A fatal error in a PhD thesis titled:
“A Generalized Variational Principle of Gravitation”
City University, London, 1982

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Abstract
I spotted a fatal error in understanding of one of the basic of mathematics used in a PhD thesis titled: “A Generalized Variational Principle of Gravitation”, City University, London, 1982. There also are two types of errors, one is a typographic error and a second one is a major error. Scanned copy of part of the thesis page (131-140) where the errors occurred is attached to this paper.

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Introduction

Errors may occur as typographic errors which are easy to spot, understand and correct, it have to be corrected in any manuscript presented to the public. In scientific researches other types of errors can occur too, but fatal errors after the thesis has been accepted with such errors in the basic mathematics used would have zero value, The PhD thesis based on such mathematical errors, too.

Discussion

As an example of these types of errors, I presented a copy of a part of a PhD thesis attached to this paper- which rely extensively on using tensors and tensor analysis- in which all these types of errors occurred.

(1) Typographic error in the thesis in pages (133-134)

In equation [4.2.29] in the second term under the integral sign in LHS a lower case latter was used for the tensor ($v^{\mu\nu}$), while an upper case letter was used ($V^{\mu\nu}$) in equation [4.2.31].

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(2) Major error in the thesis in page (134)

In equation [4.2.36] in the second term in RHS the $\frac{1}{2}$ shouldn't be there. The reason for that it is a part of the definition of Christoffel symbols of the first and second kind which are defined, respectively as

$$\Gamma_{\mu\lambda\alpha} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}}\right)$$

and,

$$\Gamma_{\mu\lambda}{}^{\eta} \equiv g^{\alpha\eta}\Gamma_{\mu\lambda\alpha} = \frac{1}{2} g^{\alpha\eta}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}}\right)$$

The $\frac{1}{2}$ should have been absorbed with the expression in the bracket in the second term in RHS of equation [4.2.35] in the definition of the Christoffel symbol of second kind to yield a correct term without the $\frac{1}{2}$ in equation [4.2.36] .

(2) Fatal error in the thesis in page (135-140)

Equation [4.2.38] and the text below it instruct one to make a change of an index with another specifically $\lambda \to \nu$.The thesis's author didn't differentiate between ***fixed*** indices and ***dummy*** indices, the indices μ and ν in LHS and RHS of equation [4.2.38] are ***fixed*** indices, they change together in name and values they take and there is no summation over them, so they play a role as labels. The other indices in RHS of equation [4.2.38] are dummy indices they could have been replaced by any other Greek letter without changing anything in the equation and there is summation over them, they are λ, η, α . The author thought that making the change of index with another would make a change in one side of the equation without making a change of the index in the other side of the equation. Even, if that is allowed, then the statement of the author that "the first term bracket cancels with the third bracket and the second cancels with fourth" in equation [4.2.38] is **wrong**. So, the statement in equation [4.2.39] is **wrong**, equation [4.2.44] is **wrong**, equation [4.2.45] is **wrong**, equation [4.2.46] is **wrong**, equation [4.2.48] is **wrong**, equation [4.2.53] is **wrong**, equation [4.2.54] is **wrong**.

Conclusion and Recommendation

It is unfortunate that such mathematical errors bypassed many people who were involved in the thesis. The author couldn't manipulate indices of tensor quantities which have lead to wrong conclusions. I hope Secretariat of Scientific affairs and college of graduate studies in universities establish a unit to assess and review PhD theses coming from ***abroad*** before acceptance.

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C - GENERALIZED EQUATIONS OF THE GRAVITATIONAL FIELD

In subsection (a) we considered the introduction of Lagrangian quadratic in R into the theory of gravitation as one way of modifying general relativity. The resulting equations are no longer those of Einstein's theory. Now instead of dealing separately with differently constructed Lagrangians we would rather assume a very general one that will allow classification for all its possible forms. This will enable us to choose from among various possible constructions the most perfect Lagrangian form which will be required to lead to a complete and self-consistent theory of gravitation that hopefully:

(1) Shares with GTR all its successes, i.e. it reduces to general relativity in the weak field areas and hence agrees with experiment and admits correct Newtonian and Minkowskian limits,

(2) does not exhibit any pathological behaviour anywhere, especially in strong-field domains,

(3) becomes amenable to quantization.

In the next chapter we will introduce our derivation based on a certain variational principle and which will yield a fourth-order in $g_{\mu\nu}$ partial differential equations, different from general relativistic equations which are of the second order in the metric tensor derivatives.

For the sake of comparison with our variation we will herein give a derivation due to C. Lanczos who first obtained this kind of generalized equation.

(i) Lanczos Variation

In his (1932) paper [23] C. Lanczos employed the Hamiltonian principle of least action [1.7.2] for the action integral [1.7.1], this would give:

$$\delta I \equiv \delta\int H d\Omega = 0 \qquad [4.2.17]$$

where function H being invariant will contain the curvature tensor $R_{\mu\nu}$ as well as the metric tensor $g_{\mu\nu}$ or their contravariant forms, i.e.

$$H = H(R_{\mu\nu}, g_{\mu\nu}) \qquad [4.2.18]$$

and, where, owing to [1.4.26] and [1.4.11] $R^{\mu\nu}$ and $g^{\mu\nu}$ are related to $R_{\mu\nu}$ and $g_{\mu\nu}$ respectively by:

$$R^{\mu\nu} = g^{\rho\mu}g^{\nu\sigma}R_{\rho\sigma} \qquad [4.2.19]$$

and

$$g^{\mu\nu} = g^{\mu\lambda}g^{\nu\gamma}g_{\lambda\gamma} \qquad [4.2.20]$$

The condition of the minimum action for the integral [4.2.17] over the volume Ω will be carried out, with assumption that $R_{\mu\nu}$ and $g_{\mu\nu}$ are, at first, independent variables.

We also consider the change $\delta R_{\mu\nu}$ in $R_{\mu\nu}$, caused by infinitesmally small change $\delta g_{\mu\nu}$ in $g_{\mu\nu}$, infinitesmally small and we denote them by:

$$\delta R_{\mu\nu} \equiv \rho_{\mu\nu} \qquad [4.2.21]$$

and

$$\delta g_{\mu\nu} \equiv \gamma_{\mu\nu} \qquad [4.2.22]$$

Then by considering the function H as the Lagrangian density:

$$H = \sqrt{g}\mathcal{L}(R) \qquad [4.2.23]$$

as it was given in [4.2.15], the variation of the action integral with respect to $g_{\mu\nu}$ will yield:

$$\int \delta(\sqrt{g}\mathcal{L})\, d\Omega = \int (\sqrt{g}\ \frac{\partial \mathcal{L}}{\partial R}\delta R + \mathcal{L}\delta\sqrt{g})\, d\Omega = 0 \qquad [4.2.24]$$

Further, by knowing from [1.4.26] that, $R = g^{\mu\nu}R_{\mu\nu}$, and by using [1.4.19] which yields

$$\delta\sqrt{g} = \frac{\sqrt{g}}{2}\, g^{\mu\rho}\delta g_{\rho\mu} \qquad [4.2.25]$$

and by denoting the derivative of $\mathcal{L}$ w.r. to R as

$$\mathcal{L}' \equiv \frac{\partial \mathcal{L}}{\partial R} \qquad [4.2.26]$$

one will obtain [4.2.24] in the form,

$$\int \sqrt{g}(\mathcal{L}' g^{\mu\nu}\delta R_{\mu\nu} + \mathcal{L}' R_{\mu\nu}\delta g^{\mu\nu} + \tfrac{1}{2}\mathcal{L} g^{\mu\nu}\delta g_{\mu\nu})\, d\Omega = 0 \qquad [4.2.27]$$

Now, by the aid of the following relation:

$$\delta g^{\mu\nu} = -g^{\mu\lambda}g^{\gamma\nu}\delta g_{\lambda\gamma} \qquad [4.2.28]$$

obtained from [1.4.17] and by using [4.2.19,21,22] integral [4.2.27] will have the general form:

$$\int \sqrt{g}(U^{\mu\nu}\rho_{\mu\nu} - U^{\mu\nu}\gamma_{\mu\nu})\, d\Omega = 0 \qquad [4.2.29]$$

where

$$U^{\mu\nu} = \mathcal{L}'(R)g^{\mu\nu}, \qquad [4.2.30]$$

and

$$V^{\mu\nu} = \mathcal{L}'(R)R^{\mu\nu} - \tfrac{1}{2}\mathcal{L}(R)g^{\mu\nu} \qquad [4.2.31]$$

Expression for $\rho_{\mu\nu}$, defined in [4.2.21] and being infinitely small change in $R_{\mu\nu}$, caused by infinitely weak deformation $\gamma_{\mu\nu}$ of $g_{\mu\nu}$, was calculated by Lanczos [24] (1923), [25] (1925).

Following Lanczos' derivation we rewrite the Ricci tensor [3.1.1].

$$R_{\mu\nu} = \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \theta_{\mu\nu} \qquad [4.2.32]$$

where

$$\theta_{\mu\nu} \equiv \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} \qquad [4.2.33]$$

will vanish if a locally inertial system of coordinate was adopted, since in this case all Γ will be zero. It will be shown also that the variation $\delta\theta_{\mu\nu}$ will give no contribution to $\rho_{\mu\nu}$.

Let us calculate the variations of $\theta_{\mu\nu}$ and Γ, i.e.

$$\delta\theta_{\mu\nu} = \delta\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} + \Gamma^{\eta}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\eta} - \delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\eta} \qquad [4.2.34]$$

where Γ is defined in [1.4.8,9] and as given in [1.7.9]

$$\delta\Gamma^{\eta}_{\mu\lambda} = \tfrac{1}{2}g^{\eta\alpha}\left(\frac{\partial\delta g_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial\delta g_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial\delta g_{\mu\lambda}}{\partial x^{\alpha}}\right) + \frac{\delta g^{\eta\alpha}}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\alpha}}\right) \qquad [4.2.35]$$

Now by using [4.2.22,28] and [1.4.8,9] one gets

$$\delta\Gamma^{\eta}_{\mu\lambda} = \tfrac{1}{2}g^{\eta\alpha}\left(\frac{\partial\gamma_{\alpha\mu}}{\partial x^{\lambda}} + \frac{\partial\gamma_{\alpha\lambda}}{\partial x^{\mu}} - \frac{\partial\gamma_{\mu\lambda}}{\partial x^{\alpha}}\right) - \tfrac{1}{2}\gamma_{\rho\alpha}g^{\alpha\eta}\Gamma^{\rho}_{\mu\lambda} \qquad [4.2.36]$$

$\gamma_{\alpha\mu}$ being a difference of two tensors will constitute a tensor and $\delta\Gamma^{\eta}_{\mu\lambda}$ is also a tensor since it transforms by the tensor law [1.4.10]. Further, by using the covariant differentiation according to [1.4.13,14] in [4.2.36] and due to the symmetry property of both $\Gamma_{\mu\nu}$ and $\gamma_{\mu\nu}$, one obtains the same expression [1.2.10] while all the terms with Γ cancel amongst themselves. We, thus, have:

$$\left.\begin{aligned}
\delta\Gamma^{\eta}_{\mu\lambda} &= \tfrac{1}{2}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\lambda} + \gamma_{\alpha\lambda;\mu} - \gamma_{\mu\lambda;\alpha}) \\
\delta\Gamma^{\lambda}_{\nu\eta} &= \tfrac{1}{2}\, g^{\lambda\alpha}(\gamma_{\alpha\nu;\eta} + \gamma_{\alpha\eta;\nu} - \gamma_{\nu\eta;\alpha}) \\
\delta\Gamma^{\eta}_{\mu\nu} &= \tfrac{1}{2}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\nu} + \gamma_{\alpha\nu;\mu} - \gamma_{\mu\nu;\alpha}) \\
\delta\Gamma^{\lambda}_{\lambda\eta} &= \tfrac{1}{2}\, g^{\lambda\alpha}(\gamma_{\alpha\lambda;\eta} + \gamma_{\alpha\eta;\lambda} - \gamma_{\lambda\eta;\alpha})
\end{aligned}\right\} \qquad [4.2.37]$$

By substituting [4.2.37] into [4.2.34] we get:

$$\left.\begin{aligned}
2\delta\theta_{\mu\nu} &= \Gamma^{\lambda}_{\nu\eta}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\lambda} + \gamma_{\alpha\lambda;\mu} - \gamma_{\mu\lambda;\alpha}) \\
&\quad + \Gamma^{\eta}_{\mu\lambda}\, g^{\lambda\alpha}(\gamma_{\alpha\nu;\eta} + \gamma_{\alpha\eta;\nu} - \gamma_{\nu\eta;\alpha}) \\
&\quad - \Gamma^{\lambda}_{\lambda\eta}\, g^{\eta\alpha}(\gamma_{\alpha\mu;\nu} + \gamma_{\alpha\nu;\mu} - \gamma_{\mu\nu;\alpha}) \\
&\quad - \Gamma^{\eta}_{\mu\nu}\, g^{\lambda\alpha}(\gamma_{\alpha\lambda;\eta} + \gamma_{\alpha\eta;\lambda} - \gamma_{\lambda\eta;\alpha})
\end{aligned}\right\} \qquad [4.2.38]$$

By making the change $\lambda \to \nu$, the first term bracket cancels with the third bracket and the second cancels with the fourth, giving for [4.2.38] the value:

$$\delta\theta_{\mu\nu} = 0 \qquad [4.2.39]$$

It will be convenient to adopt the locally inertial coordinate system all through the derivation and in addition $g_{\mu\nu}$ should be brought to its standard orthogonal form.

Now, in virtue of [4.2.22,25] and the notation

$$\gamma \equiv g^{\lambda\alpha}\gamma_{\lambda\alpha} \qquad [4.2.40]$$

we will have

$$\frac{\delta g}{g} = \gamma = \gamma^{\lambda}_{\lambda} \qquad [4.2.41]$$

and accordingly

$$\delta g^{\mu\nu} = -g^{\alpha\mu}g^{\lambda\nu}\gamma_{\lambda\alpha} = -\gamma^{\mu\nu} \qquad [4.2.42]$$

and

$$\gamma_{\lambda\nu} = g_{\lambda\mu}\gamma^{\mu}_{\nu} = g_{\lambda\lambda}\gamma^{\lambda}_{\nu} \qquad [4.2.43]$$

Then [4.2.41] and [1.4.19] yield

$$\delta\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}\,\frac{\partial\gamma}{\partial x^{\mu}} \qquad [4.2.44]$$

which could be obtained also from any of equations [4.2.37] with the use of [4.2.43].

Therefore, by taking into account [4.2.39,44], the variation $\delta R_{\mu\nu}$ of [4.2.32] will become

$$\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\,\delta\Gamma^{\lambda}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\,\delta\Gamma^{\lambda}_{\mu\nu} + \delta\theta_{\mu\nu}$$
$$= \frac{1}{2}\,\frac{\partial^{2}\gamma}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial\delta\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} \qquad [4.2.45]$$

The use of [4.2.37] and covariant differentiation will result into:

$$\delta R_{\mu\nu} = \tfrac{1}{2}\gamma_{;\mu;\nu} + \tfrac{1}{2} g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} - \tfrac{1}{2} g^{\lambda\lambda}(\gamma_{\nu\lambda;\mu;\lambda} + \gamma_{\mu\lambda;\nu;\lambda})$$

$$- \tfrac{1}{2} g^{\lambda\lambda}\left(2\gamma_{\lambda\rho} \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}} + \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda} \frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}} - \gamma_{\nu\rho} \frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}}\right) - \gamma^{\lambda\alpha} \frac{\partial}{\partial x^{\lambda}}[\mu\nu,\alpha], \qquad [4.2.46]$$

where $[\mu\nu,\alpha]$ is defined in [1.4.8], and the covariant derivatives are used instead of the ordinary ones to secure the invariant form of $\delta R_{\mu\nu}$.

Further, with the help of [1.4.37] and [4.2.43] together with the obvious relationship:

$$\gamma_{;\mu;\nu} = \gamma_{;\nu;\mu} \qquad [4.2.47]$$

expression [4.2.46] becomes:

$$\delta R_{\mu\nu} = g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} - \tfrac{1}{2}(\gamma^{\lambda}_{\nu;\lambda;\mu} - \tfrac{1}{2}\gamma_{;\nu;\mu} + \gamma^{\lambda}_{\mu;\lambda;\nu}$$

$$- \tfrac{1}{2}\gamma_{;\mu;\nu}) - \tfrac{1}{2}\gamma^{\sigma}_{\nu}R_{\sigma\mu} - \tfrac{1}{2}\gamma^{\sigma}_{\mu}R_{\sigma\nu}$$

$$+ \tfrac{1}{2}\gamma^{\lambda}_{\sigma}(R^{\sigma}_{\nu\mu\lambda} + R^{\sigma}_{\mu\nu\lambda}) - \gamma^{\lambda\alpha} \frac{\partial}{\partial x^{\lambda}} [\mu\nu,\alpha]$$

$$- \frac{g^{\lambda\alpha}}{2}\left(2\gamma_{\lambda\rho} \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}} + \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda} \frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}} - \gamma_{\mu\rho} \frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\nu\rho} \frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}}\right) . \qquad [4.2.48]$$

(ii) Lanczos Generalised Field Equations

By introducing the following notations:

$$\gamma^{\lambda}_{\nu;\lambda} - \tfrac{1}{2}\gamma_{;\nu} \equiv X_{\nu}, \qquad [4.2.49]$$

$$g^{\lambda\alpha}\gamma_{\mu\nu;\alpha;\lambda} \equiv \Box^2\gamma_{\mu\nu}, \qquad [4.2.50]$$

and the relation,

$$\gamma^{\sigma}_{\nu}R_{\sigma\mu} = \gamma_{\nu\lambda}g^{\lambda\sigma}R_{\sigma\mu} = \gamma_{\nu\lambda}R^{\lambda}_{\mu} \qquad [4.2.51]$$

resulting from [4.2.43], where

$$R^{\mu}_{\nu} = g^{\mu\sigma}R_{\sigma\nu}; \qquad [4.2.52]$$

into [4.2.48], then due to [4.2.21] one obtains;

$$2\rho_{\mu\nu} = \Box^2\gamma_{\mu\nu} - (X_{\mu;\nu} + X_{\nu;\mu}) + 2\gamma^{\lambda\sigma}R_{\mu\nu\lambda\sigma}$$

$$+ R^{\sigma}_{\mu}\gamma_{\sigma\nu} + R^{\sigma}_{\nu}\gamma_{\sigma\mu} \equiv D(\gamma_{\mu\nu}) \qquad [4.2.53]$$

where,

$$2R_{\mu\nu\lambda\sigma}\gamma^{\lambda\sigma} \equiv -g^{\lambda\alpha}\left(\gamma_{\mu\rho}\frac{\partial\Gamma^{\rho}_{\lambda\nu}}{\partial x^{\nu}} + \gamma_{\nu\lambda}\frac{\partial\Gamma^{\lambda}_{\lambda\mu}}{\partial x^{\lambda}} - \gamma_{\mu\rho}\frac{\partial\Gamma^{\rho}_{\nu\alpha}}{\partial x^{\lambda}}\right.$$

$$\left. - \gamma_{\nu\rho}\frac{\partial\Gamma^{\rho}_{\mu\alpha}}{\partial x^{\lambda}} + 2\gamma_{\lambda\rho}\frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\lambda}}\right) - 2\gamma^{\lambda\alpha}\frac{\partial}{\partial x^{\lambda}}[\mu\nu,\alpha]$$

$$- R^{\lambda}_{\mu}\gamma_{\nu\lambda} - R^{\lambda}_{\nu}\gamma_{\mu\lambda}. \qquad [4.2.54]$$

The R.H. side of [4.2.54] can be rearranged to give the L.H. side by utilizing the definition of $R_{\mu\nu\lambda\sigma}$ which is given in [1.4.27].

Furthermore, we write the following property of the "adjoint" differential expressions

$$\int d\Omega[U^{\mu\nu}D(\gamma_{\mu\nu}) - \gamma^{\mu\nu}D^{+}(U_{\mu\nu})] = \oint \qquad [4.2.55]$$

where $\oint$ is the integral over the surface containing the volume Ω, which according to Gauss vanishes by variation[26] and, D^{+} is the adjoint of D defined as:

$$D^{+}(a_{\mu\nu}) \equiv D(a_{\mu\nu}) + g^{\mu\nu}a^{\alpha\beta}{}_{;\alpha;\beta} - (g^{\alpha\beta}a_{\alpha\beta})_{;\mu;\nu} \,. \qquad [4.2.56]$$

Now, according to [4.2.55] we can transform the term with $\rho_{\mu\nu}$ in [4.2.29] and we eventually get,

$$\delta I = \int \sqrt{g}\,[\tfrac{1}{2}D^{+}(U^{\mu\nu}) - V^{\mu\nu}]\gamma_{\mu\nu}d\Omega = 0 \qquad [4.2.57]$$

Therefore, since $\gamma_{\mu\nu}$ is arbitrary, we obtain the following form for the field equation:

$$D^{+}(U^{\mu\nu}) = 2V^{\mu\nu}. \qquad [4.2.58]$$

Then, substituting for $U_{\mu\nu}$ and $V_{\mu\nu}$ from [4.2.30,31] into [4.2.56,58] yields the following 4th order in $g_{\mu\nu}$ differential equation.

$$\begin{aligned} H_{\mu\nu} \equiv\; & \mathcal{L}'''(R)(R_{;\mu}R_{;\nu} - g_{\mu\nu}R_{;\sigma}R_{;\lambda}g^{\sigma\lambda}) \\ & + \mathcal{L}''(R)(R_{;\mu;\nu} - g_{\mu\nu}\Box^{2}R) \\ & + \mathcal{L}'(R)R_{\mu\nu} - \tfrac{1}{2}\mathcal{L}(R)g_{\mu\nu} = 0, \end{aligned} \qquad [4.2.59]$$

where the covariant derivatives for the scalar curvature R are given by:

$$R_{;\mu} = \frac{\partial R}{\partial x^{\mu}} \quad , \qquad [4.2.60]$$

$$R_{;\mu;\nu} = \frac{\partial}{\partial x^\nu}\left(\frac{\partial R}{\partial x^\mu}\right) - \Gamma^\lambda_{\mu\nu}\frac{\partial R}{\partial x^\lambda} \qquad [4.2.61]$$

It is clear from [4.2.59] that the field equations are divergence-free since:

$$H^\mu_{\nu;\mu} = 0 . \qquad [4.2.62]$$

(4.3) Theory with a nonlinear Lagrangian and a limiting curvature.

The following nonlinear Lagrangian has been chosen by A. Müller et al [27];

$$\mathcal{L}(R) \equiv -\frac{R_0}{m}\left[\left(1-\frac{R}{R_0}\right)^m - 1\right], m < 1 , \qquad [4.3.1]$$

in order to fulfill certain necessary requirements like coinciding with Einstein's general relativity in the weak field area i.e. when $R \to 0$, and satisfying the asymptotic flatness.

This Lagrangian was chosen to be successful also in the strong field domain, where, in order to avoid gravitational collapse an upper limiting bound of the curvature has been postulated i.e.

$$R \leq R_0 \qquad [4.3.2]$$

As we mentioned in (0.2) the idea of the limiting bound is motivated by other field theories [28] [29].

By employing equation [4.2.59] with the Lagrangian [4.3.1], the following expressions for the scalar curvature and the digonal components; $g_{rr} = A$ and $-g_{tt} = B$ in the static isotropic metric result;