The Law of Inertia from Spacetime Symmetries

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Abstract

The law of inertia has been treated as a fundamental assumption in classical physics. However in this paper, I show that the law of inertia is completely possible to be derived, from the homogeneity of the spacetime.

1 Introduction

The law of inertia has been treated as a fundamental assumption in the area of classical physics. For example, in Newtonian mechanics, the law of inertia is a basic assumption for defining the "inertial frame," and in relativity, one use "the least time principle" to explain the law of inertia.

However, what the law of inertia tells us, is the velocity of a free particle is constant *everywhere*. Thus we should have questions, that the law of inertia is somehow related with the space (or the spacetime).

Therefore in this paper, I will show, that the law of inertia is completely possible to be derived, from the homogeneity of the spacetime. The only preliminary concepts we use, are the position and the velocity of a particle, and none of additional assumptions (such as the least time principle) is needed.

2 Time slice

First, we introduce "*time slice*." Imagine a 1-dimensional space and the time. Let there be a particle created in the origin of the spacetime. This particle is a free particle, and vanishes in 1 second after created. Now, we let a bunch of particles be simultaneously created at the origin, and let every particle has different velocities from each other (as shown in Fig. 1).

Then, we concern about worldlines of the particles. From the fact that the particles must *vanish* in time, we can say that the worldlines should *end* at certain points in the

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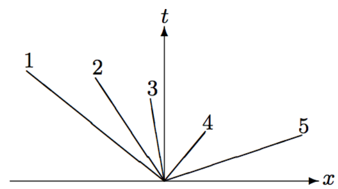

Figure 1: Worldlines of free particles, which vanish in 1 second. Numbers over the worldlines are used to identify the particles.

spacetime. Note that the worldlines described in Fig. 1 can be curves, because we didn't assume that the law of inertia is true. (And this will be revealed in Sec. 5.)

Next, we make a curve with the vanishing points of the particles, as shown in Fig. 2. And, we call this curve a *time slice*. The time slice is dependent on the vanishing time of the particle. In other words, we can obtain a different time slice from the time slice described in Fig. 2, if the vanishing time changes to some other value.

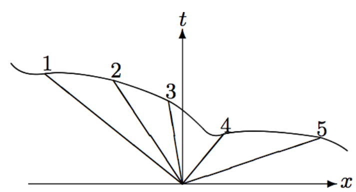

Figure 2: Time slice when the vanishing time is 1 second.

According to the special theory of relativity, the time slice described in Fig. 2 should be a hyperboloid [1]. However, in this paper we didn't assume the special relativity is true "yet." (Just like the law of inertia.) Thus, the time slice doesn't have to be a hyperboloid, and the particles don't have to be slower than the light.

From now, we discuss the symmetries in spacetime, using the time slice. Consider a situation that an observer moving with "a constant velocity" observes the time slice in Fig. 2. In a perspective of the new observer, velocities of the particles will be observed differently from the original situation, as shown in Fig. 3.

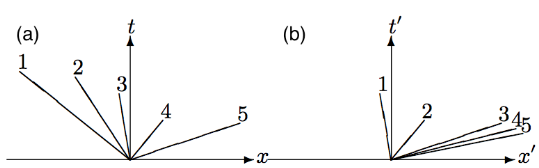

Figure 3: Worldlines of free particles, when (a) observed in a frame at rest, and (b) observed in a moving frame. Numbers over the worldlines are used to identify the particles.

On the other hand, the new observer observes *exactly the same* time slice, as the original situation. We can prove this by the following process: From the both perspective in Fig. 3, we compare two particles which have the same initial velocities, as shown in Fig. 4. And, we apply the principle of relativity by the following:

1. The two observers in Fig. 3 are being affected by "the same physical laws."
2. With the same physical laws, the same initial conditions always make the same results.
3. The two particles selected in Fig. 4, have the same initial positions (the origin) and initial velocities. (i.e. they have the same initial conditions.)
4. Thus, the two particles will vanish at the same points.

Applying this procedure to every single particle, results in the same time slices in the both situations described in Fig. 3.

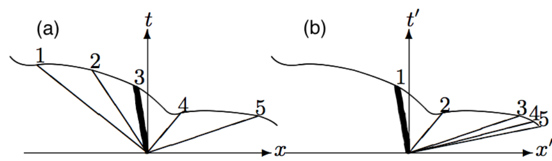

Figure 4: Time slice of a free particle, when (a) observed in a frame at rest, and (b) observed in a moving frame. Thick lines represent worldlines of particles which have the same initial velocities.

Here, we derived an invariance of the time slice, under the Lorentz boost. However, in order to discuss the invariance of a time slice, we should limit our use of the term "time slice" to the "time slice of a free particle." Now we can say, the same physical laws should result in the same time slice. For instance, the isotropy of the space will make the time slice invariant under the rotation.

3 Second derivative of a time slice: tensor $g_{\mu\nu}$

In this section, we discuss the invariance of time slice, with the aid of mathematics. From the previous section, we know that, in 1-dimensional space with time, time slice is a curve which can be described by $f(t, x) = \alpha$, where f is a function and α is a constant. And, in 3-dimensional space, we can say the time slice can be described by $f(t, x, y, z) = \alpha$.

Now the invariance of a time slice, directly goes into the invariance of the function f. For example, an invariance of a circle ($x^2 + y^2 = 1$) under the rotation, directly goes into the invariance of the function $x^2 + y^2$ under the rotation. From now, we discuss the invariance of the function f, with the symbols noted by the following:

- x^μ: μ-th component of the original coordinate system.

- x'^{μ}: μ-th component of the new coordinate system (after the coordinate transform).
- $f(x^{\lambda})$: $f(t, x, y, z)$, and $f(x'^{\lambda})$: $f(t', x', y', z')$.

First, from the function $f(x^{\lambda})$, we can make a new function $g(x'^{\lambda})$ by substituting the coordinate components; e.g. for $f(x, y) = x + y$, $x = 2x'$ and $y = 3y'$ makes $g(x', y') = 2x' + 3y'$. Since this is just a substitution, we have

$$g(x'^{\lambda}) = f(x^{\lambda}). \tag{1}$$

Then, letting $\Delta^{\lambda} \equiv x^{\lambda} - x'^{\lambda}$ and assuming Δ^{λ} is small, result in

$$g(x^{\lambda}) = g(x'^{\lambda} + \Delta^{\lambda}) \cong g(x'^{\lambda}) + \Delta^{\mu}\partial_{\mu} g(x^{\lambda}). \tag{2}$$

Now, from substituting Eq. (1) into the right-hand side of Eq. (2), we obtain

$$g(x^{\lambda}) - f(x^{\lambda}) = \Delta^{\mu}\partial_{\mu} g(x^{\lambda}). \tag{3}$$

(See Appendix A for some examples of Eq. (3).) If a function $g(x^{\lambda})$ is invariant under a certain coordinate transform, then the following goes true:

$$\Delta^{\mu}\partial_{\mu} g(x^{\lambda}) = 0. \tag{4}$$

Next, we let $\boldsymbol{\Lambda}$ be the Jacobian matrix of a coordinate transform, where $\Lambda^{\alpha}_{\mu} \equiv \partial x'^{\alpha}/\partial x^{\mu}$. And, in this section, we only discuss the "coordinate transforms *without translations*," because of the fact that the Jacobian matrix cannot describe the translation.

Now, consider a small area in the vicinity of the origin, as shown in Fig. 5. Then we

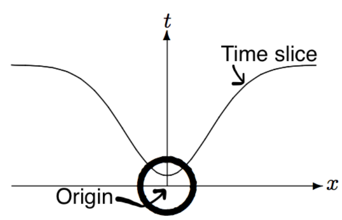

Figure 5: Time slice within the small area we consider. The area inside the thick circle represents a small area in the vicinity of the origin.

can make a power series of the function $f(x^{\lambda})$.

$$f(0^{\lambda} + x^{\lambda}) \cong f(0^{\lambda}) + x^{\mu}\left[\partial_{\mu} f(x^{\lambda})\right]_{x^{\lambda}=0^{\lambda}} + \frac{1}{2} x^{\mu} x^{\nu}\left[\partial_{\mu}\partial_{\nu} f(x^{\lambda})\right]_{x^{\lambda}=0^{\lambda}}, \tag{5}$$

where 0^{λ} denotes the zero vector, which is invariant under the coordinate transform. And, we can make a power series of $g(x'^{\lambda})$ in the same way.

$$g(0^{\lambda} + x'^{\lambda}) \cong g(0^{\lambda}) + x'^{\mu}\left[\partial'_{\mu} g(x'^{\lambda})\right]_{x'^{\lambda}=0^{\lambda}} + \frac{1}{2} x'^{\mu} x'^{\nu}\left[\partial'_{\mu}\partial'_{\nu} g(x'^{\lambda})\right]_{x'^{\lambda}=0^{\lambda}}. \tag{6}$$

Then, from $f(x^\lambda) = g(x^\lambda)$ we have $\partial_\mu f(x^\lambda) = \partial_\mu g(x^\lambda)$. Thus, we obtain

$$\big[\partial_\mu f(x^\lambda)\big]_{x^\lambda=0^\lambda} = \big[\partial_\mu g(x^\lambda)\big]_{x^\lambda=0^\lambda} = \big[\partial'_\mu g(x'^\lambda)\big]_{x'^\lambda=0^\lambda}.$$

In the same manner, we also have

$$\big[\partial_\mu \partial_\nu g(x^\lambda)\big]_{x^\lambda=0^\lambda} = \big[\partial'_\mu \partial'_\nu g(x'^\lambda)\big]_{x'^\lambda=0^\lambda} = \big[\partial_\mu \partial_\nu f(x^\lambda)\big]_{x^\lambda=0^\lambda}.$$

And, applying Eq. (4) results in

$$\begin{aligned} x^\mu \big[\partial_\mu f(x^\lambda)\big]_{x^\lambda=0^\lambda} - x'^\mu \big[\partial'_\mu g(x'^\lambda)\big]_{x'^\lambda=0^\lambda} \quad &= \quad (x^\mu - x'^\mu)\big[\partial_\mu g(x^\lambda)\big]_{x^\lambda=0^\lambda} \\ &= \Delta^\mu \big[\partial_\mu g(x^\lambda)\big]_{x^\lambda=0^\lambda} \\ &= 0. \end{aligned}$$

Now, we obtain the following by subtracting Eq. (6) from Eq. (5):

$$x^\mu x^\nu \big[\partial_\mu \partial_\nu g(x^\lambda)\big]_{x^\lambda=0^\lambda} - x'^\mu x'^\nu \big[\partial_\mu \partial_\nu g(x^\lambda)\big]_{x^\lambda=0^\lambda} = 0. \tag{7}$$

Next, we consider the covariant derivatives:

$$\begin{aligned} g_\mu \quad &\equiv \quad \nabla_\mu g(x^\lambda) = \partial_\mu g(x^\lambda), \\ g_{\mu\nu} \quad &\equiv \quad \nabla_\mu \nabla_\nu g(x^\lambda) = \partial_\mu g_\nu - \Gamma^\kappa_{\mu\nu} g_\kappa. \end{aligned}$$

And, from the assumption that x^λ is small, we have $x'^\alpha = \Lambda^\alpha_\mu x^\mu$, thus Eq. (7) becomes

$$x^\mu x^\nu \big(\big[g_{\mu\nu}\big]_{x^\lambda=0^\lambda} - \Lambda^\alpha_\mu \Lambda^\beta_\nu \big[g_{\alpha\beta}\big]_{x^\lambda=0^\lambda}\big) = 0.$$

Therefore, we obtain

$$\big[g_{\mu\nu}\big]_{x^\lambda=0^\lambda} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \big[g_{\alpha\beta}\big]_{x^\lambda=0^\lambda}. \tag{8}$$

Here, we have shown that a second derivative of a function $g(x^\lambda)$, "a tensor $g_{\mu\nu}$," is *invariant* under the coordinate transform, if the function is invariant under the same coordinate transform.

4 Homogeneity of the spacetime and Lorentz transformation

However, the translation cannot be described by Eq. (8). Thus, from now, we discuss the invariance of the tensor $g_{\mu\nu}$, *under the translation.* First, consider a free particle created at the origin. A time slice of this particle can be described by the following:

$$g(x^\lambda) = \alpha. \tag{9}$$

Next, consider a free particle, created at a different point. Then, from the homogeneity of the spacetime, "Time slice of the new particle should be the translated curve of Eq. (9)." Thus, we have

$$g(x^\lambda - D^\lambda) = \alpha \tag{10}$$

for the time slice of a particle from a different point, where D^λ denotes a vector field satisfying $\nabla_\kappa D^\lambda = 0$.[1]

We already showed that, if the function $g(x^\lambda)$ is invariant under a coordinate transform, then the tensor $g_{\mu\nu}$ is invariant as well. Thus, we can apply the same thing to *the translation*: "The tensor $g_{\mu\nu}$ should be invariant, if the function is nothing but translated."

We will examine this argument by covariant differentiating the right-hand side of Eq. (10) twice.

$$\begin{aligned} \nabla_\mu \nabla_\nu g(x^\lambda - D^\lambda) &\cong \nabla_\mu \nabla_\nu \left[g(x^\lambda) - D^\alpha \partial_\alpha g(x^\lambda) \right] \\ &= \nabla_\mu \nabla_\nu g(x^\lambda) - D^\alpha \nabla_\mu \nabla_\nu g_\alpha \\ &= g_{\mu\nu} - D^\alpha \nabla_\mu g_{\nu\alpha}. \end{aligned}$$

And we resulted in the fact that a condition $\nabla_\kappa g_{\mu\nu} = 0$ allows our argument to stand.

Therefore, from now, we will say that the homogeneity of the spacetime always guarantees

$$\nabla_\kappa g_{\mu\nu} = 0. \tag{11}$$

And here, we have completely discussed, the coordinate transforms which act on the function $g(x^\lambda)$ (or the time slice).

Now, we derive the Lorentz transformations from the results we have obtained. First, we let $\Gamma^\kappa_{\mu\nu} = 0$ and apply Eq. (11) for the homogeneity of the spacetime. Then, we can say that every single component of $g_{\mu\nu}$ is a constant. Thus we let

$$g_{\mu\nu} = \begin{pmatrix} T & u_1 & u_2 & u_3 \\ u_1 & X & u_4 & u_5 \\ u_2 & u_4 & Y & u_6 \\ u_3 & u_5 & u_6 & Z \end{pmatrix} \tag{12}$$

[1]The vector field satisfying $\nabla_\kappa D^\lambda = 0$ always exists, because of the local flatness of the spacetime.

and we consider the rotation matrices, denoted by the following:

$$
\begin{aligned}
(R_1)^\mu{}_\nu &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta_1 & 0 \\ 0 & -\theta_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
(R_2)^\mu{}_\nu &\cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & -\theta_2 & 1 \end{pmatrix}, \\
(R_3)^\mu{}_\nu &\cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & \theta_3 & 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{13}
$$

Now, substituting Eqs. (12) and (13) into Eq. (8) results in

$$
u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 0, \qquad X = Y = Z.
$$

Next, considering only 1-dimensional space with time, we can say that every single component of Λ^α_μ should be a constant.[2] Thus we let

$$
g_{\mu\nu} = \begin{pmatrix} T & 0 \\ 0 & X \end{pmatrix}, \quad \Lambda^\alpha_\mu = \begin{pmatrix} \xi & \eta \\ \zeta & \omega \end{pmatrix}, \tag{14}
$$

and substituting Eq. (14) into Eq. (8), with using $\det \boldsymbol{\Lambda} = 1$ from Eq. (8) results in

$$
g_{\mu\nu} = T \begin{pmatrix} 1 & 0 \\ 0 & -A \end{pmatrix}, \quad \Lambda^\alpha_\mu = \frac{1}{\sqrt{1 - Av^2}} \begin{pmatrix} 1 & Av \\ v & 1 \end{pmatrix},
$$

where T is a constant, and $A \equiv -X/T$, $v \equiv \zeta/\omega$. We can let $A = 1/c^2$ to obtain the Lorentz transformation.

Here, we derived the Lorentz transformation using the homogeneity (Eq. (11)), the isotropy (Eq. (13)), and the principle of relativity (Eq. (14)). This accords with the previous researches saying that the Lorentz transformation can be derived by assuming only three symmetries [2, 3]. Therefore, we can conclude that Eq. (11) exactly means the homogeneity of the spacetime.

Furthermore, we discuss the time slice in the usual spacetime (where the Lorentz transformation goes true). First, we can obtain the tensor $g_{\mu\nu}$ in 3-dimensional space with time.

$$
g_{\mu\nu} = T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/c^2 & 0 & 0 \\ 0 & 0 & -1/c^2 & 0 \\ 0 & 0 & 0 & -1/c^2 \end{pmatrix}. \tag{15}
$$

[2] If $\Gamma^\kappa_{\mu\nu} = 0$, then $\partial_\kappa \Lambda^\alpha_\mu = 0$.

And, from $g_{\mu\nu} = \partial_\mu \partial_\nu g(x^\lambda)$ we have

$$g(x^\lambda) = \iint g_{\mu\nu} dx^\mu dx^\nu .$$

Also, from the fact that the components of $g_{\mu\nu}$ are constants, we obtain

$$g(x^\lambda) = \begin{cases} \frac{1}{2} g_{\mu\nu} x^\mu x^\nu & \text{if } \mu = \nu, \\ g_{\mu\nu} x^\mu x^\nu & \text{if } \mu \neq \nu. \end{cases} \tag{16}$$

Now, we have the result by substituting Eq. (15) into Eq. (16):

$$g(x^\lambda) = T \left(t^2 - \frac{x^2}{c^2} - \frac{y^2}{c^2} - \frac{z^2}{c^2} \right) . \tag{17}$$

Equation (17) means that the time slice of a free particle is a hyperboloid.

5 Motion of a free particle: Law of inertia

According to Sec. 2, time slice should be dependent on the vanishing time of a particle. However, Eq. (9) doesn't seem to be dependent on such a parameter. Thus, from now, denoting the vanishing time by τ, we describe the time slice of a free particle by the following:

$$g(x^\lambda) = \alpha(\tau). \tag{18}$$

Eq. (18) has an important information about the position of the particle: Time slice is a "location that the particle vanishes." Thus, *The particle should be located at the time slice, at the time it vanishes.* Now we have a clue to the motion of a free particle. And, from now, we derive the equation of motion of a free particle: the law of inertia.

First, we assume that $\alpha(\tau) = \tau^2$ (the other cases will be considered in Appendix B), and Differentiate Eq. (18) twice by τ.

$$g_\kappa(x^\lambda) \left(\frac{d^2 x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) + g_{\mu\nu}(x^\lambda) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 2. \tag{19}$$

Unlike Eq. (18), There is no τ in the right-hand side of Eq. (19). That means the position x^μ and the velocity $dx^\mu/d\tau$ should satisfy Eq. (19), regardless of the particle's proper time τ.

One should note that differentiating Eq. (18) by the other parameter (other than the proper time τ) is meaningless, because of the fact that the quantity $dx^\mu/d\tau$ is the velocity, only if τ means the proper time.

Next, consider a free particle from the different point. Its time slice can be denoted by Eq. (10), and we obtain the following by differentiating Eq. (10) twice by τ:

$$\begin{aligned}\overline{\nabla}_\kappa g(X^\lambda - D^\lambda)\left(\frac{d^2X^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right)&\\ +\overline{\nabla}_\mu\overline{\nabla}_\nu g(X^\lambda - D^\lambda)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} &= 2,\end{aligned}\tag{20}$$

where $\overline{\nabla}_\kappa$ means "covariant differentiation by X^κ." (e.g. $\overline{\nabla}_\kappa g(X^\lambda) = \partial g(X^\lambda)/\partial X^\kappa$.)

And, from the assumption that D^λ is small, we have

$$\begin{aligned}\overline{\nabla}_\kappa g(X^\lambda - D^\lambda) &\cong \overline{\nabla}_\kappa\left[g(X^\lambda) - D^\alpha\frac{\partial g(X^\lambda)}{\partial X^\alpha}\right]\\ &= \overline{\nabla}_\kappa g(X^\lambda) - D^\alpha\overline{\nabla}_\kappa\overline{\nabla}_\alpha g(X^\lambda)\\ &= \overline{\nabla}_\kappa g(X^\lambda) - D^\alpha\overline{\nabla}_\alpha\overline{\nabla}_\kappa g(X^\lambda),\end{aligned}\tag{21}$$

$$\overline{\nabla}_\mu\overline{\nabla}_\nu g(X^\lambda - D^\lambda) \cong \overline{\nabla}_\mu\overline{\nabla}_\nu g(X^\lambda).\tag{22}$$

($\nabla_\mu\nabla_\nu g(x^\lambda) = \nabla_\nu\nabla_\mu g(x^\lambda)$ can be derived from $\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$.) Also,

$$\begin{aligned}\overline{\nabla}_\kappa g(X^\lambda) &= \left[\nabla_\kappa g(x^\lambda)\right]_{x^\lambda = X^\lambda} = \left[g_\kappa(x^\lambda)\right]_{x^\lambda = X^\lambda}\\ &= g_\kappa(X^\lambda),\\ \overline{\nabla}_\mu\overline{\nabla}_\nu g(X^\lambda) &= \left[\nabla_\mu\nabla_\nu g(x^\lambda)\right]_{x^\lambda = X^\lambda}\\ &= g_{\mu\nu}(X^\lambda).\end{aligned}$$

Now, we obtain the result by substituting Eqs. (21) and (22) into Eq. (20):

$$\begin{aligned}g_\kappa(X^\lambda)\left(\frac{d^2X^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) + g_{\mu\nu}(X^\lambda)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}&\\ - D^\alpha g_{\alpha\kappa}(X^\lambda)\left(\frac{d^2X^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) = 2.&\end{aligned}\tag{23}$$

Both of Eqs. (19) and (23) describe the motions of free particles.

Then, we let the two free particles coincide at a certain point r^λ. Now the positions of the particles are determined, thus Eq. (19) becomes

$$g_\kappa(r^\lambda)\left(\frac{d^2x^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) + g_{\mu\nu}(r^\lambda)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 2.\tag{24}$$

And, Eq. (23) becomes

$$\begin{aligned}g_\kappa(r^\lambda)\left(\frac{d^2X^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) + g_{\mu\nu}(r^\lambda)\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}&\\ - D^\alpha g_{\alpha\kappa}(r^\lambda)\left(\frac{d^2X^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\right) = 2.&\end{aligned}\tag{25}$$

In the meanwhile, the two particles are free particles. Thus, *the only parameter we can distinguish them is "the velocity."* Therefore, if we let there be a free particle, which is "created at the point r^λ," then the following goes true: (The particle from the origin will be called 'A', the particle from the other point will be called 'B', and the particle from r^λ will be called 'C'.)

1. If A and C have the same velocities, then we cannot distinguish them.
2. Thus, if A and C have the same velocities, then the motion of C will satisfy Eq. (24).
3. And, if B and C have the same velocities, then the motion of C will satisfy Eq. (25).
4. Therefore, the motion of C will satisfy either of Eq. (24) or Eq. (25), depending on its initial velocity.

However, we want to figure out the motion of C, *regardless of its initial velocity.* Thus, we have to find an equation of motion, which satisfies both of Eqs. (24) and (25). Then, if we let $dr^\mu/d\tau$ be the velocity of C, then we can obtain the equation of motion of C, at the point r^λ, by subtracting Eq. (25) from Eq. (24).

$$D^\alpha g_{\alpha\kappa}(r^\lambda)\left(\frac{d^2r^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dr^\mu}{d\tau}\frac{dr^\nu}{d\tau}\right)=0. \tag{26}$$

And, Eq. (26) must be true for an arbitrary vector field D^α. Therefore,

$$g_{\alpha\kappa}(r^\lambda)\left(\frac{d^2r^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}\frac{dr^\mu}{d\tau}\frac{dr^\nu}{d\tau}\right)=0. \tag{27}$$

Here we derived the equation of motion of a free particle. If we let $\Gamma^\kappa_{\mu\nu}=0$, then we can obtain the law of inertia.

$$\frac{d^2r^\kappa}{d\tau^2}=0.$$

6 Conclusion

In this paper, we have derived the equation of motion of a free particle (Eq. (27)), from the homogeneity of the spacetime (Eq. (11)). Now, we review the result in a qualitative view. Equation (27) means the velocity $dr^\mu/d\tau$ is being parallel transported along the path of the particle. In other words, Eq. (27) says that the velocity of the particle is always constant along its path [4].

Thus, we can say,

- Equation (27): The velocity of a particle is constant *everywhere.*

What we can find is, Eq. (27) means a "constancy" of some physical quantity *everywhere*, which is in accord with the argument we made in Sec. 2: Equation (11) means the homogeneity of the spacetime.

Appendix A

Consider a function $f(x,y) = x^2 + y^2$, and let x', y' be the new coordinate components after a rotation of angle φ. Then we obtain

$$\begin{cases} x' = x\cos\varphi + y\sin\varphi, \\ y' = -x\sin\varphi + y\cos\varphi, \end{cases} \qquad \begin{cases} x = x'\cos\varphi - y'\sin\varphi, \\ y = x'\sin\varphi + y'\cos\varphi. \end{cases}$$

And, from $\Delta^\mu = x^\mu - x'^\mu$ we have $\varphi^2 \approx 0$, thus

$$\begin{cases} x' \cong x + \varphi y, \\ y' \cong -\varphi x + y, \end{cases} \qquad \begin{cases} x \cong x' - \varphi y', \\ y \cong \varphi x' + y'. \end{cases} \tag{28}$$

Next, using Eq. (28), we change the variables of $f(x,y)$ from x, y into x', y', and obtain $g(x',y') \cong x'^2 + y'^2$. Thus we have $g(x,y) = x^2 + y^2$. Also, from Eq. (28) we have

$$\begin{cases} \Delta^1 = x - x' = -\varphi y, \\ \Delta^2 = y - y' = \varphi x, \end{cases}$$

and we substitute them into Eq. (3).

$$\begin{aligned} \text{(LHS)} &= g(x,y) - f(x,y) = (x^2 - y^2) - (x^2 - y^2) \\ &= 0, \\ \text{(RHS)} &= \Delta^\nu \partial_\nu g(x,y) = \Delta^1 \frac{\partial g(x,y)}{\partial x} + \Delta^2 \frac{\partial g(x,y)}{\partial y} \\ &= -\varphi y \cdot 2x + \varphi x \cdot 2y \\ &= 0. \end{aligned}$$

Here, we showed an example of Eq. (3).

And, for another example, consider a function $F(x,y) = x + y$ and apply Eq. (28).

$$\begin{aligned} G(x',y') &\cong x'(1+\varphi) + y'(1-\varphi), \\ G(x,y) &= x(1+\varphi) + y(1-\varphi). \end{aligned}$$

Then, substituting them into Eq. (3) results in

$$\begin{aligned} \text{(LHS)} &= G(x,y) - F(x,y) \\ &= [x(1+\varphi) + y(1-\varphi)] - (x+y) \\ &= \varphi x - \varphi y, \\ \text{(RHS)} &= \Delta^\nu \partial_\nu G(x,y) = \Delta^1 \frac{\partial G(x,y)}{\partial x} + \Delta^2 \frac{\partial G(x,y)}{\partial y} \\ &= -\varphi y \cdot (1+\varphi) + \varphi x \cdot (1-\varphi) \\ &= \varphi x - \varphi y. \end{aligned}$$

Equation (3) also stands for this example.

Appendix B

Let $\alpha(\tau) = \tau^N$, where N is a natural number. Then, Eq. (9) becomes

$$g(x^\lambda) = \tau^N, \tag{29}$$

and we have to differentiate Eq. (29) N times by τ, in order to make τ vanish in the right-hand side. First, we assume $\Gamma^\kappa_{\mu\nu} = 0$ and $N = 3$. Then we have

$$g_\kappa(x^\lambda)\frac{d^3x^\kappa}{d\tau^3} + 3g_{\mu\nu}(x^\lambda)\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau} = (\text{constant}),$$

and obtain an equation of motion of a free particle, by using the same method.

$$\frac{d^3r^\kappa}{d\tau^3} = 0. \tag{30}$$

And for $N = 4$, we have

$$\frac{d^4r^\kappa}{d\tau^4} = 0,$$

then we have an equation of a free particle when $\alpha(\tau) = \tau^N$.

$$\frac{d^Nr^\kappa}{d\tau^N} = 0. \tag{31}$$

Next, we integrate Eq. (31) N times by τ.

$$r^\kappa = C^\kappa\tau^{N-1} + \cdots, \tag{32}$$

where C^κ is a constant. Then, we substitute Eq. (32) into Eq. (16).

$$\begin{aligned} g(x^\lambda) &= \begin{cases} \frac{1}{2}g_{\mu\nu}C^\mu C^\nu\tau^{2N-2} + \cdots & \text{if } \mu = \nu, \\ g_{\mu\nu}C^\mu C^\nu\tau^{2N-2} + \cdots & \text{if } \mu \neq \nu. \end{cases} \\ \therefore g(x^\lambda) &\sim \tau^{2N-2}. \end{aligned} \tag{33}$$

And, comparing Eqs. (29) and (33) results in

$$N = 2.$$

Here, we showed that $\alpha(\tau) = \tau^2$ is the only possible case.

Finally, we review this result in a qualitative viewpoint. A particle's initial condition is determined by the position and the velocity. However, Eq. (30) tells us "a time derivative of the acceleration is zero," which means *we cannot determine the acceleration, because the "initial acceleration" is undetermined.* And we cannot determine the position and the velocity neither, because of the fact that the acceleration is undetermined.

From the same reason, Eq. (31) can only be valid when $N = 2$, because it is the only case that the "initial acceleration" is determined. If $N \neq 2$, then the position of a free particle will not be determined, and the time slice will remain undetermined as well.

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