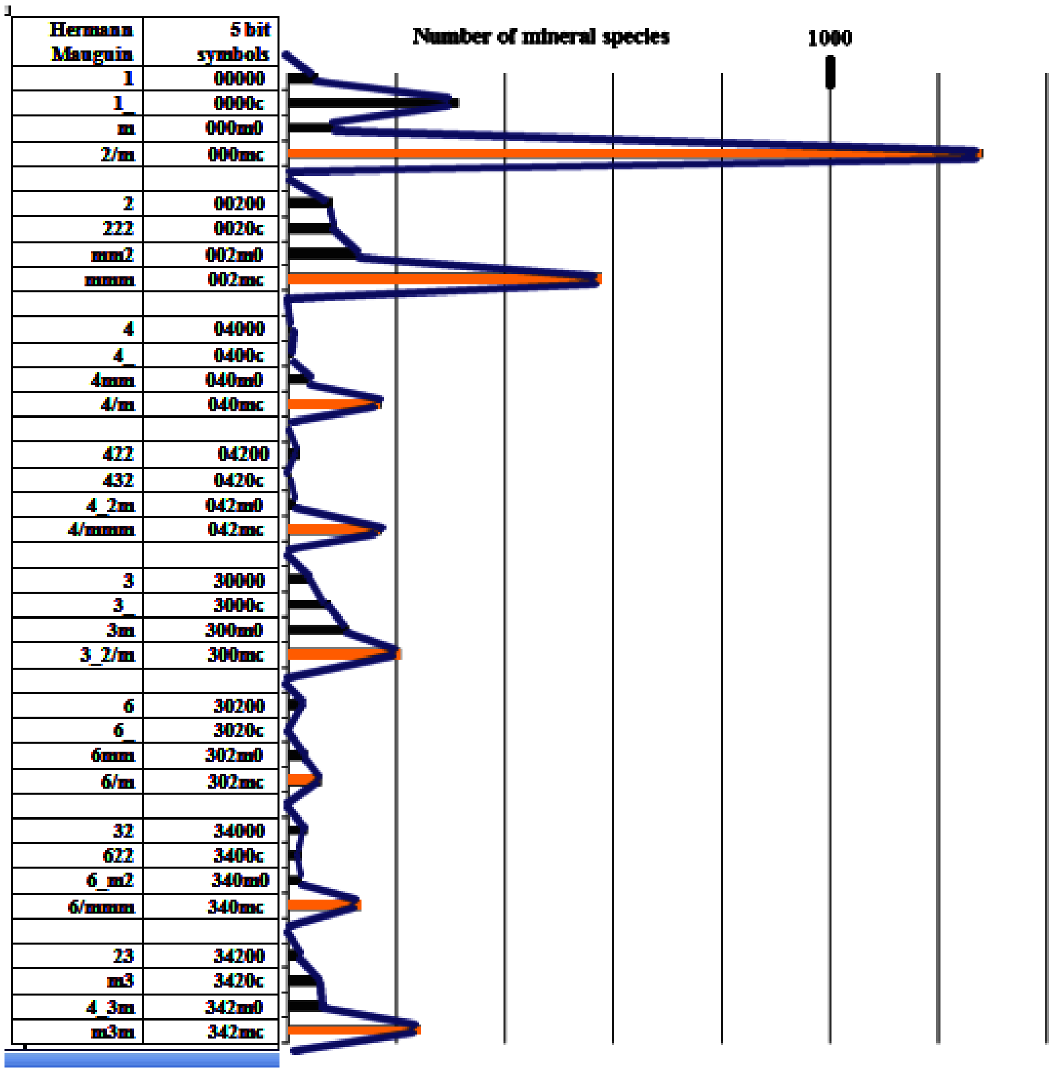


Fig. 6 Number of mineral species for each class

Observing the chart there is a strong analogy with the “eight groups of four”, that was not wanted nor specifically sought.
It is also noted that in each group of four, the class with the highest number of symmetry elements (*), or rather, of bits, is also one that possess the largest number of mineral species.
The graph of the number of mineral species contained in each class can also be thought in this other way :
the frequency of the different point groups in minerals [8].
This fact makes me realize that it is possible to consider sequences as "words", of which 3, 4, 2, m, c, are the "letters".
Thus, it is possible to calculate its statistical frequency, information content, and so on [9].

The obtained classification lacks some interpretations.
Mainly, I repeat, they are the following.

1 -classes like 222 and others in a similar position have a bit c while the class lacks symmetry "center". It is therefore reiterated that bit c is not the center. One could hypothesize that these classes have the “maximum possible symmetry, even if there is no symmetry center"”.
The 222 class has 3 binary axis at 90 degrees. This means that for each point lying on the orthogonal planes x=0, y=0 , z=0 the property “center” holds true.

2 - to class 4_ I assigned a bit c while it does not have symmetry c nor symmetry 4.This shows that the bits can not be and do not coincide with the generators. Similarly, class 6_.
On the contrary, it can not be denied that symmetries 4 and c are required to achieve class 4_. The only special fact, so to speak, is that these symmetry operations are the following:
individually they do not hold true, only their product holds true.
Similarly, class 6_.

3 - the bit sequences have a clear meaning up to 34000. From 34000 onwards, and in particular for pair 34 and triplet 342, the meaning is not crystal-clear, though it is intuitive.

(*) We could call it the holohedric class of the group, from the Greek: ὅλος = all, and ἕδρα = face. But it is not always the case, for example, the gyroidal class has the maximum number of symmetries in his group.
On the contrary this class posses very few mineral species.

CONCLUSIONS

The 32 existing crystal classes are classifiable with 5 bits representing successive symmetry properties.
The 32 classes are organized into eight groups of four classes. The first class of each group has only rotational symmetries. The eight allowed rotational symmetries are:
a class with no symmetries;
a class where axis 2 is present;
two classes where axis 4 is present;
four classes where axis 3 is present.
For each class, the additional operations c, m and c & m are added. Operation c has a meaning related to inversion over a center, in the sense that if there is the "center" property there is bit c, but the reverse is not necessarily true. The operation m always has the meaning of symmetry with respect to one plan.

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APPENDIX 1
The cubic system.

Consider the cubic class 23, which has only rotational symmetries that are 2 and 3.
Starting from this class we can verify that there is a group of four cubic classes according to the announced scheme.
Indeed, the cubic case is extremely illuminating with regard to: 1-the confirms and 2-unnecessary complications.
We start from the minimal symmetries characterizing the classes, the "generators", taken from the Internet.

Often they are written in a complicated way, and the demonstration is not obvious.
Example, from [10]:

23	**34200**	**2y, 3-111**
m3	**3420c**	**2y, 3-111, c**
4_3m	**342m0**	**3-111, c.4z, m(1-10)**
m3m	**342mc**	**3-111, 4z, c, m(1-10)**

Fig. 7 Generators of four cubic classes, taken from [10].

In fact, with help from Bilbao Server, it is noteworthy that it is legitimate to employ them as follows:

23	**34200**	**2z,2y, 3111**
m3	**3420c**	**2z,2y, 3111, c**
4_3m	**342m0**	**2z,2y, 3111, m110**
m3m	**342mc**	**2z,2y, 3111, m110,c**

Figure 8 as in Figure 7 above, but with generators written differently

So there is exactly a class of “rotations only”, the class 23, axis 2 orthogonal to cube faces and axis 3 along to the vertices. The other classes are generated by adding symmetries c, m and mc.
This is at the same time a confirmation that the true cubic classes are these four, with class 432 shifted into the tetragonal system.

Instead, with regard to bit sequences 34200, 34200, etc., the meaning is not clear to me, though it is intuitive.
This means, as I have already said, that I have not yet been able to interpret the triplet 342.
Generally speaking, it may be assumed that these classes have, among the indispensable symmetries, more than one axis 2 and the matrix 4 that with repeated applications generates an axis 3111.
However, it should be noted that for example class 23 does not have any symmetry A4 and therefore this shows again that bit 4 can not be and can not coincide with a generator.

APPENDIX 2
The rotational symmetry 3.

Let’s write down four classes with rotational symmetry 3 together with the four initial classes, 00, 0c, m0, mc.
The generators taken from the Bilbao Server are

1	**00000**	
1_	**0000c**	**c**
m	**000m0**	**my**
2/m	**000mc**	**2y,c**

Fig. 9 generators of four classes (triclinic & monoclinic), by Bilbao Server.

3	**30000**	**3z**
3_	**3000c**	**3z, c**
3m	**300m0**	**3z, m110**
3_2/m	**300mc**	**3z, 2(1-10), c**

Fig. 10 generators of four classes with axis 3, by Bilbao Server.

With a change of generators you can rewrite it like this

1	**00000**	
1_	**0000c**	**c**
m	**000m0**	**m**
2/m	**000mc**	**m,c**

Fig. 11 as in the previous figure 9, but with generators written and / or interpreted differently

3	**30000**	**3**
3_	**3000c**	**3,c**
3m	**300m0**	**3,m**
3_2/m	**300mc**	**3,m,c**

Fig. 12 as in the previous figure 10, but with generators written and / or interpreted differently

The last class 3_2 / m does not have symmetry 2 or 2y or 2 (010) among her symmetries.
Only 2 (1-10) 2 (120) and 2 (210) are written (always from the Bilbao Server), but it is obviously the same to assume 2y because it is a pure reference issue. Crystals do not read books, so they think to have a center and a binary axis perpendicular to the 3z axis.
In conclusion this group of four classes is ok.

APPENDIX 3
General discussion

Instead of continuing on individual cases I prefer to make a general survey.
With the help of Bilbao Server we can proceed.
The Bilbao Server has the data for the 32 classes or 32 point groups. For each class we can find:
-the symmetries, namely all the symmetries of the class, written down as 3 x 3 matrices;
-the group generators, i.e. the matrices by which all the other matrices can be obtained.

We have to analyze eight groups.

The 1st group

monohedron **no symmetry** **1**	**parallelohedron** **1_**	**dome** **m**	**prism** **2/m**

Fig. 13 the first group

The first group has no problem. The generators are
1, which means bit 00000
1_, which means 0000c
my, which means 000m0
2y, 1_ which means 2 and c, but obviously we can assume m and c, so that in bit 000mc.

Let's analyze the second group

sphenoid **2**	**rhombic disphenoid** **222**	**rhombic pyramid** **mm**	**rhombic dipyramid** **mmm**

Fig. 14 the 2nd group

Generators are
2y which means 00200
2z, 2y which means 0020c with the notes on crystal class 222
2z, my which means 0002m0
2z, 2y, 1_ Instead of the couple 2y, 1_ (axis + center) we assume my + center, which means 002mc.

The 3rd group

tetragonal pyramid **4**	**tetragonal disphenoid** **4_**	**ditetragonal pyramid** **4mm**	**tetragonal dipyramid** **4/m**

Fig. 15 3rd group

The generators by Ref. [4] are:

4z, 2z but 4z it's enough, so in bit 04000
4z_, 2z ie 0400c with the notes on the 4_ class
4z, 2z, my ie 040m0
4z, 2z, 1_ which means 040mc, because (2z+ center) means mz
So this group is ok.

Etc etc.
The analysis goes on in the same way.
The final result is this table. The 8 groups, 32 crystal classes, can be (are) classified by 5 bit.
Note that, obviously, I use the word "group"not to be confused with "group" in "point group".
To summarize the situation in a simple way and with a visual "graphic" message, I added to the Fig. 16 table a legend, made in explanatory drawings.
In vertical I have reported the different possible combinations of axes.
Horizontally, the sequence 0 c m mc.
The rotation classes that so to say "originate" each group are highlighted in yellow.
To recall the meaning of the presence of the "c" bit I used a symbol with three orthogonal axes

	axes	c center	m plane	c center + m plane
None	**monohedron no symmetry** **1**	**parallellohedron** **1_**	**dome** **m**	**prism** **2/m**
2	**sphenoid** **2**	**rhombic disphenoid** **222**	**rhombic pyramid** **mm**	**rhombic dipyramid** **mmm**
4	**tetragonal pyramid** **4**	**tetragonal disphenoid** **4_**	**ditetragonal pyramid** **4mm**	**tetragonal dipyramid** **4/m**
4 2	**tetragonal trapzohedron** **422**	**gyroid** **432**	**tetragonal scalenohedron** **4_2m**	**ditetragonal dipyramid** **4/mmm**
3	**trigonal pyramid** **3**	**rhombohedron** **3_**	**ditrigonal pyramid** **3m**	**hexagonal scalenohedron** **3_2m**
3 2	**hexagonal pyramid** **6**	**trigonal dipyramid** **6_**	**dihexagonal pyramid** **6mm**	**hexagonal dipyramid** **6/m**
3 2	**trigonal trapezohedron** **32**	**hexagonal trapezohedron** **622**	**ditrigonal dipyramid** **6_m2**	**dihexagonal dipyramid** **6/mmm**
3 2	**tetartoid** **23**	**diploid** **m3**	**hextetrahedron** **4_3m**	**hexoctahedron** **m3m**

Fig. 16 the 32 classes, eight groups of four

APPENDIX 4
A tentative interpretation.

What is the physical fact?
Indeed, a binary numbering can not generate a physical fact.
It must be a physical or geometric fact, which generates a binary numbering.
It can be assumed that binary numbering originates from the following situation.
Consider an object, or a crystal, in which there are only two possible symmetries, m and c.
We can have the combinations:
00 no symmetry is present
0c there is the c symmetry
m0 there is the m symmetry
mc there are both symmetries m and c
Then
with m & c there are 4 cases
(which can be compared to 4 among the first classes of the triclinc / monoclinic systems).

At this point a new feature appears in addition to m, c:
2, a symmetry of binary rotation.
With this new symmetry, all the previously investigated cases (4 cases) will happen or may happen, having in addition this new feature (*).
So
with 2 there are a total of 4 cases.

There is now a new property: the 4.
This new feature will occur along with all the cases already studied, which were four with the 2, and four without the 2.
Total 8 cases.
So with 4 you generate 8 cases.

Now there is a further feature: the 3, a ternary rotation symmetry.
This symmetry must appear, along with all the situations that have been already studied, which are 16:
eight with 4, and eight without 4.
Total: 16 cases.
So with 3 you generate 16 cases.

This is necessarily a binary numbering. And following the logic shown here I believe that the possibility of a mathematical discussion opens.
As you can see, the hypothesis is to have 8 classes dominated by a quaternary axis, and 16 from a ternary axis.

(*) The new symmetry adds exactly to the old symmetries, so both of them apply. Of course, the new symmetry, the 2 in this case, may be incompatible with the old symmetries. In this case, the old symmetries can not be added exactly, but only "at their best."

This in turn corresponds to
- to treat class 432 crystals as if they were tetragonal
- to treat the crystals of the remaining 4 cubic classes as if they were trigonal.

It may seem weird but it's pretty obvious.
It's just a matter of point of view.
Just as a rhombus can be thought of "as a cube stretched along one space diagonal, this direction is conventionally called rhombohedral axis" (Hitzer, [6]), similarly, inversely, a cube can be thought of "as a rhombohedron flattened along the rhombohedral axis".

This all, i.e.the way in which 32 classes are born from 5 bits, may be summarized in the Figg. 17, 18, 19. The 32 classes are grouped as eight groups of four. The internal labels (numbers and letters, such as example 4z, m) show the minimum symmetries of each class. This Rule must be fulfilled: RULE 1: the new (pink) classes must have (or must have at their best) also the symmetries of the green classes.

None	**c**	**m**	**m,c**
2z	**2z, c**	**2z, m**	**2z, m, c**
4z	**4z,c**	**4z,m**	**4z,m,c**
4z,2y	**4z,2y,c**	**4z,2y, m**	**4z,2y, m,c**
3z	**3z,c**	**3z,m**	**3z,m,c**
3z,2z	**3z,2z,c**	**3z,2z,m**	**3z,2z,m,c**
3z,2y	**3z,2y,c**	**3z,2y, m**	**3z,2y,m,c**
3111, 2z,2y	**3111, 2z,2y,c**	**3111, 2z,2y,m**	**3111, 2z,2y,m,c**

Fig. 17- The pink, axis 2 classes must have (or must have at their best) also the symmetries of the green classes.

None	**c**	**m**	**m,c**
2z	**2z, c**	**2z, m**	**2z, m, c**
4z	**4z,c**	**4z,m**	**4z,m,c**
4z,2y	**4z,2y,c**	**4z,2y, m**	**4z,2y, m,c**
3z	**3z,c**	**3z,m**	**3z,m,c**
3z,2z	**3z,2z,c**	**3z,2z,m**	**3z,2z,m,c**
3z,2y	**3z,2y,c**	**3z,2y, m**	**3z,2y,m,c**
3111, 2z,2y	**3111, 2z,2y,c**	**3111, 2z,2y,m**	**3111, 2z,2y,m,c**

Fig. 18- The pink, axis 4 classes must have (or must have at their best) also the symmetries of the green classes.

None	**c**	**m**	**m,c**
2z	**2z, c**	**2z, m**	**2z, m, c**
4z	**4z,c**	**4z,m**	**4z,m,c**
4z,2y	**4z,2y,c**	**4z,2y, m**	**4z,2y, m,c**
3z	**3z,c**	**3z,m**	**3z,m,c**
3z,2z	**3z,2z,c**	**3z,2z,m**	**3z,2z,m,c**
3z,2y	**3z,2y,c**	**3z,2y, m**	**3z,2y,m,c**
3111, 2z,2y	**3111, 2z,2y,c**	**3111, 2z,2y,m**	**3111, 2z,2y,m,c**

Fig. 19- The pink, axis 3 classes must have (or must have at their best) also the symmetries of the green classes.

To this Rule we must add this other rule we already found:
RULE 2: all the four classes of a group share the same rotational symmetries of the first one.

APPENDIX 5

The frequency of point groups in minerals.

Quottion from [11]: “It is known that one of the most prominent and least explained empirical generalizations based on the facts of modern crystallography consists in a striking preference of one space groups for crystal structures over the others ... etc.”

In the case under examination we have seen that a maximum of mineral species occurs every 4 crystal classes.

I report here the graph with the number of mineral species for each class, or for each "word".

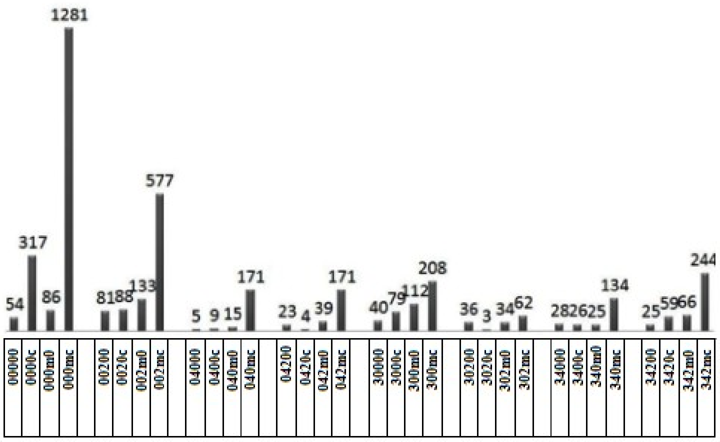
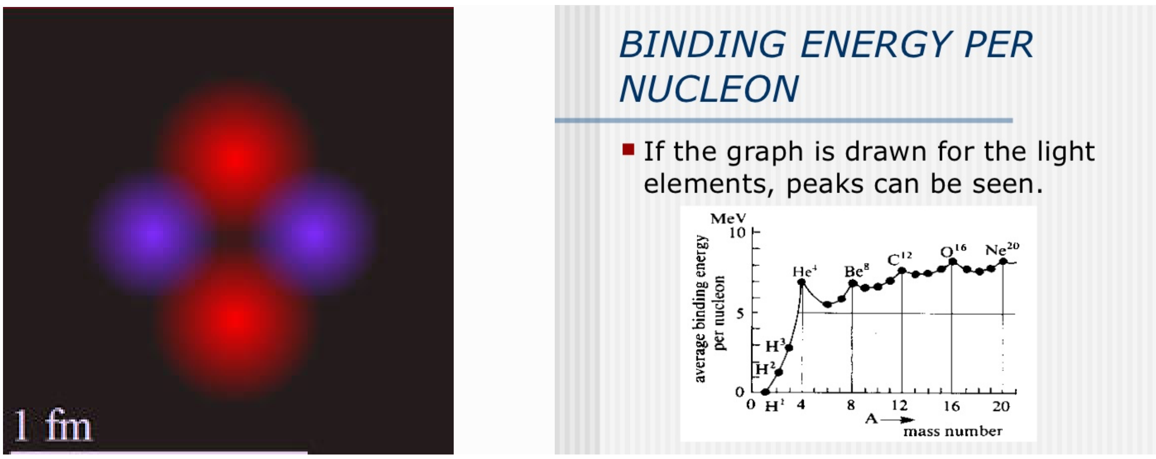


Fig. 20- number of mineral species per crystal class / word

The maximum frequency that occurs in groups of four is also present in nature in other cases.

In the "alpha cluster" model, for example, the nuclei present the maximum of bending energy in groups of four, due (it is hypothesized) to the maximum stability presented by the configuration of the alpha particle or the helium nucleus. (Figures 21a, 21b).



Figg. 21 A - Helium nucleus cloud, source https://en.wikipedia.org/wiki/Helium-4....
B - binding energy per nucleon, source
https://www.slideshare.net/cjordison/the-structure-of-the-nucleus-07

We could think of this analogy. In the case currently under examination we might think that a configuration having both m & c symmetries (Fig. 22) presents the maximum stability compared to "m only" or "c" or "none" only.

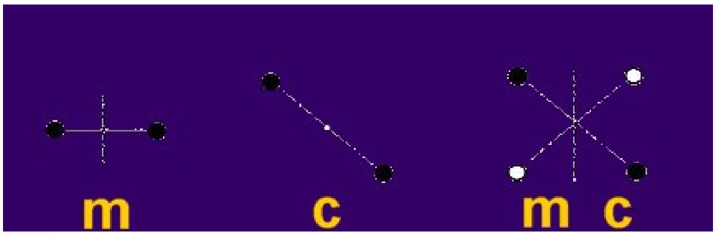


Fig. 22- Fig. 22 - symmetries m, c, mc for which I assume increasing stability

Let us assume, for example, the following probabilities of the various sequences / words 0000, 0000c, 000m0 etc (take the maximum equal to 100):

Probability to have m & c = (100%)
Probability to have m only, or c only = reduced to 10%
Probability to have “no symmetry” only= reduced to 1%

Let these probabilities further reduced by a multiplicative factor 0.2 if there is also some rotational symmetry (0.5 if the symmetry is binary).

With these simple assumptions the following probabilities are obtained for the various sequences / words

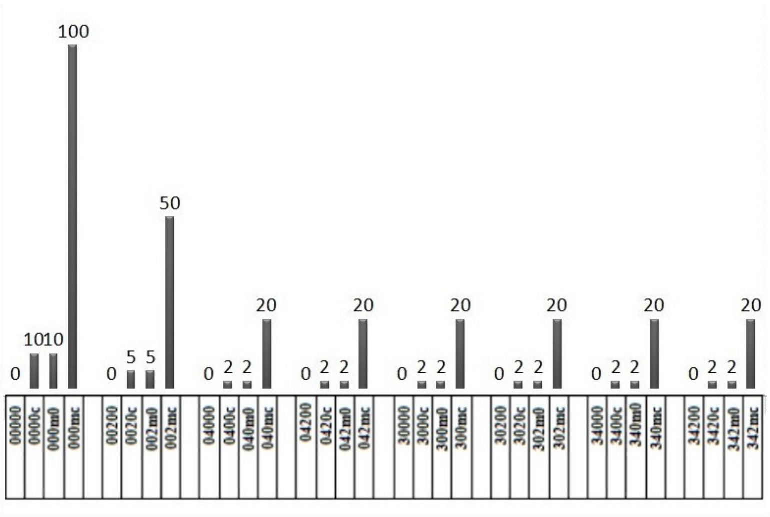


Fig. 23- Relative frequency of each crystal class / word under simple assumptions