An approach for analysing time dilation in the TSR (v5, 2017-11-13)

Per Hokstad, phokstad@gmail.com

Abstract. We investigate time dilation in the theory of special relativity, and discuss some basic questions and controversies. Starting out from the Lorentz transformation, we stress that the location of clocks used for time readings is essential for the magnitude of the observed time dilation. We focus on three observational principles: 1) A reference frame may apply a single clock (SC); 2) It may apply multiple clocks (MC); or 3) There is a completely symmetric situation between the two reference frames. We specify two types of simultaneity. First, there can be a direct comparison of clock readings at identical positions, ('basic simultaneity'). Further, there is simultaneity 'in the perspective' of any reference frame. By also using an auxiliary reference frame – in combination with symmetry considerations – we also define 'simultaneity at a distance'; becoming an essential concept in our investigations. Throughout there is a focus on symmetry. A main statement is to stress that the Lorentz transformation has a fully symmetric solution with respect to the two reference frames: We will always observe that a single clock on any reference frame goes slower than the passing clocks on the other frame. This is mutual for both frames. We use the obtained framework to give a thorough discussion of the travelling twin paradox, and arrive at a conclusion regarding the twins' ages, which deviates from the prevalent view regarding this example.

Key words: Time dilation, Lorentz transformation; Dingle's question; observational principle; symmetry; simultaneity; auxiliary reference frame; travelling twin.

1 Introduction

The present work explores the concept of time dilation within the theory of special relativity (TSR). Chapter 2 presents a rather limited literature review; mainly of some references that present the TSR for non-experts. We also introduce main questions discussed in this paper, and provide specifications of basic concepts.

The material of Chapter 3 is rather well known, but starts out to by giving a somewhat modified version of the Lorentz transformation. We further pinpoint the importance of the *observational principle*, that is, the specification of which clocks to apply for the required time comparisons. The chapter provides a unified framework for the various observational principles, and points out the most interesting special cases.

Chapter 4 gives a thorough discussion on the common statement that the 'moving clock goes slower'. We prefer an alternative formulation to avoid misconceptions. The fundamental question is rather which clocks we single out for time comparisons.

One can always specify 'basic simultaneity', referring to events that occur at the same instant and same place. In Chapter 5 we also see how the introduction of an *auxiliary reference frame* – in combination with symmetry requirements - can provide a useful tool also to specify 'simultaneity at a distance'.

Overall, the first chapters provide a tool for investigating time dilation within the framework of the TSR. In Chapter 6 we apply the approach to give a lengthy discussion of the 'travelling twin' example; claiming to give a logical conclusion regarding the twins' ages. Our result will deviate from the solution usually given in the literature. So in total, the given approach will challenge the current narrative on time dilation and simultaneity in the TSR. We here argue that in a symmetric situation, we should also give a symmetric definition of 'simultaneity at a distance'. A failure to do so seems to have caused some confusion in the past.

The focus of the present paper is on the mathematical modelling, rather than on the physical, and the main objective is to investigate the logical implications of the Lorentz transformation. The findings should have some implications regarding our conception of relative time and simultaneity within the framework of the TSR; as we illustrate by the travelling twin paradox.

2 Background and basic assumptions

We review some basic literature and a couple of web sites. Based on these works we point out some questions that do not seem fully settled. We also provide the basic assumptions and concepts of the present work.

2.1 Problem formulation

First, I find the literature somewhat ambiguous regarding the very interpretation of *time dilation*. For instance, how should we interpret the common statement: 'Moving clock goes slower'? Many authors apply the expression 'as seen' by the observer on the other reference system, perhaps indicating that it is an apparent effect, not a physical reality, but without elaborating on the interpretation of 'as seen'. Others stress that 'everything goes slower' on the 'moving system', not only the clocks; truly stating that the time dilation represents a physical reality also under the conditions of TSR (*i.e.* no gravitation *etc*.). On the other side Giulini (2005; Section 3.3) states: 'Moving clocks slow down' is 'potentially misleading and should not be taken too literally'. However, the expression 'not be taken too literally' is not very precise. So in what sense – and under which precise conditions – is time dilation to be considered a true physical phenomenon?

Regarding the concept of time dilation, I also miss a more precise discussion of the multitude of time solutions (depending on location) offered by the Lorentz transformation. Some authors treats this (*e.g*. Mermin 2005), but most authors do not elaborate on this. In the present work we will look at the total picture of *all* expressions for time dilation.

As pointed out *e.g.* on Pössel's web site (Special relativity) the phenomenon of time dilation stems from the fact that clocks of the two systems have to be compared at least twice, so it cannot be the same two clocks being compared. Since movement is relative, however, an interesting question is how to decide which system (clock) is moving. Mermin (2005) states that what 'moves' is decided by which clocks are chosen to be synchronized. This seems to be is in line with the views of the present work: the procedure of clock synchronization and clock comparison decides which reference system has the time 'moving faster/slower'.

When reference frames are moving relative to each other, the definition of *simultaneity* becomes crucial. The convention seems to define simultaneity across reference frames by use of light rays, but this hardly maintains the symmetry, (*cf*. Hokstad 2016, Chapter 6). Now we can course always consider the 'basic simultaneity'; *i.e*. simultaneity of events occurring at the same instant *and* same location. In addition, we can consider the simultaneity of events from the 'perspective' of a certain reference frame: As all clocks on a specific reference frame are synchronized, the events that occur at any location, where the clock of this reference frame show the same time, are all simultaneous *in the perspective of this reference frame.* We want to explore the potential also of this type of simultaneity.

Also the question of *symmetry* is essential. The TSR actually describes a symmetric situation for the two systems/observers moving relative to each other, but the literature does not seem to be completely consistent regarding this; some references describe situations apparently involving some asymmetry. For instance - when discussing the travelling twin paradox – (Hamilton's Homepage) clearly describes a symmetric situation, while (Feynman 2008) does not.

Actually, the so-called travelling twin paradox, (*e.g*. Feynman 2008; Mermin 2005; Smith 2015; Hamilton's Homepage) is well suited to highlight the above dilemmas. And even if the main-stream conclusion is that the travelling twin actually ages less than the earthbound twin, there still seem to be some opponents. The so-called Dingle's question, Dingle (1972) raises the important question of symmetry: "Which of the two clocks in uniform motion does the special theory require to work more slowly? McCausland (2008) presents the full question and discusses it at length. I find this an important question, and it is rather surprising that according to McCausland (2008) it has so far not been given a satisfactory answer.

Now some authors also question the validity of the TSR and the Lorentz transformation, (*e.g.* see McCausland 2011; Phipps 2013; Robbins 2013). In particular, McCausland (2011) reviews various controversies on the topic (related to H. Dingle) during several decades, and also gives a lot of references. Robbins (2013) also treats the Bergson-Einstein controversy, dating back to 1922. Further, Phipps (2013) presents a harsh critique of the TSR modelling from the standpoint of a physicist; for instance stating that the theory fails to include causes of the relative motion and thus will not capture the inherent asymmetry of the phenomenon. The scope of the present work, however, is more restricted: We will derive logical and consistent consequences of the Lorentz transformation; thus, accepting the validity of the TSR as a premise.

2.2 Basic assumptions and concepts

We base the discussions on the standard theoretical experiment: Two co-ordinate systems (reference frames), K and K_v (pointing in the same direction) are moving relative to each other at speed, v. We consider just one space co-ordinate, (x-axis), and investigate the relation between space and time parameters, (x, t) on system K and the corresponding parameters (x_v, t_v) on the system K_v. Thus, we have the following 'basic simultaneity': At time t and position x on K we observe that time equals t_v and position equals x_v on the system K_v. We base our discussions on the Lorentz transformation, including the following specifications:

- There is a complete *symmetry* between the two co-ordinate systems, K and K_v; the systems being identical in all respects.
- On both reference frames there is an arbitrary number of identical, synchronized clocks, located at any positions where it is required to measure time.
- At time $t = t_v = 0$, clocks at the location $x = 0$ on K and location $x_v = 0$ on K_v are synchronized. This represents the defining starting point, from which all events are measured: the 'point of initiation'.
- We will choose the *perspective* of one of the systems, (here usually K), and refer to this as the *primary* system. We consider this to be 'at rest', (as the other reference frame move relative to it at velocity, c). Time on this 'primary' system is at any position, x given as a constant, $t(x) \equiv t$, independent of x, (all clocks being synchronized). In contrast, at a certain time, t on the primary system, the observed time, t_v on other ('secondary' system(s), here K_v), will depend on the location where the clock reading is carried out. When there are several reference frames, we are free to choose any one as the primary.
- Throughout SC will refer to a reference frame utilizing a 'single clock' (or 'same clock'), for the time comparisons with other reference frames; and similarly MC will refer to a reference frame utilizing 'multiple clocks' (at various locations) for time comparisons.

3 The Lorentz transformation and special cases

We here present the Lorentz transformation, and further investigate a variant of it, which we find useful.

3.1 The standard formulation

The Lorentz transformation represents the fundament for our discussion of time dilation. Note that we introduce a change of the standard notation. Rather than the usual t' and x' we will write t_v and x_v. Then the Lorentz transformation takes the form

$$t_v = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(\frac{v}{c})^2}} \tag{1}$$

$$x_v = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \tag{2}$$

Thus the position, x_v corresponds to (has the same location as) x when the clocks at this positions show time t and t_v, respectively. The formulas include the length contraction along the x-axis (Lorentz factor):

$$k_x = \sqrt{1-(\frac{v}{c})^2} \tag{3}$$

3.2 An alternative formulation

Taking the perspective of *K,* we may at any time *t* choose an 'observational position' equal to $x = wt,$ (for an arbitrarily chosen *w*). By inserting $x = wt$ in (1) we directly get that time on K_v at this position equals:

$$t_v(w) = \frac{1-\frac{vw}{c^2}}{\sqrt{1-(\frac{v}{c})^2}} t \tag{4}$$

Thus to pinpoint the dependence on *w* we here -and when appropriate- write $t_v(w)$ rather than t_v. The new time dilation formula (4) will – for a given time, *t* on the primary system, *K* - give the time, $t_v(w)$ on the secondary system, K_v, as a linear, decreasing function of *w*. We can now also introduce

$$\gamma_v(w) = (1-\frac{vw}{c^2})\,/\sqrt{1-(\frac{v}{c})^2} \tag{5}$$

as the 'generalized time dilation factor', valid for any location, (any $w=x/t$), *i.e.* any observational principle. Thus, we can write (4) as

$$t_v(w) = \gamma_v(w)\,t.$$

Fig.1 provides an illustration of this time dilation formula. Here we give time in the perspective of *K*. So the figure illustrates an instant when time equals *t* all over this reference frame. The horizontal axis gives the 'position' $w = x/t$ at which the clock measurements are carried out. The vertical axis gives the actual clock readings. As stated, time on *K* equals *t* at any 'position', *w*; see horizontal line. Further, the clock readings on K_v at this instant, that is $t_v(w)$, depend on *w*; see decreasing straight line. The three special cases of *eqs.* (9), (13) and (8) are inserted; corresponding to $w = 0$, w' and *v,* respectively.

This illustration given in Fig. 1 is rather fundamental for the interpretation of relative time. Being a direct consequence of the Lorentz transformation, it is of course well-known. For instance, (Mermin 2005) gives a thorough discussion on this relation, focusing on how the exact expression for $t_2' - t_1'$ depends on $x_2 - x_1$. But perhaps they do not fully utilize the potential of this general relation.

In particular we note that the result (4), and the accompanying Fig. 1, is significant for our understanding of the concept 'simultaneity'. As all clocks on *K* are synchronized, we say that all events on *K,* (at any position *x*), occurring at the same time, *t,* are simultaneous *in the perspective of reference frame K*. So in the perspective of *K* also the event that time on K_v equals $t_v(w) = \gamma_v(w)t$ at the position corresponding to $x = wt$ on *K,* are simultaneous to these events, ('basic simultaneity'). However, in the perspective of K_v we of course have quite another result, and so we clearly see the well-known result that the simultaneity in TSR is relative and depends on the perspective of the observer.

Now, in analogy to letting $x = wt$, we also define a w_v so that $x_v = w_v \cdot t_v = w_v \cdot t_v(w)$. By inserting both $x = wt$ and $x_v = w_v \cdot t_v$, in (2), we will after some manipulations obtain

$$w_v = \frac{x_v}{t_v(w)} = \frac{w-v}{1-\frac{w}{c}\frac{v}{c}} \tag{6}$$

So equations (4), (6) represent an alternative version of Lorentz transformation, here expressed by parameters (*t, w*) rather than (*t, x*). The equation (6) has a direct interpretation. According to standard results of TSR *e.g.* (Einstein 2004; Feynman 2008; Giulini 2005; Mermin 2005) the velocities v_1 and v_2 sums up to *v,* given by the formula

$$v = v_1 \oplus v_2 \stackrel{\text{def}}{=} \frac{v_1+v_2}{1+\frac{v_1}{c}\frac{v_2}{c}} \tag{7}$$

So by defining the operator $\oplus$ this way, eq. (6) actually says that $w_v = w\oplus(-v)$, or equivalently (as we easily derive), $w_v\oplus v = w$, thus, clearly interpreting *w* and w_v as velocities along the *x* - axis.

Note that we do not need to think of *w* as a velocity; rather as a way to specify a certain position $x = wt$ on *K,* representing the location of the clocks being applied at time *t*. However, we will later see that it can also be fruitful to interpret *w* as the velocity of a third observational reference frame.

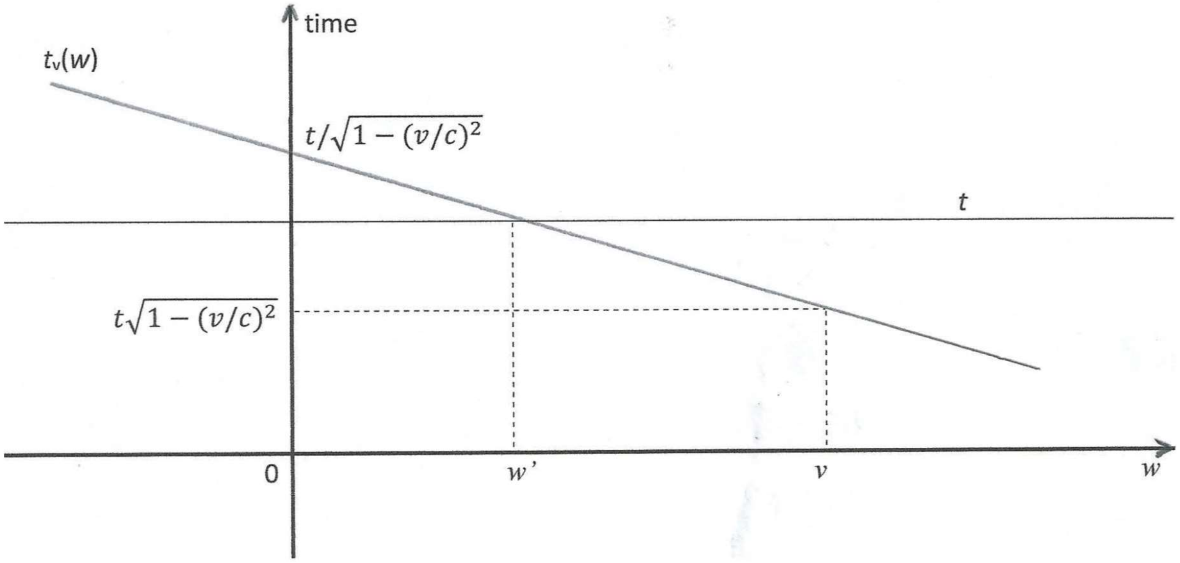

Figure 1. *Time dilation.* We have the perspective of *K,* and time all over *K* equals, *t.* Time, $t_v(w)$,on K_v given as a function of *w,* see *eq.* (4); (where *w=x/t* gives the 'position' on *K*).

3.3 Standard special cases

Focusing on time, *eq.* (4), there are various interesting special cases. First, if a specific clock located at $x_v = 0$ on K_v is compared with the passing clocks on *K.* These clocks on *K* must have position $x = vt$, and thus we choose *w=v* and get the relation

$$t_v(v) = t\sqrt{1-(v/c)^2} \tag{8}$$

Which equals the 'standard' time dilation formula. Further, when a specific clock at position $x = 0$, on *K* is used for comparisons with various passing clocks on K_v, we must choose *w*=0 and get

$$t_v(0) = t\,/\sqrt{1-(v/c)^2} \tag{9}$$

as the relation between *t* and t_v. The two special cases (8), (9) are specified in Fig. 1, and we also return to these in Chapter 4.

Two other standard special cases are obtained by choosing $w = c$ and $w = -c$, respectively. First

$$t_v(c) = \frac{1-v/c}{\sqrt{1-(\frac{v}{c})^2}}t = \frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}t \tag{10}$$

We could observe this result by applying two clocks on both system: One at $x = 0$ and one at $x = ct$ on *K*; and similarly, one at $x_v = 0$ and one at $x_v = ct_v(c)$ on K_v. Now eq. (10) is valid when a light ray is emitted in the positive direction ($x > 0$); *i.e. c* having the same direction as the velocity *v,* as seen from *K*. Similarly, emitting light in the negative direction, (choosing $x = -ct$), gives the well-known result:

$$t_v(-c) = \frac{1+v/c}{\sqrt{1-(\frac{v}{c})^2}}t = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}t \tag{11}$$

The results (10), (11) seems essentially applied for two ways light flashes, *i.e.* 'round trips', (*e.g.* Hokstad 2016).

3.4 The symmetric case

Now consider a fifth special case of the Lorentz transformation, (4), (6). We ask which value of w (and thus w_v) would results in $t_v(w)= t$. We easily find that this equality is obtained by choosing $w=w'$, where

$$w' = \frac{c^2}{v}\left(1 - \sqrt{1-\left(\frac{v}{c}\right)^2}\right) = \frac{v}{1+\sqrt{1-(v/c)^2}} \tag{12}$$

By this choice of w we also get $w_v = -w'$. This means that if we consistently consider the positions where simultaneously $x = w't$ and $x_v = -w't_v = - w't$, then no time dilation will be observed at these positions. In other words (*cf.* Fig. 1):

$$t_v(w') \equiv t \tag{13}$$

At this position $x_v = -x$, and so we see it as the midpoint between the origins of the two reference frames; thus, providing a nice symmetry. Note that when we choose the observational principle, (12), then absolutely everything is symmetric, and it should be no surprise that we get $t_v = t$.

Also note that we could give w' a nice interpretation. Recalling the definition of the operator $\oplus$ in eq. (7) for adding velocities in TSR, ($v = v_1 \oplus v_2$), it is easily verified that when w' is given by (12), then we get $w' \oplus w' = v$. So when (the origins of) K and K_v have speed w' and $-w'$, respectively, relative to the chosen point of observation, then the relative speed between K and K_v becomes v. Note that we also could link this observational point to an auxiliary reference frame, see Chapter 5.

4 "The moving clock": SC *vs.* MC

In this chapter we investigate the special cases (8) and (9) in more detail, focusing on the concepts SC (single clock) and MC (multiple clocks),

In (8), the specific clock at position $x_v \equiv 0$ on K_v at time t passes the location $x = vt$ (on K). So on K_v we just apply a single clock (SC) for the time comparisons, and in this case we say that K_v operates as a SC system. So for these SC time readings, $t_v(v)$ of *eq.* (8) we now write

$$t_v^{SC} \stackrel{\text{def}}{=} t_v(v) \tag{14}$$

Thus, *eq.* (8) becomes

$$t_v^{SC} = t\sqrt{1-(\frac{v}{c})^2}$$

Further, in (9) we follow a clock at $x \equiv 0$ on K, and at this position we make comparisons with various clocks on K_v as they pass along. We now let MC indicate a reference frame utilizing multiple clocks, and so write

$$t_v^{MC} \stackrel{\text{def}}{=} t_v(0) \tag{15}$$

Then *eq.* (18) equals:

$$t_v^{MC} = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}t$$

But these two cases are closely linked. When K_v operates as SC then K becomes MC and *vice versa.* Thus the two symmetric results, (8), (9) could be presented in a compact form as

$$t^{SC} = t^{MC}\sqrt{1-(v/c)^2} \tag{16}$$

Actually, this is much more than an effective way to write the two *eqs.* (8) and (9). By *eq.* (16) we stress that (8) and (9) actually represent the same result. So correctly understood (16) is much more informative than (8) and (9). This specifies which of the two systems we operate as SC, and which as MC.

Here we should add a note on our notation, which now differs from that of the previous chapters. For instance *eq.* (16) illustrates the following notation: There is a SC reference frame, K^{SC}; so we utilizes just one clock on this system, giving time readings, t^{SC} on the clock located at its origin, $x^{SC} = 0$. Further, t^{MC} is the time on a MC reference frame, K^{MC}. We would denote the positions of its time measurements x^{MC}. We have to admit that we will apply both sets of notations from now on, but refer to Appendix A for a summary on the notation for time parameters.

Further, note that we in (16) have dropped the subscript, v on *both* time parameters. This just means that (16) is valid irrespective of which reference frame is chosen as 'primary'. But, if needed, we could add a subscript v to either t^{SC} or t^{MC}, to indicate which of the systems we choose as the secondary system.

Before we leave (16) some comments are relevant. First we stress that observers on both reference frames will agree on this result (16). Thus, I find it rather misleading to apply the phrase 'as seen' regarding the clock reading on 'the other' system; which is a formulation used by some authors. The time readings are objective, and all observers (observational equipment) on the location in question will 'see' the same thing. The main point is rather that observers at *different reference frames* will not agree regarding simultaneity of events.

Secondly, we have the formulation 'moving clock goes slower'. It is true that an observer on a reference frame (K^{MC}), observing a *specific clock* (on K^{SC}) passing by, will see this clock going slower when it is compared to his own clocks. So in a certain sense this confirms the standard phrase 'moving clock goes slower'. However, we could equally well take the perspective of the single clock, considering this to be at rest, implying that the clocks on K^{MC} are moving. The point is definitely not that clock(s) on K^{SC} are moving and clocks on K^{MC} are not. Rather, we could look at the symmetry of the situation: We are starting out with two clocks at origin, moving relative to each other. Then the decision on which of the two clocks we will compare with a clock on the other system, (a decision that can be interchanged at random!), will decide which of the two clocks comes out as the slow one.

So, first of all, none of the clocks are more moving than the other. Further, it is the observational principle that decides which of the two clocks initially at the origin, which we observe to move slower. Therefore, I find the talk about the 'moving clock' rather misleading.

Further, this choice on which reference frame shall apply just a single clock is obviously crucial, and it introduces an asymmetry between the two reference frames.

We note that the insight as provided by *eq.* (16) is in no way new. Our concepts SC and MC for instance correspond to the concepts 'proper' and 'improper' time used by (Smith 2015). In particular, *eq.* (16) equals *eq.* (3-1) of that book. However, this reference has perhaps not fully utilized the potential of this relation.

We should further stress one fact. It is not required to point at one reference frame to be SC (having 'proper' time), and the other to be MC (having 'improper' time). Actually, we can specify any clock at one of the two reference frames; and if we decide to follow this clock, we will find that it goes slower than the passing clocks on the other reference frame. Thus, we may at the same time have clock(s) on *both* reference frames observed to 'go slower'. In this respect also *eq.* (16) could be misleading. We do not *have* to point at one reference frame to be SC and one to be MC. The equations just says that if we follow a specific clock, we will observe that this goes slower than the passing clocks on the other reference frame.

This point is essential, and in my opinion it also gives an answer to Dingle's question, (see Dingle 1972; McCausland 2008), *cf.* Section 2.1. It is not the case that the clock(s) on *one* of the two reference frames go(es) slower than the clock(s) on the other, (as indicated by the Dingle's question). We could very well choose to follow *both* the two clocks being at the origin at time 0; which will give that both reference frames have a clock 'going slower'. (We note that this fact is also most relevant when we study the travelling twin example; see Chapter 6 below.) So, the result on time dilation is actually fully symmetric with respect to the two reference frames! The question is not which reference frame has a clock that 'goes slower'; it is rather which observational principle we have chosen. This fully demonstrates that it is rather inappropriate to apply the statement 'moving clock goes slower'.

Actually, Professor Dingle in his later work claimed that the TSR was inconsistent; see thorough discussion by McCausland (2011). However, I do not find this argumentation very convincing, as - according to McCausland (2011) - Dingle again seems to focus on the apparent inconsistency of our *eqs.* (8), (9), rather than discussing the interpretation of the more relevant *eq*. (16).

A final comment: There might be a drawback with the presentation given in Fig. 1 that it does not provide a symmetry between the two reference frames. Everything is seen in the perspective of one of them (*K*). However, we solve this in the next chapter by introducing an auxiliary reference frame.

5 Using an auxiliary reference frame of symmetry

In this Chapter we elaborate further on the special cases of *eqs.* (8) and (9), (or rather (16)). We still have two 'main' reference frames moving at relative speed, *v*. However, we now want to treat them symmetrically, and rather than referring to *K* and $K_{v'}$, we will denote them K_1 and K_2. In addition, we introduce an auxiliary reference frame, denoted *K,* and we will use this new *K* as our primary reference frame, and so make our observations 'in the perspective' of *K*..

In order to get the speed between K_1 and K_2 to equal *v* (in a symmetric way), we let K_1 have speed –w' with respect to *K,* and let K_2 have speed *w'* with respect to *K,* where *w'* is defined in (12). As explained in Section 3.4 this implies that the speed between K_1 and K_2 becomes *v.*

Now we can apply the relation (16) between the auxiliary reference frame, *K* and the two reference frames K_1 and K_2. However, we must also specify the observational principle.

First, we let the auxiliary reference frame operate as SC, and so both the secondary reference frames K_1 and K_2 are MC; that is we write (16) as

$$t_v^{MC}\sqrt{1-(v/c)^2} \;=\; t^{SC}, \quad \text{both for } v = -w' \text{ and } v = w' \tag{17}$$

In practice, this means that we always read the clocks of the three reference frames at the origin of the primary reference frame, *K*; (recall *eq.* (15) or consult Appendix A regarding notation). Thus, the MC references frames K_1 and K_2 at any moment use different clocks. The two relations in (17) now directly give

$$t_{-w'}^{MC} = t_{w'}^{MC} = t^{SC}/\sqrt{1-(w'/c)^2} \tag{18}$$

The important thing here is that $t_{-w'}^{MC} = t_{w'}^{MC}$, the clock readings of K_1 and K_2 at the origin of *K* are identical. This of course is just the result obtained in Section 3.4, and therefore is of limited interest.

However, if we now let the auxiliary reference frame operate as MC, and both K_1 and K_2 be SC, the result becomes more interesting. Then single clocks at the origins of K_1 and K_2 are at any time compared with various clocks along *K*. Then, instead of (17), (18) we get

$$t_v^{SC}/\sqrt{1-(v/c)^2} \;=\; t^{MC}, \quad \text{both for } v = -w' \text{ and } v = w' \tag{19}$$

implying

$$t_{-w'}^{SC} = t_{w'}^{SC} = t^{MC}\sqrt{1-(w'/c)^2} \tag{20}$$

Again, the most important result is that $t_{-w'}^{SC} = t_{w'}^{SC}$. These are now the 'simultaneous' clock readings at the origins of our two reference frames K_1 and K_2 (moving relative to each other at speed, *v*).

We illustrate all these results in Fig. 2, which provides an analogy to Fig. 1. While Fig. 1 presented time dilation between two reference frames, taking the perspective of one of them, Fig. 2 gives a symmetric picture with respect to two such reference frames. We do this by introducing a third, auxiliary reference frame, *K,* and take the perspective of this new one. So Fig. 2 gives a snapshot of the time measurements at an instant when all clocks on *K* read time *t*; see horizontal line marked *t*

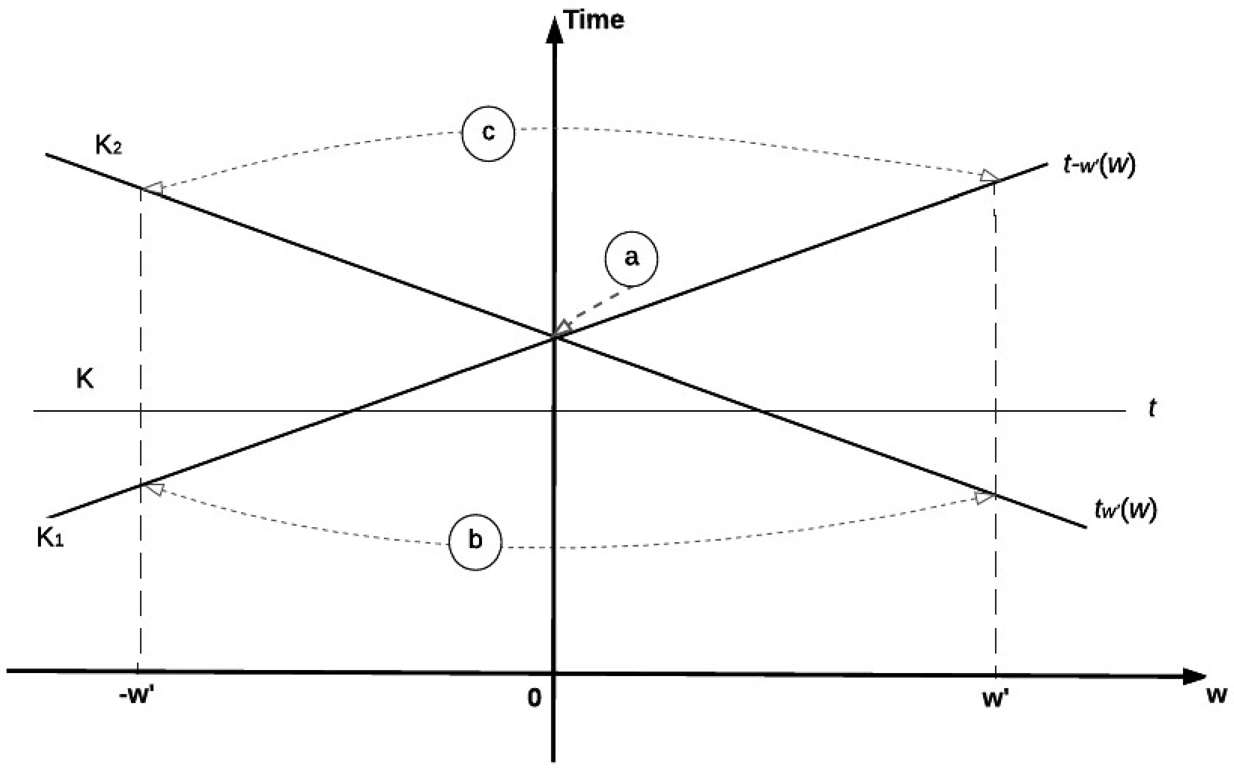

Figure 2. Time measurements in the perspective of the auxiliary reference frame, *K,* where the reference frames K_1 and K_2 have speed -*w'* and *w',* respectively, relative to *K.*

The parameter, *w* (horizontal axis) refers to the 'positions' (*w =x/t*) on the auxiliary reference frame, *K.* The reference frames, K_1 and K_2 move relative to *K* at speed –*w'* and *w'*, respectively. Further, the lines $t_{-w'}(w)$ and $t_{w'}(w)$ give the time measured on clocks at K_1 and K_2, respectively; as a function of *w* (and time *t*) on *K*. So we observe that Fig. 2 applies the 'generic' notation, $t_{w'}(w)$, rather than t^{SC} and t^{MC}, as each reference frame can be seen both as SC and MC, depending on application; (again we refer to *eqs*. (14), (15) and Appendix A regarding this notation).

We focus on three 'positions' on *K*, *i.e. w* equal to -*w',* 0 and *w'*, respectively. These three values correspond to the origins of the reference frames, K_1, *K* and K_2, respectively.

The letter *a* in the figure indicates the simultaneous clock readings of reference frames K_1 and K_2, observed at the origin of *K*, (see (18)). At this position the clocks on K_1 and K_2 show the same time, and are simultaneously located at the same location, *w*=0; so we are actually just referring to 'basic simultaneity'. For these measurements the reference frame *K* is a SC system, and its clock will appear slower than the corresponding clocks on K_1 and K_2: we observe the line *t* falling below the point *a*.

Next, the two points marked with *b* give identical time readings on K_1 at the position –*w'* and on K_2 at the position *w'*. So in this situation the SC time readings at the origins of K_1 and K_2, that is, $t^{SC}_{-w'} = t_{-w'}(-w')$ and $t^{SC}_{w'} = t_{w'}(w')$, are identical according to (20). These origins have moved apart after time 0; so the events of these two clock readings are not simultaneous, neither in the perspective of K_1 nor in that of K_2.

However, *eq.* (20) tells that *in the perspective of the auxiliary reference frame* we have two simultaneous events. Now simultaneity in the perspective of the auxiliary reference frame is a quite weak form of simultaneity. However, when we also have this complete symmetry of the positions of the observations, the result becomes interesting, (and not very surprising). Actually, we will postulate that this specific 'simultaneity at a distance (or 'simultaneity by symmetry') represents a valid form of simultaneity. In view of the required symmetry, this actually seems a rather week assumption; we could even consider it to be a consequence of this overall symmetry.

In conclusion, this seems to be the most significant result obtained by using the auxiliary reference frame: We manage to establish a (kind of) simultaneity of events at K_1 and K_2 'at a distance' (*when they no longer are at the same position*). We claim that it is a key question in a proper handling of time dilation to achieve this. See further discussions in the next chapter.

Finally, in Fig. 2 we also see clock readings corresponding to the letter *c*. These exhibit the same type of symmetry/simultaneity as the points *b*, and the only difference is that the time readings at *c* will *not* correspond to the origins of the two main reference frames. A numerical example related to Fig. 2 is given in Appendix C, utilizing all points *a, b* and *c*.

The observant reader might realize that the experimental set-up given here is well suited for handling the travelling twin paradox; which we discuss in the next chapter.

6 The travelling twin

We now utilize the framework of the previous chapters to analyse the so-called travelling twin example, which goes back to Langevin (1911). As stated for instance in Mermin (2005) the travelling twin paradox shall illustrate that two identical clocks, initially in the same place and reading the same time, can end up with different readings if they move apart from each other and then back together.

6.1 The numerical example

Mermin (2005, Chapter 10) gives the following numerical example: "If one twin goes to a star 3 light years away in a super rocket that travels at 3/5 the speed of light, the journeys out and back each takes 5 years in the frame of the earth. But since the slowing-down factor is $\sqrt{1-(3/5)^2}=4/5$ the twin on the rocket will age only 4 years on the outward journey, and another 4 years on the return journey. When she gets back home, she will be 2 years younger than her stay-at-home sister, who has aged the full 10 years." So the claim is that the referred difference in ageing occurs during the periods of the journey with a constant speed; *i.e.* under the conditions of the TSR, (ignoring the acceleration/deceleration periods). After all the whole argument relies on the Lorentz transformation! Thus, our discussion will restrict to the periods of constant velocity. Now throughout this chapter we let

t = time on the clock of the earthbound twin

t_v = time on the clock of the travelling twin

The distance between earth and the 'star' equals x = 3 light years, and the rocket has speed, $v = (3/5)c$, giving $\sqrt{1-(v/c)^2}=4/5$. It follows that in the reference frame of the earth/star, the rocket reaches the star at time, $t = x/v$ = 5 years. And the clock on the rocket is then located on $x_v = 0$, corresponding to $x = vt$, and thus the Lorentz transformation gives that at the arrival at the star this clock reads $t_v = t\cdot\sqrt{1-(v/c)^2}=4$ years; so obviously, t_v/t =0.8 at the star (travelling twin); and the argument is also valid for the return travel.

This is a rather convincing argument. It does follow from the Lorentz transformation that the returning clock shows 8 years when he/she returns. However, recalling the discussion of Chapters 3-5 the case is perhaps not that straightforward, and since we have made no assumption of asymmetry regarding the periods of constant velocity, we seem to have a true paradox.

Thus, we will not question the clock of the travelling twin, but take a new look at the clock of the earthbound twin, trying to look at the total situation. First, we observe - following the notation of Chapter 4 - that the above presentation describes the travelling twin as a 'SC system', and thus the earthbound twin is located on a 'MC system'. So, actually we could just look at *eq.* (16) to obtain the above result. It is well known (*cf.* Appendix B) that this is related to the length contraction: Seen from the perspective of the travelling twin, the distance between earth and the star does not equal $x = 3$ light years but just $x_v = 3\cdot 0.8 = 2.4$ light years; fully 'explaining' the reduction in travelling time.

Now we have the question (which is frequently asked): Could we not similarly describe the situation as the travelling twin being located on a MC system, and the earthbound twin on a SC system (which would then have one clock located on the earth). If we insist on the symmetry of the situation, the answer must be yes. Thus, we simply assume that there is also a reference frame of the travelling twin with the required number of clocks. Say, he is equipped with rockets at appropriate and fixed distances from his own rocket, all moving with constant speed in the same direction as himself, and all equipped with a synchronized clock showing the same time, t_v. Whether this is practically feasible is not relevant here. We are referring to the model of the TSR, and point out what this theory tells about clock readings, *if* we provide such an arrangement.

By making this assumption, we could consider the earthbound twin as travelling back and forth along the reference frame of the travelling twin. This will now give that the one way travelling time of the earthbound twin is 4 years; while arguing as above the time passed for the travelling twin equals 5 years. (Actually, we have here described the essence of the Dingle's question.) Due to this symmetry of results, we find it required to proceed with the discussion. Now we provide both a lengthy and a short argument. We first take the lengthy. But those fully familiar with the discussion of Chapter 5 might go directly to Section 6.3.

6.2 The lengthy argument

We include this section to contemplate on the described phenomenon with an open mind, without directly utilizing the simultaneity result of the previous chapter. We first follow up on the possibility of treating the earthbound twin as a SC system. Thus, he just applies his clock at the earth for time comparisons with the various travelling clocks; (therefore the reference frame of the travelling twin is equipped with several clocks at various locations). So now $t_v = t^{SC}$ and $t = t^{MC}$, and *eq.* (16) gives the result, $t_v / t = 1/0.8 = 1.25$ (for the observations on the earth) whatever instant we consider after departure. So, now of course it is the clock of the earthbound twin that always 'goes slower'.

We want the earthbound twin to make an observation of the clocks exactly 'at the instant' when his traveling twin arrives at the star. However, the core of the problem is that we actually do not fully know how to define this moment on the earth. The Lorentz transformation does not seem to give a definite answer regarding the simultaneity of events 'at a distance'. So now let us consider various options regarding the moment at which he should choose for observing the clocks; (both his own clock positioned on the earth, and the one passing by belonging to the reference frame of his travelling twin):

1. *Perspective of the travelling twin*. At the moment when the travelling twin arrives to the star, the clock on his rocket shows 4 years. Then all clocks on the reference frame of the travelling twin show time, $t_v = 4$ years. So this is also the case for the clock which as this moment is passing the earth, *i.e.* at $x = 0$. Thus, the clock of the earthbound twin at this instant shows time $t = t_v\cdot 0.8 = 3.2$ years.
2. *Perspective of the earth/star system*. At the instant when the twin arrives at the star, the time of the earthbound system at this location equals $t = 5$ years. (The earthbound twin could verify this by also installing a clock at the 'star'.) When he performs a clock comparison at the earth (at $x = 0$) at

this moment, it gives that $t_v = t{\cdot}1.25 = 6.25$ years for the travelling clock which passes the earth at this moment.

3. *Symmetric solution,* (*Perspective of the auxiliary system, cf.* Ch. 5). The above two cases demonstrate that the two twins completely disagree about which event at the earth is simultaneous with the travelling twin's arrival at the star; (which of course is obvious also from *e.g.* Figs. 1 and 2). But now consider the moment when the clock on the earth shows $t = 4$ years and the passing 'travelling clock' shows $t_v = t{\cdot}1.25 = 5$ years. This instant obviously occurs in between the previous two moments, and represents a moment being completely symmetric to the event of the twin's arrival at the star (regarding clock readings). And more important: It is the instant when the earthbound twin have carried out a 'travel' equivalent to the distance of the travelling twin. There is a complete symmetry! (From Ch. 5 we have that this corresponds to the perspective of the auxiliary reference frame; see Fig. 2).

In summary, when the earthbound twin represents a 'SC system', and thus, carry out the clock comparison at the earth, we always get $t_v/t = 1.25$; *i.e.* it is the clock on the earth that 'goes slower'. We summarize the findings in Table 1. The three 'perspectives' in some way all 'correspond to' the arrival of the travelling twin at the star, and thus, demonstrate the problem we have to define the 'simultaneous' event on the earth.

So how should we conclude regarding the time (clock readings) at the turning of the rocket? When now the information of Table 1 is available, let us assume that the earthbound twin is in charge. He could control his twin's travel by sending a light signal to the star, which on arrival initiates the return of his travelling twin. How should he do this to be sure the signal arrives at the right moment? One possibility that he might consider is to send a signal that arrives at the star when his earthbound clock shows 5 years (*i.e.* perspective 2 in Table 1). The problem is that at this moment the clock on the travelling twin system passing the earth shows 6.25 years. Thus, one could suspect the travelling twin when he returns have aged 12.5 years (and not 8). The reason being that if he turns when his twin's signal reaches the rocket, he may have travelled a longer distance than the intended 3 light years. A similar objection applies to using perspective 1.

Actually, if the earthbound twin should be in charge, I guess the following strategy should be the most ingenious. Knowing about the length contraction, he will know that the travelling twin will observe a travelling distance to the star that equals just 2.4 light years. So the earthbound twin will adopt option 3: he sends a signal ordering to turn, such that the travelling twin will receive this signal when the clock on his own earthbound system shows 4 years, (*cf.* 'perspective 3' of Table 1). In my view the only logical and consistent result is that this is the signal that reaches the travelling twin at the moment when he arrives at the star.

Following this option, we conclude that at the local time when each of the twins now consider to be the turning of the rocket, the twins will agree on the following facts: Their own clock shows 4 years, and the adjacent clock on the other system shows 5 years. So by the direct measurements, they observe that the other twin at this moment apparently has aged more than himself by a factor 1.25. This gives a completely symmetric and consistent answer to the paradox.

Table 1. Various clock readings (light years) at/on the earth, *potentially* 'simultaneous to' the arrival of the travelling twin at the star; (at that location we have $t_v = 4$, t=5).

Location of time reading	'Perspective'		
	1.Travelling twin	2.Earthbound twin	3.Symmetric
Travelling twin system (***t***$_v$)	4	6.25	5
Earth/star system (***t***)	3.2	5	4

Following this argument, the clocks of both twins show 4 years at the point of return. The same argument applies for the return travel, and we should conclude that by the reunion *both* clocks show 8 years.

Another way to put it. We can choose between three options, (all in apparent agreement with the Lorentz transformation):

1. The travelling twin being on a SC system giving travelling times 8 years for him and 10 for the earthbound, *eq.* (16);
2. The earthbound twin is located on a SC system, giving 8 years for him and 10 for the travelling *eq* (16);
3. The symmetric solution, *eq.* (20), treating *both* systems as SC, giving a total duration of 8 years for both twins.

To me this choice is easy. In addition to the symmetry, it is an important point that options 1 and 2 just follow the clock of *one* twin, while we in option 3 follow *both* clocks.

6.3 The short argument and further comments

Another rather direct argument for our answer to the twin paradox goes as follows. By the start of the travel the two clocks are at the same location, and they are then synchronized. Afterwards they move relative to each other at constant speed, and for symmetry reasons we should then conclude that when the two clocks show the same time, then we have 'simultaneous events'. In particular – as we have strongly argued – the event that the clock of the travelling shows 4 years (and he is going to return) is 'simultaneous with the event that the clock of the earthbound twin shows 4 years. So by return they have both aged 8 years.

Finally, our argument could apply an auxiliary reference frame as introduced in Chapter 5. We directly apply the simultaneity in the perspective of the auxiliary system; that is 'simultaneity at a distance', *cf.* Fig. 2. According to this result, the event (t_v = 4 years, t =5 years) at the star is simultaneous in this sense to the event (t = 4 years, t_v = 5 years) on the earth. So by following both twins, and thus considering *both* systems as SC, the conclusion is that both twins have aged 4+4=8 years when they meet again. We follow both clocks in a symmetric way, from the moment when they depart (having basic simultaneity) to the moment when they are again united (again basic simultaneity). The time is reduced, compared to the result of 10 years, since SC observers experiencing a length contraction, *cf.* Appendix B. We also refer to Appendix C for a numerical calculation related to Figure 2, based on the above example.

This result requires that the return travels for both twins have the same duration as their outward travel. However, since they are traveling the same distance, at exactly the same speed, giving the same length contraction, their clocks should measure the same duration for the outwards and return travels.

One might still raise the question: since the simultaneity by the various perspectives obviously differ so wildly, why should we trust the perspective of the auxiliary reference frame introduced here, and not the result of any of the other reference frames that we could come up with? Our answer is simply that this is the only perspective exhibiting the desired symmetry between the two 'main' reference frames. Can we come up with an asymmetry here, requiring a certain asymmetry; we would choose an auxiliary system to accommodate for this. So far I have seen no such asymmetry. Actually, when we – as in the present work - focus on the full symmetry of the situation, it would be rather meaningless to claim that one ages slower than the other.

There are some authors searching for asymmetry to explain a difference in ageing. For instance Feynman (2008), Smith (2015) argue that the acceleration required for the travelling twin when he turns, is the reason why we can distinguish between the two twins, thus, introducing an asymmetry, and therefore infer that one of them ages slower. Others introduce the concept of polar and equatorial clocks see references and discussion in McCausland (2011, Appendix II). I find these attempts rather *ad hoc* and not very convincing.

After all, the claimed magnitude of the difference in ageing is calculated using the Lorentz transformation, which actually exhibits symmetry. It seems that some try to defend an asymmetric solution to a symmetric mathematical framework by adding some apparently irrelevant assumptions outside the scope of the mathematical framework. Actually, it is my claim that - when treated correctly - the Lorentz transformation does give a symmetric result.

So what is wrong with the standard argument of the twins' ageing. It is obviously about simultaneity. The standard narrative seems implicitly to assume that the arrival of the twin at the star occurs 'simultaneously' with the earthbound twin having aged 5 years. I disagree with this claim. The Lorentz transformation tells that the clock of the earthbound system, *which is located at the 'star'*, shows 5 years by the arrival of the traveling twin. However, that does not imply that we can say that the earthbound twin has aged 5 years 'at the same time'. In my understanding, one cannot infer such simultaneity from the TSR.

A similar objection applies to the argumentation given in Smith (2015, Chapter 6), which also claims that the returning twin has actually aged less. However, the various arguments here starts out with statements like, 'He [the earthbound twin] thinks the whole trip took T seconds', (where T corresponds to 5 years in our case), 'the earth twin knows the outbound trip took $T/2$ seconds', 'the whole trip takes a time T'. So it seems he starts out by making an assumption that must lead to the desired conclusion, and we have a circle argument. A crucial question is the totality of information in principle available for the earthbound twin, as discussed in the previous section; *cf.* Table 1. Then we should not take for granted 'what he thinks', and he will not necessarily 'know' that the whole trip takes 10 years. As many authors discussing this he seems to lack a more holistic view, considering the various, apparently contradictory observations we might have. As we have seen, clock readings (time) depend on location, and simultaneity depends on the chosen perspective (reference frame). So we have to be careful, and the argumentation of Smith (2015) seems too simplistic.

Now one could say that also the solution presented here is somewhat paradoxical as it involves apparently contradictory observations for events 'at a distance'. However, this paradox is seemingly inherent in the Lorentz transformation. By accepting this as a model for how the world is 'working', I find the solution presented here consistent and logical.

7 Summary and conclusions

There still seems to be some open questions regarding the interpretation of time dilation between two reference frames moving at constant speed v relative to each other. We start out from the Lorentz transformation, and in order to have a precise basis for the discussions, make the following specifications:

- There is a complete *symmetry* between the two reference frames.
- Each reference frame is equipped with a number of synchronized clocks, (at any required position).
- We may take the perspective of one reference frame and specify this to be the *primary* one; (being 'at rest', and having the same clock reading at any location).
- *Basic simultaneity* of events at different reference frames will refer to events occurring simultaneously at the same location, ('simultaneity by location').
- We introduce an *auxiliary reference frame,* and in combination with symmetry considerations, use this to define 'simultaneity at a distance', or equivalently 'simultaneity by symmetry'.
- An investigation of time dilation includes a specification of the *observational principle;* that is, we specify the location of the clocks that are used for time comparisons between the reference frames.
- In particular a reference frame may use 'single clock' (SC) observations; *i.e.* the same clock is used for time comparisons with the other reference frame. Similarly, in 'multiple clock' (MC) observations we apply several clocks along the x- axis of the reference frame. In situations that are more complex (*cf.* travelling twin), we may consider both types of observations on both frames.

- We do not use the expression '*as seen'* (from the other reference frame). Observers (observational equipment) on both reference frames agree on the time readings; as they are carried out 'on location'.
- We do not describe time dilation by the expression 'moving clock goes slower'. It is the observational principle that matters.

The Lorentz transformation gives the 'standard' time dilation expression as a special case of, but we choose to write this as

$$t^{SC} = t^{MC}\sqrt{1-(v/c)^2}$$

This is a mutual relation between the two reference frames, and tells that when we follow a single clock (SC) on one of the reference frames, this clock goes slower than the passing clocks (MC) on the other frame by a factor $\sqrt{1-(v/c)^2}$. This formulation is more informative than the potentially misleading phrase 'moving clock goes slower'. There is no reason to see the single clock to be moving and the other clocks not. In general the specification of which system(s) act as SC and which act as MC is crucial for how to comprehend this phenomenon.

We get another special case by permanently performing clock comparisons at the midpoint between the origins of the two reference frames; which gives identical clock readings. This observational principle is symmetric with respect to the two reference frames. So when we otherwise get different clock readings (in a symmetric situation), this is caused by an asymmetry in the chosen observational principle.

Our main statement is that the Lorentz transformation indeed has a symmetric solution. This seems contrary to the views of many authors, and implicitly assumed in the so-called Dingle's question (asking 'which of the two clocks goes more slowly'). Actually, *both* reference frames will observe that a single clock on the 'other system*'* goes slower. Thus, a search for the one reference frame where time goes slower is indeed in vain.

We apply the given framework to analyse again the travelling twin paradox; (disregarding the acceleration periods), and present a symmetric experimental set-up, where we follow the single clocks of *both* twins. Then we arrive at the solution that the age of *both* twins is reduced by a factor $\sqrt{1-(v/c)^2}$, as compared to the travelling time calculated for a stationary reference frame. The case is that from any SC perspective one will observe a length contraction on the 'other system', and therefore also a time reduction. Thus, any single clock 'goes slower' when compared to the passing clocks on the 'other system'. However, there is a symmetry! By specifying a reference frame for each twin, we see that they will both observe this phenomenon.

Actually, under the conditions of complete symmetry it would be rather meaningless to claim a true time dilation, causing different ageing at the two reference frames. Therefore, it would be required to identify conditions (departures from symmetry) that could cause time dilation to represent a physical reality. I cannot see a proper identification of such conditions in the travelling twin example. In order to observe a difference in ageing, we probably have to introduce *e.g.* gravitation/acceleration effects, and cannot argue by the STR.

The problem with standard arguments on the travelling twin paradox seems to be a failure to handle properly the 'simultaneity at a distance'. If we want to specify a symmetric situation, also the definition of simultaneity (at a distance) should be symmetric. If the situation is not to be symmetric, the modelling should reflect this, without introducing some ad hoc assumptions (to arrive at an asymmetric solution). The present paper insists on symmetry, and we introduce a special auxiliary reference frame to obtain simultaneity at a distance (in a symmetric way).

Such symmetry considerations should generally be valid, when we deal with phenomena described by the TSR. Thus, it should affect our understanding of time in the context of TSR.

Another comment regarding our findings: An observer moving relative to a reference frame where an event takes place could be a rather unreliable observer. Various observational principles will provide him with different results. Thus, one should be careful to let such an 'outside' observer define the phenomenon, without taking an overall view and properly consider his own position.

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Appendix A Some notation

We apply a diverse notation regarding time. This Appendix provides a summary.

1. Time on the primary reference frame; also used when we do not want to specify whether we refer to a primary or a secondary system:

t Generic time; being constant all over the reference frame.

t^{SC} Clock reading at the origin (x=0) of this reference frame; thus, applying only a single clock.

t^{MC} Clock reading at the position corresponding to the origin of the 'other' (secondary) reference frame; thus, applying several clocks.

2. Time on the secondary reference frame, moving relative to the primary at speed, v:

t_v Generic time on this system; without specifying the position

$t_v(w)$ Time at the position which corresponds to $w = x/t$, (where x is position *on the primary system*).

$t_v^{SC} = t_v(v)$ Clock reading at the origin of this reference frame; thus, applying only a single clock.

$t_v^{MC} = t_v(0)$ Clock reading at the position corresponding to the origin of the 'other' (primary) reference frame; thus, applying several clocks.

Appendix B A note on length contraction

The interpretation of x and x_v in (1) and (2) is rather straightforward. The position x_v on K_v corresponds exactly to the position x on K at an instant where the clock located at x_v shows time t_v and the clock at x shows time t.

This means that x and x_v could have the following interpretation. We again consider a 'SC system', K_v moving at relative speed along a system, K. Now let a distance, x be marked out on K in the same direction as this movement. As known, the time measured on K_v for its single clock to pass this distance will imply that – as measured from K_v – the length of the distance x equals

$$x_v^{SC} = x\sqrt{1-(\frac{v}{c})^2} \tag{B.1}$$

So now x_v^{SC} equals the length of x on K 'as seen from' K_v. (In order to utilize the clock reading to observe x_v^{SC} one first have to establish the relative speed, v, between the reference frames.) Therefore, this length contraction (B.1) corresponds exactly to the time dilation observed for a single clock moving relative to a fixed distance on the other reference frame. Thus, anyone on K_v observes the distance travelled to be shorter, and so the time required to travel this distance will be observed to be shorter (both on K and K_v). Therefore, the length contraction and time dilation are indeed two aspects of the same phenomenon.

Appendix C Example of numerical calculations

Now to familiarize a little further with the argument regarding the 'simultaneity at a distance' (in the perspective of the auxiliary reference frame) we elaborate on the numerical example in Section 6. As an illustration, we use Fig. 2, and consider this to represent the situation when the travelling twin has reached his point of destination. From the above numerical values, we will from (12) get $w' = c/3$ (= $5v/9$), being the speed between the auxiliary system and a twin. Further, we get that the 'slowing down factor' related to this speed equals $\sqrt{1-(w'/c)^2} = \frac{2}{3}\sqrt{2} \approx 0.94$. Using this, the time readings of the three reference frames are:

i. The auxiliary reference frame (primary). Time is constant, t, (see horizontal line in Fig. 2).
ii. Travelling twin. Time is a function of w: $t_{-w'}(w) = (\sqrt{2}/4) \cdot (3 - w/c)t$.
iii. Earthbound twin. Time is a function of w: $t_{w'}(w) = (\sqrt{2}/4) \cdot (3 + w/c)t$.

In Fig. 2 we now let the observational point b have value 4 years, (*i.e.* the clock readings of the travelling twin at his arrival), corresponding to $w = w' = c/3$. This gives $t = 3\sqrt{2} \approx 4.24$ years. Inserting this above, we in Fig.2 also get that point a corresponds to 4.5 years and point c corresponds to 5 years. So all values are in full agreement with those used in the example.

The time $t = 3\sqrt{2} \approx 4.24$ years of the auxiliary system seems less relevant. The main role of the auxiliary reference frame is to allow us to treat the reference frames of *both* twins as SC, and further to establish 'simultaneity at a distance' with respect to these.