About Physical Inadequacy of the Three-Dimensional Navier-Stokes Equation for Viscous Incompressible Fluid.

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ABSTRACT. This paper deals with the analysis of physically possible constructions of a viscous incompressible fluid model. Physical principles that allow to create the only possible construction of this model were found. The new model does not use new constants that characterize properties of the fluid and coincides with the Stokes model only in the plane case. Within the framework of this model, new equations for fluid motion were obtained. The new equations coincide with Navier-Stokes system in the plane case, but do not coincide in the three-dimensional one. The model makes it possible to see why the three-dimensional Navier-Stokes equations cannot physically adequately describe fluids motion, and obliquely confirms the finite time for the existence of its regular solutions.

In paper [1] it was shown that the equation similar in several key properties to three-dimensional Navier-Stokes equation for incompressible fluid, has a smooth solution which blows up in finite time. It was also shown that the problem of Navier-Stokes equation in the three-dimensional case cannot be solved by analysis methods existing today.

Almost a mystique situation: the Navier-Stokes equation must describe real fluids; which behavior has a certain set of properties. These properties should be visible during the analysis of the equation but this does not happen. More precisely, it happens only for the plane case of fluid motion, but not for the three-dimensional one [2], [3]. Suspicion occurs that Navier-Stokes equation have some defect. And this defect has physical nature and arises only in the three-dimensional case.

Three physical laws are the basis of Navier-Stokes equation for viscous incompressible fluid. Two of them are fundamental: mass and impulse conservation laws. Third law is empirical – the Newton’s viscous friction law. If Navier-Stokes equation has a defect, it can only originate from the viscous friction law. The generalized viscous friction law determines the dependence between the stresses tensor

$$\Pi = \begin{Vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{Vmatrix}$$

and the deformation rates tensor

$$\Phi = \begin{Vmatrix} \varepsilon_x & \frac{1}{2}\theta_{xy} & \frac{1}{2}\theta_{xz} \\ \frac{1}{2}\theta_{yx} & \varepsilon_y & \frac{1}{2}\theta_{yz} \\ \frac{1}{2}\theta_{zx} & \frac{1}{2}\theta_{zy} & \varepsilon_z \end{Vmatrix} \tag{1}$$

Generally, the question of mathematical dependence of tensors Π and Φ was investigated, for example in [4], where based on Stokes postulates the following formula was received.

$$\Pi = \alpha I + \beta\Phi + \gamma\Phi^2 \tag{2}$$

Here I - is a unit tensor, α , β , γ - are scalar functions of principal invariants I_1, I_2 , I_3 of deformation rates tensor Φ. Functions α , β , γ can be chosen in an arbitrary way, the result is a fluid with a specific set of viscous properties. The assumption of linear dependence between tensors Π and Φ was accepted for real Newton's fluids. For incompressible viscous fluids, this dependence was formulated as:

$$\Pi = -pI + 2\mu\Phi \tag{3}$$

This is Stokes model of viscous incompressible fluid. The author in [4] gave the following assessment of this assumption: «…The fact that this is a hypothesis should be clearly understood: it is not to be derived from experiments, nor can it be proved by abstract reasoning; if results obtained on the basis of this hypothesis agree with experiments, then of course so much the better for the hypothesis and our faith in its validity».

Following formulas follows from (3)

$$\begin{aligned} \sigma_x &= -p + 2\mu\varepsilon_x; & \tau_{xy} &= \tau_{yx} = \mu\theta_{xy} \\ \sigma_y &= -p + 2\mu\varepsilon_y; & \tau_{xz} &= \tau_{zx} = \mu\theta_{xz} \\ \sigma_z &= -p + 2\mu\varepsilon_z; & \tau_{yz} &= \tau_{zy} = \mu\theta_{yz} \end{aligned} \tag{4}$$

Using formulas (4), mass and energy conservation laws, one can define the function of energy dissipation D

$$D = \mu\left(2\varepsilon_x^2 + 2\varepsilon_y^2 + 2\varepsilon_z^2 + \theta_{xy}^2 + \theta_{xz}^2 + \theta_{yz}^2\right) \tag{5}$$

Function D is an invariant of deformation rates tensor Φ, as it is stated through its principal invariants.

$$D = 2I_1^2 + 4I_2$$

Going back to formula (3) one can note that it is possible to prove physically, but physical principles included in this proof will require formula modification.

Two assumptions, not one, were made when formula (3) was developed. The first – about linear dependence between Π and Φ, it looks obvious. The second – was not obviously formulated, but it exists and is hidden in the speculations. So, from (2) based on the first assumption, the formula (3) was obtained, where coefficient **2** was set in from the beginning with no explanations why. But it has a strong reason, because only in this case shear stresses, described with formulas (4) will give value corresponding to the Newton's scheme within the viscous tension law.

$$\tau = \mu \frac{V_0}{h} = \mu \frac{\partial V_x}{\partial y}$$

But in this scheme the fluid motion is plane, while formulas (4) describe the three-dimensional motion. The question is: **how adequate is the assumption that shear stress in the plane–parallel fluid motion will not change with applying additional deformations in orthogonal planes?** This assumption is doubtful and should be checked. Then formula (3) should be stated as following:

$$\Pi = -pI + 2\mu\beta\Phi \tag{6}$$

Where β, according to (2) may be dimensionless scalar function of the principal invariants I_1, I_2, I_3 of the deformation rates tensor Φ. In case of plane motion, function β should take the value equal to one, formula (6) must correspond to the Newton's scheme. Formulas for stresses in this case will look like this:

$$\begin{aligned}
\sigma_x &= -p + 2\mu\beta\varepsilon_x; \qquad & \tau_{xy} &= \tau_{yx} = \mu\beta\theta_{xy} \\
\sigma_y &= -p + 2\mu\beta\varepsilon_y; \qquad & \tau_{xz} &= \tau_{zx} = \mu\beta\theta_{xz} \\
\sigma_z &= -p + 2\mu\beta\varepsilon_z; \qquad & \tau_{yz} &= \tau_{zy} = \mu\beta\theta_{yz}
\end{aligned} \tag{7}$$

And function of dissipation D will look like this:

$$D = \mu\beta\left(2\varepsilon_x^2 + 2\varepsilon_y^2 + 2\varepsilon_z^2 + \theta_{xy}^2 + \theta_{xz}^2 + \theta_{yz}^2\right) \tag{8}$$

The problem of dependence between tensors Π and Φ can be also looked from a different perspective, which will allow to disclose the following: in formula (6) the value β is a function of the principal values e_1, e_2, e_3 of the deformation rates tensor Φ.The value of function β in dependence of correlation between e_1, e_2, e_3 values is restricted in the closed interval $1 \le \beta \le 4/3$. For the plane motion, $\beta = 1$ and formulas for stresses (7) will coincide with (4). Since principal values e_1, e_2, e_3 may be expressed through the principal invariants I_1, I_2, I_3, value β can be considered as a function of principal invariants I_1, I_2, I_3 of tensor Φ, which is in full compliance with (2). From the other side, function β is fully defined by velocities field, therefore continuously depends on coordinates, i.e. $\beta = \beta(x, y, z)$. Important to mention that function β is an invariant of deformation rates tensor Φ.

It is also important to mention that the formula (6) is a new model of the exactly Newtonian fluids. This model has a linear dependence between the tensors Π and Φ, and completely corresponds to the Newton's scheme mentioned above. Later it will be shown that the Stokes model (3) leads to absurd physical consequences.

Let's analyze tensor Φ in principal axis, let's name these axis X, Y, Z, in this case it will look like this:

$$\Phi = \begin{Vmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{Vmatrix}$$

where $\varepsilon_x = e_1$, $\varepsilon_y = e_2$, $\varepsilon_z = e_3$, and e_1, e_2, e_3 are the principal values of the tensor. Let's choose from three deformation rates the one with the highest absolute value. Let it be ε_y for determinacy, define value

$$\varepsilon_m = |\varepsilon_y| \tag{9}$$

and then define the following values

$$\eta_x = \varepsilon_x / \varepsilon_m \qquad \eta_y = \varepsilon_y / \varepsilon_m \qquad \eta_z = \varepsilon_z / \varepsilon_m \tag{10}$$

Value η_y will be equal to 1 or -1, let's consider that always $\eta_y = -1$ (this will not affect the final results), so the compression always occurs along axis Y. Then the tension will always occur along axis X and Z, values η_x and η_z will be always positive. In addition to this, due to incompressibility of fluid if we define $\eta_z = \eta$, then $\eta_x = 1 - \eta$, we get the following result

$$\eta_x = 1 - \eta, \qquad \eta_y = -1, \qquad \eta_z = \eta \tag{11}$$

From this it is clear, that by varying one parameter $0 \le \eta \le 0.5$ it is possible to analyze the full spectrum of possible correlations between values ε_x, ε_y, ε_z. Further it is convenient to add parameter ξ - a measure of deformation condition triaxiality, defining it as following

$$\xi = 2\eta \tag{12}$$

Value $\xi = 0$ will correspond to plane fluid motion (no triaxiality) and value $\xi = 1$ will correspond to the highest triaxiality level; $\eta_x = 0.5$, $\eta_y = -1$, $\eta_z = 0.5$.

According to formulas (8), (9), (10), (11), (12) let's define the dissipation D

$$D = 2\mu\beta\varepsilon_m^2\left((1-\eta)^2 + (-1)^2 + \eta^2\right) = \mu\beta\varepsilon_m^2(4 - 2\xi + \xi^2) \tag{13}$$

Using formulas (7), (10), (11), (12) let's calculate the stresses

$$\sigma_x = -p + \mu\beta\varepsilon_m(2-\xi); \qquad \tau_{xy} = 0$$

$$\sigma_y = -p - 2\mu\beta\varepsilon_m; \qquad \tau_{xz} = 0$$

$$\sigma_z = -p + \mu\beta\varepsilon_m\xi; \qquad \tau_{yz} = 0$$

Among three stresses σ_x, σ_y, σ_z, let's define the highest $\sigma_{\max}$ and the lowest $\sigma_{\min}$, considering the sign. One can straight away see that $\sigma_{\max} = \sigma_x$ and $\sigma_{\min} = \sigma_y$. Let's analyze the value

$$\sigma(\xi) = \sigma_{\max} - \sigma_{\min} = \sigma_x - \sigma_y = \mu\beta\varepsilon_m(4-\xi) \tag{14}$$

By physical meaning this is the measure of viscous forces which occur in the fluid as a reaction to its deformation, caused by deformation rate ε_m with triaxiality parameter ξ. We will speak about value $\sigma(\xi)$ in the plural, the stresses, because this value characterizes the total viscous stresses level. It is important to mention that value σ is an invariant of tensor Φ, because it is definitely defined by principal values e_1, e_2, e_3, it follows from the procedure of its obtaining.

Let's define $\beta = 1$ in formulas (13) and (14), hence the Stokes model will be analyzed. Let's look at the dependencies $D(\xi)$ (see Figure 1) and $\sigma(\xi)$ (see Figure 2).

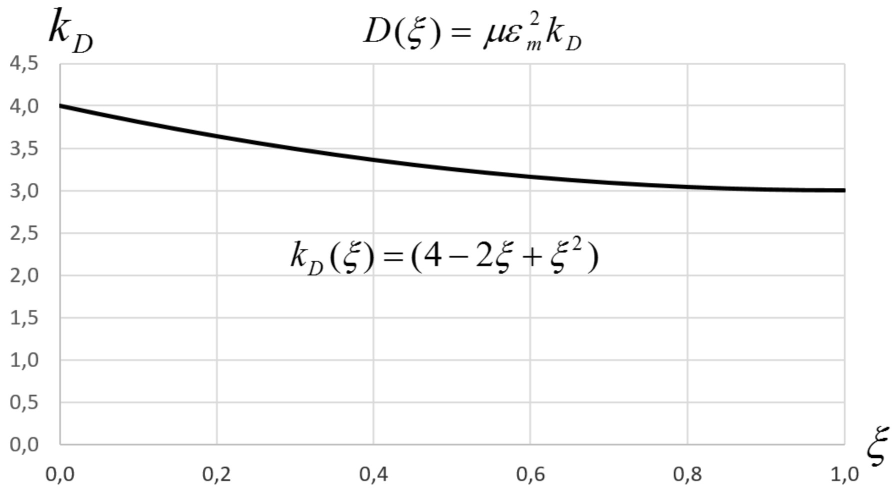

Figure 1. Dependence $D(\xi)$ for the Stokes model.

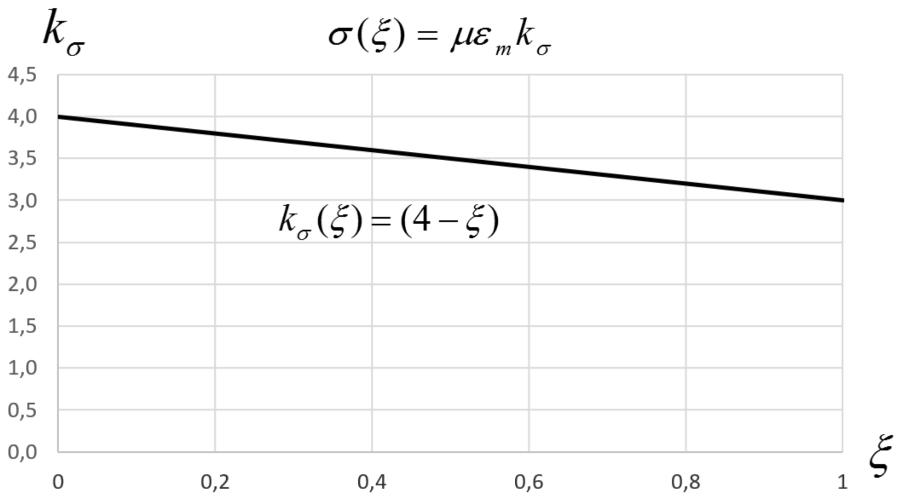

Figure 2. Dependence $\sigma(\xi)$ for the Stokes model.

Figure 1 shows that dissipation D is a monotonically decreasing function of ξ and Figure 2 shows that the same type of dependence exists in stresses. From physical point of view similar type of dependence is a classic example of "energy hole", and figuratively speaking "the fluid will be always falling into this hole".

Real Newtonian fluids cannot have dissipation function $D(\xi)$ of this type!

So, there is a variant of fluid deformation $\xi = 1$ with the lowest dissipation energy and with the lowest stresses level for any value ε_m. Energetically, this is the most beneficial case of fluid deformation. Because of that the fluid will tend to move the way that the value of its triaxiality ξ would differentiate from 1 in the least possible way in every point of space. In case the area of fluid motion is not initially the three-dimensional fluid source (sink) where in all area $\xi = 1$, then generally it would be impossible to realize this type of fluid motion. Which type of motion will this contradiction lead to? It is obvious that any plane motion $\xi = 0$, will inevitably become a three-dimensional $\xi > 0$. However, the contradiction does not disappear in this case, since at each point of space $\xi \neq 1$. Probably the general picture of the motion can be described as follows: plane or simply laminar fluid motion will be possible only during a short initial phase. As value ξ of these motions are close to zero, the pattern of fluid motion will be quickly destroyed with the course of time. Any initial motion (except the source) will be quickly turbulized. Whereas the level of turbulence with the course of time will become massively smaller and the structure of fluid motion will be like a "quantum foam". Reynolds scheme, according to which the turbulence occurs when the Reynolds number reaches the critical value, does not fit to this picture.

This is contrary to experience!

The main conclusion is: the Stokes model of viscous incompressible fluid, cannot physically adequately describe fluid behavior in tree-dimensional case of motion.

It's clear that, dependence $D(\xi)$ for real Newtonian fluids cannot have minimums, otherwise turbulence will always inevitably occur. The only possible way to escape physically absurd results is to accept the hypothesis about dissipation independence from the triaxiality parameter ξ, so

$$D(\xi) = const \tag{15}$$

In this case, there are no energetically beneficial cases of fluid deformation conditions and dissipation is no longer a parameter controlling the character of fluid motion. Value of constant in formula (15) is known, as for plane fluid motion dissipation is known $D(0) = 4\mu\varepsilon_m^2$, then

$$D(\xi) = 4\mu\varepsilon_m^2 \tag{16}$$

Thus, arises a model of viscous incompressible fluid with constant dissipation. Now using formulas (13) and (16) one can define function β

$$\beta(\xi) = \frac{4}{4 - 2\xi + \xi^2} \tag{17}$$

The function $\beta(\xi)$ diagram is shown on the Figure 3.

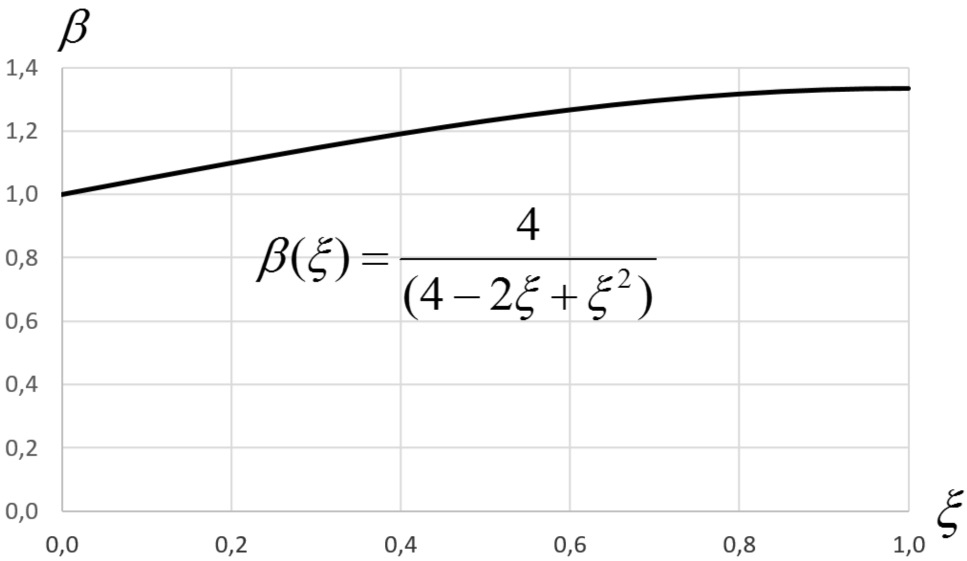

Figure 3. Function $\beta(\xi)$ diagram.

As previously mentioned if $D(\xi) = const$ the character of fluid motion cannot be controlled by dissipation, in this case the parameter, which controls the character of fluid motion, may only be viscous stresses. Using formulas (14), (17) let's define the stresses.

$$\sigma(\xi) = \mu\varepsilon_m \frac{4(4-\xi)}{(4 - 2\xi + \xi^2)}$$

The function $\sigma(\xi)$ diagram is shown on the Figure 4.

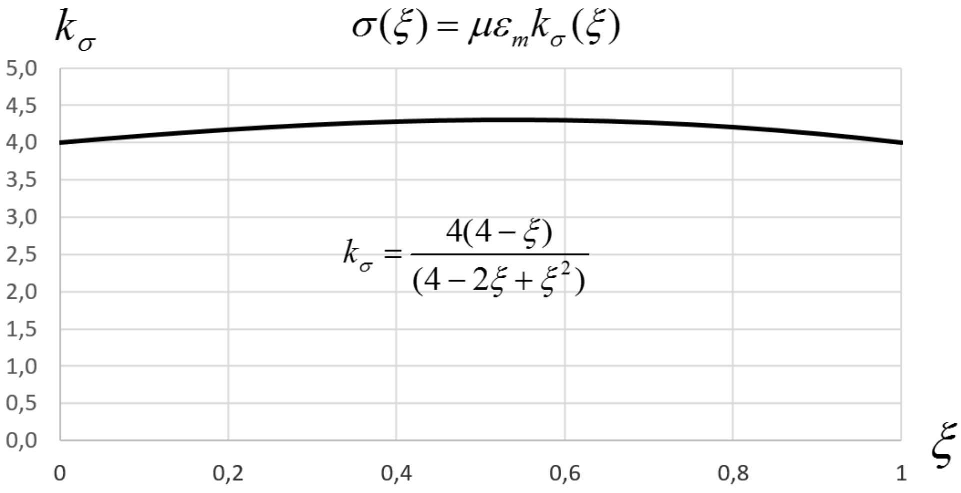

Figure 4. Dependence $\sigma(\xi)$ for the model of constant dissipation.

On the stresses diagram (see Figure 4) there is a maximum in $\xi = 0.5$. Maximum value of stresses exceeds minimum (with $\xi = 0$ and $\xi = 1$) by approximately 7,7%. This diagram allows to obliquely analyze viscous fluid behavior in this case.

Let's assume that there is a laminar fluid motion with value of ξ close to zero, in which a small local perturbation happened. Value ξ increases in the area of perturbation, which leads to viscous stresses increase according to the dependence shown on Figure 4. The fluid somehow resists and tries to calm down the perturbation. But this character of fluid motion will remain till value ξ is less than 0.5 (rising section of diagram). In the contrary case $\xi \geq 0.5$ (decreasing section of diagram) turbulence will occur. In this situation, everything will depend from correlation between viscous and inertia forces. If the inertia forces are less than the forces of viscosity, turbulence will fade, in the contrary case, turbulence will develop. A classical scheme of turbulence development occurs when reaching the critical Reynolds number. It is visible that the existence of increasing section in $\sigma(\xi)$ dependence allows to postpone the moment of turbulence development by minimizing initial perturbation level. It is also clear that fluid motion will be resistant to infinitely small perturbations as for turbulence development ξ must be more than 0.5. If turbulence occurs it will not be able to stop, as in turbulence area value of $\xi \to 1$ is in the decreasing section of dependence $\sigma(\xi)$ and the transition to the increasing section relates to overcoming the barrier with $\xi = 0.5$…

If in the derivation of Navier-Stokes equations instead of formulas for stresses (4) one uses (7), new equations different form Navier-Stokes will be received. Left parts of those equations will be fully corresponding to Navier-Stokes system, differences will be only in the right parts of equations, which are as following:

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\beta\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + 2\nu\frac{\partial \beta}{\partial x}\frac{\partial V_x}{\partial x} + \nu\frac{\partial \beta}{\partial y}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) + \nu\frac{\partial \beta}{\partial z}\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_y}{\partial x}\right)$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\beta\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right) + 2\nu\frac{\partial \beta}{\partial y}\frac{\partial V_y}{\partial y} + \nu\frac{\partial \beta}{\partial z}\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right) + \nu\frac{\partial \beta}{\partial x}\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\beta\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right) + 2\nu\frac{\partial \beta}{\partial z}\frac{\partial V_z}{\partial z} + \nu\frac{\partial \beta}{\partial x}\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right) + \nu\frac{\partial \beta}{\partial y}\left(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}\right)$$

It is important to mention that for the plane fluid motion those equations will be fully corresponding to Navier-Stokes system. One cannot but hope that strong mathematical research of this equations will show that they have properties which are described above for moving fluid, and which cannot be found in Navier-Stokes system.

And now we can highlight the following: in modified equations in components containing viscosity it is always present in combinations $\nu\beta$ and $\nu\partial\beta / \partial x_i$. If the fluid motion starts to be different from plane, values of functions β and $\partial\beta / \partial x_i$ start to increase and the fluid viscosity is somehow increasing. Moreover, if function β is limited at its maximum, the functions $\partial\beta / \partial x_i$ do not have such limits. As a rough approximation one can consider that modified equations are Navier-Stokes equations where in the event of triaxiality in plane fluid motion considerable increase of viscosity happens. Curiously fact is that a variant of modified Navier -Stokes equations with increasing viscosity was earlier proposed by Ladyzhenskaya O.A. [2]. In equations proposed by her, increase of viscosity happens in the areas with larger velocity gradients and regulated by a small constant ε (*additional characteristic of viscous fluid properties*). However, according to her own saying: "... equation system requires thorough full analysis from the physical phenomenon point of view which occurs in liquid media. Also, coefficient ε requires definition. But from mathematical point of view the system (at least by today) has the advantage that it has a proven unique solvability "in general" for initial-boundary value problems and range of other properties, which are assumed for Navier-Stokes equations, but cannot be proven."

In equations proposed by Ladyzhenskaya O.A., the effect of viscosity increase occurs also in the plane case, however, no problems occur with the plane motion even without this effect. This effect is required only for the three-dimensional equation system (dissipation increases) then it is possible to find all necessary properties which are searched for Navier-Stokes system for about one hundred years. One cannot help but remember the words of Confucius about difficulties of finding a black cat in a dark room, especially when it's not there…

REFERENCES

1. Terence Tao. Finite Time Blowup for an Averaged Three-Dimensional Navier-Stokes Equation. http://arxiv.org/pdf/1402.0290v2.pdf
2. Ladyzhenskaya O.A. Mathematical questions of viscous incompressible fluid dynamics. Moscow, 1970. (Ладыженская О. А. Математические вопросы динамики вязкой несжимаемой жидкости. – М.: Наука, 1970).
3. Ladyzhenskaya O.A. The sixth problem of the Millennium: Navier-Stokes Equations, Existence and Smoothness. Uspekhi Matematicheskikh Nauk Journal 2003. Vol.58, issue 2. (Ладыженская О. А. Шестая проблема тысячелетия: уравнения Навье-Стокса, существование и гладкость. Успехи математических наук. 2003. Т. 58, выпуск 2 (350), с. 45-78).
4. Serrin J. Mathematical Principles of Classical Fluid Mechanics. Berlin-Gottingen-Heidelberg, 1959.