Topics in space-time, gravity and cosmology

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Abstract

We derive the Poincaré model of the Lobachevsky geometry from the Fermat principle. The Lobachevsky geometry is in relativity interpreted as the Lobachevsky-Beltrami-Fok velocity space geometry of moving particles. The relation of this geometry to the decay of the neutral π-meson is considered. The generalization of the Lobachevsky geometry is performed and the new angle of parallelism is derived. Then, we determine nonlinear transformations between coordinate systems which are mutually in a constant symmetrical accelerated motion. The maximal acceleration limit follows from the kinematic origin. Maximal acceleration is an analogue of the maximal velocity in special relativity. We derive the dependence of mass, length, time, Doppler effect, on acceleration as an analogue phenomena in special theory of relativity. Next we apply the derived nonlinear Lorentz group to the so called Thomas precession. The total quantum energy loss of binary is caused by the production of gravitons emitted by the rotation motion of binary. We have calculated it in the framework of the Schwinger theory of gravity for the situation where the gravitational propagator involves radiative corrections. We also derive the finite-temperature gravitational Cherenkov radiation involving radiative corrections. The graviton action in vacuum is generalized for the medium with the constant gravitational index of refraction. From this generalized action the power spectral formula of the Cherenkov radiation of gravitons is derived in the framework of the Schwinger theory at zero and nonzero temperature. The next text deals with non-relativistic quantum energy shift of H-atom electrons due to

Gibbons-Hawking thermal bath. The seventh chapter deals with gravity as the deformation of the space time and it involves the light deflection by the screw dislocation. In conclusion, we consider the scientific and technological meaning and the perspectives of the results derived. Some parts of the complex are published in the reputable journals.

Key words: Geometry, the Fermat principle, light ray trajectories, optics, the Poincaré model of the Lobachevsky geometry, Beltrami model, Lobachevsky-Beltrami-Fok theory, generalized Lobachevsky geometry, optical black hole, mass shift, Thomas precession, Binary, gravitons, energy shift, deformation, dislocations.

PREFACE

The discovery of Lobachevsky had a great impact on the development of various parts of mathematics. The Lobachevsky discovery permeates throughout the Poincaré and Beltrami entire remarkable creation.

The Euclid geometry is based on the five postulates, which are in contemporary language in the following form.

1. Each pair of points can be joined by one and only one straight line segment.

2. Any straight line segment can be indefinitely extended in either direction.

3. There is exactly one circle of any given radius with any given center.

4. All right angles are congruent to one another.

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

The fifth postulate can easily be seen to be equivalent to the following parallel postulate. "Given a line and a point not on it, there is exactly one line going through the given point that is parallel to the given line".

For two thousand years mathematicians attempted to deduce the fifth postulate of the Euclid geometry from the four simpler postulates. In each case one reduced the proof of the fifth postulate to the conjunction of the first four postulates forming in a such way the geometrical theorem and not axion.

They tried to prove fifth postulate by reductio ad absurdum taking the negation of fifth postulate as a hypothesis and to get a contradiction with the rest of the axioms. However the desired contradiction did not show up. Some people including Gauss, Bolyai and Lobachevsky guessed that the specific negation of the fifth postulate opens a door into a vast unexplored territory rather then leads to the expected dead line. Lobachevsky was the first one who shared this opinion with public and explored some parts of the new geometry which he called imaginary geometry (IG). However, the components remained highly speculative until Beltrami in 1868 found some models of the Lobachevsky geometry, which proved that the Lobachevsky geometry is consistent and so can be treated on equal footing with Euclidean one. Finally Hilbert in 1899 put things in order by modernising Euclidean axiomatic method and clarifying the logical structure of non-Euclidean geometries.

The starting point was the analytical statement that basic terms, such as "point", "line" and "plane" are undefined and could just as well be replaced with other terms without affecting the validity of results. Despite this change in terms, the proof of all four theorems would still be valid, because correct proofs do not depend on diagrams; they depend only on stated axioms and the rules of logic. Thus, geometry is a purely formal exercise in deducing certain conclusions from certain formal premises. Mathematics makes statements of the form "if ... then". And according to Bertrand Russell philosophy of mathematics, every mathematical statement is non-existential. Or, it does not say anything about the physical meaning, or, physical truthfulness of the statement.

Lobachevsky was one who deserted geometrical reasoning from intuition and spatial experience, stopped asking whether or not usual axioms of geometry are true and came to the notion of mathematics as playing with axioms. Why Lobachevsky and not others? The obvious reason for it is that fifth postulate looked for others dubious to begin with.

However, it may be easy to see that we are able to construct the Euclidean geometry using the alternative approach. Namely, from definition of an area and a volume. We know, that the point A and B can be joined by the infinite number of lines going from A to B. However, there is the shortest distance between A and B forming the segment of the straight line going from A to B. The segment AB of a straight line can be prolongated in the direction AB, or BA in order to generate the straight line. The prolongation can be easily performed by the following construction. If point C is a such point that it does not lie on the segment AB, then if we define $AB = c$, $BC = a$, $CA = b$, then the area P of the triangle ABC is given by the **Heron formula**

$$P = \sqrt{s(s-a)(s-b)(s-c)} \quad s = \frac{1}{2}(a+b+c). \tag{1}$$

The segment of a straight line is a triangle with zero area. If $P = 0$, then point C lies between A and B, or, on the prolongation of AB. If $P \neq 0$, the point C does not lies on the prolongation of AB. The prolongation can be repeated infinite times to create the straight line. The Euclid plane is formed by three points. It means, if the point C is not on the prolongation of line AB, then ABC is an triangle and triangle is the element of Euclid plane. The Euclidean sheet constructed in such a way is the mathematical object composed from infinite number of triangles. In other words, the Euclidean plane can be constructed by the triangulation. Adding the point D to the system of ABC we get tetrahedron with triangles sides d_{ik} and then it is valid the following **Theorem:** The volume of the tetrahedron $ABCD$ is given by the Euler formula:

$$288V_E^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} \\ 1 & d_{21} & 0 & d_{23} & d_{24} \\ 1 & d_{31} & d_{32} & 0 & d_{34} \\ 1 & d_{41} & d_{42} & d_{43} & 0 \end{vmatrix}, \tag{2}$$

where

$$d_{ik} = (x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2, \tag{3}$$

where x_i, y_i, z_i are Cartesian coordinates of point with index i = 1, 2, 3, 4. Then, if a point D is placed in the space in such a way that $V_E = 0$ then point D lies in the plane ABC. At the same time, it is possible to construct all points of the Euclid plane using the Euler formula with $V_E = 0$. If point D has a such position with regard to triangle that $V_E \neq 0$, then point D belongs to the 3-dimensional space. We can see that using the above operations, we can construct total Euclidean geometry, where the Lobachevsky definitions cannot be involved in the system of our construction elements. The Poincaré invention is only the model of the Lobachevsky geometry in the half plane.

At present time we know, that we have three geometries. Namely: Euclidean one, where the sum of the angles in a triangle is 2π, spherical geometry, where the sum of the angles in triangle is greater than 2π and the geometry of the pseudosphere (i.e. the surface with the Lobachevsky geometry) with the sum of the angles in the triangle on such psedosphere is less than 2π.

The area of the triangle in the Lobachevsky geometry is

$$S = r^2(\pi - \alpha - \beta - \gamma), \tag{4}$$

where r is the radius of a sphere and formula (4) follows from the the spherical triangle area $S = r^2(\alpha + \beta + \gamma - \pi)$, after application of the Beltrami operation $r \longrightarrow ir$.

From the formula (3) it is evident that in the Lobachevsky geometry similar triangles do not exist. Now we can put a question, if the Lobachevsky geometry is the physical one, or, if it is only mathematical construct without physical meaning. We show later that there is the physical meaning of the Lobachevsky geometry with the experimental consequences.

By discovering the group of transformations not altering the Maxwell-Lorentz equations, Poincaré introduced the notion of four-dimensional space-time exhibiting pseudo-Euclidean geometry. This concept of geometry was later developed by Minkowski with the line element.

$$ds^2 = (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2. \tag{5}$$

This element is the basic mathematical object of the special theory of relativity with postulates: 1) Principle of constant light velocity, 2) Maxwell equations, 3) Invariance of physical laws in inertial systems.

In case of noninertial systems the principle 1) cannot be accepted. Mandelstam wrote in his book (Mandelstam, 1972): ... *"special relativity theory cannot answer the question, how a clock behaves when moving with acceleration and why it slows down, because it does not deal with reference systems moving with acceleration".*

According to Einstein - within the framework of special relativity theory there is no place for a satisfactory theory of gravity. Free motion of a test body in an arbitrary reference system takes place along a geodesic line of Minkowski space. So, there is no force in Einstein theory and if we want to accept the notion of force, then we must deduce this phenomenon from the Einstein theory.

In the article Geometry and experiment, Einstein wrote: *"The issue of whether this continuum has an Euclidean, Riemannian, or, any other* *structure is a physical issue, which can only be settled by experiment, and not an issue of convention concerning a choice of simple experience..."* (Einstein, 1924).

On the other hand, there are alternative theories to Einstein gravity constructed in the Minkowski space-time, where the Riemann metric is not involved. One such theory is the vector theory of gravitation as the analog of the electromagnetic theory of Maxwell. It is based on the continuation of the Heaviside theory of gravitation. Other theory is the Schwinger quantum theory of gravity, where gravity force is caused by gravitons as an analog of quantum electrodynamic which is based on photons and electrons and their interactions. We consider the application of this theory to the emission of gravitons by binary. The other alternative theory of gravity is based on the explanation of gravity in such a way that massive body causes the deformation of space-time, which is the origin of the gravitational force.

In the **first chapter** the Poincaré model of the Lobachevsky geometry is derived from the Fermat principle. The Lobachevsky geometry can be physically interpreted as the Lobachevsky-Fok velocity space geometry of moving particles. The relation of this geometry to the decay of the neutral π-meson is considered. The generalization of the Lobachevsky geometry is performed and the new angle of parallelism is derived (Pardy, 2013). The light confined circularly in the optical medium is defined as the optical black hole. The existence of the centrifugal force acting on the photon is discussed.

In the **second chapter** we determine nonlinear transformations between coordinate systems which are mutually in a constant symmetrical accelerated motion. The maximal acceleration limit follows from the kinematical origin. Maximal acceleration is an analogue of the maximal velocity in special relativity. We derive the dependence of mass, length, time, Doppler effect, on acceleration as an analogue phenomena in special theory of relativity.

The **third chapter** deals with the application of the derived nonlinear transformations between coordinate systems in mutually constant symmetrical accelerated motion to the so called Thomas precession.

The **fourth chapter** deals with the total quantum loss of energy caused by production of gravitons emitted by the binary system. The effect is calculated in the framework of the Schwinger theory of gravity for the situation with the gravitational propagator involving radiative corrections.

The **fifth chapter** concerns the finite-temperature gravitational Cherenkov radiation involving radiative corrections. The graviton action in vacuum is generalized for the medium with the constant gravitational index of refraction. From this generalized action the power spectral formula of the Cherenkov radiation of gravitons is derived in the framework of the source theory at zero and nonzero temperature.

The **sixth chapter** deals with energy shift of H-atom electrons due to Gibbons-Hawking thermal bath. The electromagnetic shift of energy levels of H-atom electrons is determined by calculating an electron coupling to the Gibbons-Hawking electromagnetic field thermal bath. Energy shift of electrons in H-atom is determined in the framework of non-relativistic quantum mechanics.

The **seventh chapter** deals with gravity as the deformation of the space time and it involves the light deflection by the screw dislocation. We also explain the geometrical meaning of the metric of space-time and discuss the problem of the dimensionality of space.

The **last chapter** is conclusion and discussion, where we consider the scientific and technological meaning and the perspectives of the results derived in our contribution.

1 From Fermat optical theorem to the Lobachevsky-Fok space-time

1.1 Introduction

The Fermat optical theorem is based on the the physical trajectories. Trajectories of elementary particles in the Wilson chamber, ATLAS in LHC, or in the further terrestrial and cosmical detectors are the basic ingredients of physics of elementary particles and cosmical rays. No elementary particle can exist without its trajectory. While in the particle physics the trajectories of particle are determined by their parameters as mass, charge, spin, velocity and by the influence of the magnetic and electric fields on its motion, some trajectories of bodies are also determined as a result of cybernetic and physical laws. We apply here the notion of trajectory also in geometry to define not only straight line, but all geometrical curves. We follow here the article by author (Pardy, 2013).

The Fermat optical theorem states that the trajectory of light from point A to B in the optical medium is the trajectory performed during the minimal time. At the same time the trajectory of the optical ray from point A to point B with reflection on the mirror in the reflection point C is also performed during the minimal time. This principle can be generalized for an arbitrary number of reflection points.

The trajectory of light passing from point $A(x_1, y_1)$ to point $B(x_2, y_2)$

can be determined by the variational principle (Lavrentyev et al., 1950). It is the mathematical formulation of the Fermat principle and it states that the minimal time T of light passing from point $A(x_1, y_1)$ to point $B(x_2, y_2)$ is the result of the minimization of the functional

$$T(y, y') = \int_{x_1}^{x_2} \frac{ds}{v(y)} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{v(y)} dx, \tag{1}$$

where $v(y)$ is the velocity of light.

The functional $T(y, y')$ is the solution of the Euler-Lagrange equations with $y' = dy/dx$:

$$\frac{\partial T}{\partial y} - \frac{d}{dx}\left(\frac{\partial T}{\partial y'}\right) = 0. \tag{2}$$

If $v = Ay$, the solutions of eq. (2) are the circles forming the Poincaré model of the Lobachevsky geometry:

$$(x - C)^2 + y^2 = r^2. \tag{3}$$

Let us remark that the above method can be applied for determination of the trajectory of light in the stratified medium, or, in medium with reflections on the boundary.

1.2 The Poincaré optical model of the Lobachevsky geometry

The Lobachevsky geometry is the integral part of the general geometry called non-euclidean geometry, or, hyperbolic geometry. The name non-Euclidean was used by Gauss to describe a system of geometry which differs from Euclid's in its properties of parallelism. Such a system was developed also independently by Bolyai in Hungary and Lobachevsky in Russia, many years ago. Another system, differing more radically from Euclid's, was suggested later by Riemann in Germany and Schlafli in Switzerland. The subject was unified by Klein, who gave the names parabolic, hyperbolic, and elliptic to the respective systems of Euclid, Gauss-Bolyai-Lobachevsky, and Riemann-Shlafli (Coxeter, 1998).

The substantial mathematical object in the Lobachevsky geometry is the angle of parallelism defined by Lobachevsky as follows. Given a point P and a line q. The intersection of the perpendicular through P let be Q and $PQ = x$. The intersection of line p passing through P, with q, let be R and $QR = k$. Then, the angle RPQ for perpendicular distance x

$$\Pi(x) = 2 \tan^{-1} e^{-x/k}. \tag{4}$$

is known as the Lobachevsky formula for the angle of parallelism (Coxeter, 1998; Lobachevsky, 1914).

The Poincaré model of the Lobachevsky geometry is the physical model of the optical trajectories in a medium with the velocity of light $v = Ay$.

According to Hilbert (Hilbert, 1903; McCleary, 1994), it is not possible to realize the Lobachevsky geometry globally on surface with the constant negative curvature. The Beltrami realization of the Lobachevsky geometry is only partial one. The famous Russian mathematician Ostrogradsky never acknowledged the Lobachevsky geometry.

Following Bukreev (1951), we can investigate the Lobachevsky geometry and the Poincaré model of it using the pseudosphere (2D manifold) with metric

$$ds^2 = du^2 + e^{\frac{2u}{r}} dv^2. \tag{5}$$

By relations

$$v = x, \quad re^{-\frac{u}{r}} = y, \tag{6}$$

we get the line element as

$$ds^2 = \frac{r^2}{y^2}(dx^2 + dy^2), \tag{7}$$

which was used during the application of the Fermat principle of the minimal time.

The transformation (6) is the conformal mapping of the pseudo-spherical abstract surface (2-dimensional continuous differentiable manifold) into the upper Poincaré half plane in the Cartesian coordinates x, y. As an analogue to this situation it is possible to consider the conformal transformation of the 4-dimensional Einstein-Riemann gravitational manifold to the 4-dimensional Cartesian coordinates x, y, z, t of space-time. Let us still remark that there are many inversion transformation from the Cartesian Poincaré metric to the 2-dimensional manifold ds^2, to form the integral part of the optical models of the Lobachevsky geometry.

The trajectory of light in the Poincaré model is a trajectory passing from $A(x_1, y_1)$ to $B(x_2, y_2)$ and determined by the minimal time from $A(x_1, y_1)$ to $B(x_2, y_2)$. It is the result of the minimum of the functional (1).

The Poincaré circles (pseudo-straight lines) in his model are analogue of the straight lines in the Euclidean geometry.

The theorems following from the metric (7) (Bukreev, 1951) are valid in the Poincaré model of the Lobachevsky geometry:

Theorem 1: Only one half-circle passes through two points A, B in the Poincaré plane.

Theorem 2: The curvilinear segment AB in the Poincaré plane is of the shortest length.

Theorem 3: The parallels are two half-circles with the intersections on the x-axis.

Theorem 4: If point $A \notin q$, then there are $q_1 \parallel q$, $q_2 \parallel q$ passing through A, with $q_1 \neq q_2$.

Theorem 5: If point $A \notin q$, $q_1 \parallel q, q_2 \parallel q$, then q_1, q_2 divide the Poincaré plane in four different sectors I, II, III, IV.

Let us remark that the optical distance between point A and B is not equivalent to the mechanical distance realized by the nonelastic flexible fibre as the shortest distance between point A and B. The Poincaré model of geometry where the light velocity is $v = Ay$ is the interaction model of light with the optical medium.

It is elementary to see that if we define the Poincaré problem on a sphere, then we get so called spherical Poincaré model of the Lobachevsky geometry.

1.3 The Lobachevsky angle of parallelism from trigonometry

It is well known that Beltrami showed that the Lobachevsky trigonometry is the spherical trigonometry with the imaginary radius of the sphere. Or, $r \to ir$. Then instead of the trigonometrical cosine and sine relation on sphere,

$$\cos\frac{a}{r} = \cos\frac{b}{r}\cos\frac{c}{r} + \sin\frac{b}{r}\sin\frac{c}{r}\cos A, \tag{8}$$

$$\frac{\sin A}{\sin a/r} = \frac{\sin B}{\sin b/r} = \frac{\sin C}{\sin c/r}, \tag{9}$$

$$\cos A = -\cos B\cos C + \sin B\sin C\cos(a/r) \tag{10}$$

where r is the radius of sphere and a, b, c are lengths of sides of the triangle on the sphere and A, B, C are corresponding angles, the following relations of the Lobachevsky imaginary pangeometry follows from the Beltrami operation $r \to ir$:

$$\cosh\frac{a}{r} = \cosh\frac{b}{r}\cosh\frac{c}{r} - \sinh\frac{b}{r}\sinh\frac{c}{r}\cos A \tag{11}$$

and

$$\frac{\sin A}{\sinh\frac{a}{r}} = \frac{\sin B}{\sinh\frac{b}{r}} = \frac{\sin C}{\sinh\frac{c}{r}}, \tag{12}$$

$$\cos A = -\cos B\cos C + \sin B\sin C\cosh\frac{a}{r}. \tag{13}$$

Now, we are prepared to derive the Lobachevsky function $\Pi(a)$, where a is BC in the Lobachevsky triangle ABC and the angle B is $\angle B = \pi/2$ and $\angle C = \Pi(a)$.

We have from (13)

$$1 = \sin\Pi(a)\cosh\frac{a}{r}. \tag{14}$$

On the other hand,

$$\cos\Pi(a) = \sqrt{1-\sin^2\Pi(a)} =$$

$$\sqrt{1-\frac{1}{\cosh^2\frac{a}{r}}} = \frac{\sqrt{\cosh^2\frac{a}{r}-1}}{\cosh\frac{a}{r}} = \frac{\sinh\frac{a}{r}}{\cosh\frac{a}{r}} = \tanh\frac{a}{r}. \tag{15}$$

Then, with $\tan\Pi = \sin\Pi/\cos\Pi$, we have

$$\tan^2\frac{\Pi(a)}{2} = \frac{1-\cos\Pi(a)}{1+\cos\Pi(a)} =$$

$$\frac{1-\tan\frac{a}{r}}{1+\tan\frac{a}{r}} = \frac{\cosh\frac{a}{r}-\sinh\frac{a}{r}}{\cosh\frac{a}{r}+\sinh\frac{a}{r}} = \frac{e^{-\frac{a}{r}}}{e^{\frac{a}{r}}} = e^{-\frac{2a}{r}}. \tag{16}$$

Or,

$$\tan\frac{\Pi(a)}{2} = e^{-\frac{a}{r}}. \tag{17}$$

The last formula is the famous one for the Lobachevsky angle $\Pi(a)$.

Let us remark that the angle of parallelism is immediately related to the decay of the neutral π-meson to two gamma-photons, detected by the coincidence experimental method (Steinberger, et al., 1950). Or,

$$\pi^0 \quad\rightarrow\quad \gamma+\gamma. \tag{18}$$

The angle between velocities of the gamma photons in the rest system of neutral meson is evidently π. However according to the special theory of relativity the angle is transformed in the laboratory system according to the Lorentz transformation and it is smaller than π. It is equivalent to the statement that the Lobachevsky angle Π is smaller than $\pi/2$, or, $\Pi < \pi/2$. Such experiment can be considered as the confirmation of the Lobachevsky geometry in the elementary particle physics. Similarly the decay of the neutral η-meson $\eta^0 \to \gamma + \gamma$, axion $A^0 \to \gamma + \gamma$, or, the Higgs boson decay $H^0 \to \gamma + \gamma$, are the confirmation of the Lobachevsky geometry in the elementary particle physics and at present time can be tested in CERN.

The statement which is valid for the decay channel of the π^0-meson is valid by analogy also for all decay channels described in the Review of the Particle physics (Amsler et al., 2008).

1.4 The generalized Lobachevsky geometry

Theorem: The generalized Lobachevsky formulas for triangles in generalized Lobachevsky geometry follow from the spherical formulas (8), (9), (10) by transformation $r \to r + i\varrho$:

$$\cos\varphi_a \cosh\chi_a + i\sin\varphi_a \sinh\chi_a =$$

$$[\cos\varphi_b \cosh\chi_b + i\sin\varphi_b \sinh\chi_b][\cos\varphi_c \cosh\chi_c + i\sin\varphi_c \sinh\chi_c] \quad +$$

$$[\sin\varphi_b \cosh\chi_b + i\cos\varphi_b \sinh\chi_b][\sin\varphi_c \cosh\chi_c + i\cos\varphi_c \sinh\chi_c]\cos A, \quad (19)$$

$$\frac{\sin A}{\sin\varphi_a \cosh\chi_a + i\cos\varphi_a \sinh\chi_a} = \frac{\sin B}{\sin\varphi_b \cosh\chi_b + i\cos\varphi_b \sinh\chi_b} = \frac{\sin C}{\sin\varphi_c \cosh\chi_c + i\cos\varphi_c \sinh\chi_c}, \quad (20)$$

$$\cos A = -\cos B\cos C \quad + \sin B\sin C[\cos\varphi_a \cosh\chi_a + i\sin\varphi_a \sinh\chi_a], \quad (21)$$

where

$$\varphi_a; \varphi_b; \varphi_c; = \frac{ar}{r^2+\varrho^2}; \quad \frac{br}{r^2+\varrho^2}; \quad \frac{cr}{r^2+\varrho^2} \quad (22)$$

and

$$\chi_a; \chi_b; \chi_c; = \frac{a\varrho}{r^2 + \varrho^2}; \quad \frac{b\varrho}{r^2 + \varrho^2}; \quad \frac{c\varrho}{r^2 + \varrho^2}. \tag{23}$$

and ϱ is the new parameter of the new triangle on the 2D manifold and A, B, C are corresponding triangle angles.

It follows fro eq. (21) that when A, B, C are real quantities, then it is necessary to be $\sin\varphi_a = 0$, or, $\varphi_a = l\pi, l = 1, 2, 3....$ Similarly, $\sin\varphi_b = 0$, or, $\varphi_b = m\pi, m = 1, 2, 3...$, $\sin\varphi_c = 0$, or, $\varphi_c = n\pi, n = 1, 2, 3....$ It means that from the generalized Beltrami operation $r \to r + i\varrho$, the quantization of the generalized Lobachevsky geometry of the 2D-manifold follows.

Now, we can derive the generalized Lobachevsky function $\Pi(a)$, where a is BC in the generalized Lobachevsky triangle ABC, $\angle B = \pi/2$ and $\angle C = \Pi(a)$.

We have from eq. (21) for the generalized rectangular triangle:

$$1 = \sin\Pi(a)\cosh\chi_a, \tag{24}$$

where

$$\chi_a = \frac{a\varrho}{r^2 + \varrho^2} = \frac{\varrho}{r}\varphi_a = \frac{\varrho}{r}l\pi; \quad l = 1, 2, 3... \tag{25}$$

We have from eq. (24):

$$\tan\frac{\Pi(a)}{2} = e^{-\chi_a} \tag{26}$$

It is evident that in the limiting case $\varrho \to 0$, we get the Euclidean angle $\Pi(a) = \pi/2$. While the original Lobachevsky angle $\Pi(a)$ was confirmed in decay of the neutral π-meson, the generalized Lobachevsky angle $\Pi(a)$ is expected to be confirmed in the high energy physics by experiments in CERN and it is not excluded that the new geometry will be revealed in the Little Bang (Dusling et al., 2011) if performed by LHC in CERN, or, in the vicinity of the galactical nucleus.

1.5 The Lobachevsky-Beltrami-Fok velocity space

We know from the special theory of relativity that the relative velocity of two particles with velocities $\mathbf{v}_1, \mathbf{v}_2$ is given by the formula (Landau et al., 1987) (c =1, for the velocity of light):

$$\mathbf{v}' = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}}{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}. \tag{27}$$

The last formula can be easily transformed for $\mathbf{v}_1 = \mathbf{v}$ and $\mathbf{v}_2 = \mathbf{v} + d\mathbf{v}$ to get new differential form which can be considered as the length element in the velocity space, where v_1, v_2, v_3 are so called the Beltrami coordinates (Fok, 1955; Kagan, 1947; ibid., 1948). Or,

$$dl_v^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1-v^2)^2} =$$

$$\frac{dv^2}{(1-v^2)} + \frac{v^2}{(1-v^2)}(d\theta^2 + \sin^2\theta d\varphi^2), \tag{28}$$

where θ and φ are polar and the azimutal angles of the velocity $\mathbf{v}$ in the spherical coordinate system.

Using the substitution $v = \tanh\chi$, we get the line element in the velocity space as

$$dl_v^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2). \tag{29}$$

The last line element is from the geometrical point of view the element of the Lobachevsky space with the constant negative Gauss curvature.

The Lobachevsky space follows from the spherical element (Landau et al. 1987)

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{30}$$

if we replace in the spherical metric the variable r by $r = a\sinh\chi$ and $a \to ia$. Then

$$dl_v^2 = a^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{31}$$

The area of the sphere is $A = 4\pi a^2 \sinh^2\chi$ and volume V goes to infinity for r goes to infinity. So the Lobachevsky abstract space is identical with the Friedmann solution of the Einstein equations with the negative curvature.

Let us remark that if we perform transformation $r \to ir + \rho$ in formula (30) where we write $r^2 = r.r = r.r^*$, where $r^* = \rho - ir$, in the form

$$dl^2 = \frac{dr^2}{1 - \frac{r^2+\rho^2}{a^2}} + (r^2 + \rho^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{32}$$

which is the elementary generalization of the Lobachevsky geometry line element.

Now, if we perform the substitution

$$r^2 + \rho^2 = a^2 \sin^2 \chi, \tag{33}$$

then the element (32) can be transformed into the form:

$$dl^2 = \frac{a^4 \sin^2 \chi d\chi^2}{a^2 \sinh^2 \chi - \rho^2} + a^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \tag{34}$$

It may be easy to see that the last formula is adequate to the metric of the exotic cosmology with the new geometrical term ρ which should not be identified with the Einstein cosmological constant Λ.

1.6 The geodesic line in the Lobachevsky-Beltrami-Fok space of velocities

If we put $d\mathbf{v} = \dot{\mathbf{v}}dt$, for the line element in the Lobachevsky space we get from eq. (28) relation

$$\left(\frac{dl_v}{dt}\right)^2 = \frac{\dot{\mathbf{v}}^2}{1 - v^2} + \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{(1 - v^2)^2} = 2F, \tag{35}$$

where the symbol $2F$ is introduced by definition.

Then we write (Fok, 1955):

$$dl_v = \int_{t_1}^{t_2} \sqrt{2F} dt, \tag{36}$$

where $L = \sqrt{2F}$ is the Lagrange function, or

$$L = \sqrt{\frac{\dot{\mathbf{v}}^2}{1 - v^2} + \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{(1 - v^2)^2}}. \tag{37}$$

The geodetic line from time t_1 to time t_2 is the solution of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}_k}\right) - \frac{\partial L}{\partial v_k} = 0; \quad k = 1, 2, 3, \tag{38}$$

or,

$$\frac{d}{dt}\left(\frac{1}{\sqrt{2F}}\frac{\partial F}{\partial \dot{v}_k}\right) - \frac{1}{\sqrt{2F}}\frac{\partial F}{\partial v_k} = 0; \quad k = 1, 2, 3. \tag{39}$$

We use here the parameter t which is an arbitrary parameter. The interpretation of it as time is of course possible. We choose it in such a way that

$$\frac{dF}{dt} = 0; \quad F = const. \tag{40}$$

Then equations (39) will be equivalent to

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}_k}\right) - \frac{\partial F}{\partial v_k} = 0; \quad k = 1, 2, 3. \tag{41}$$

As the function F is not the function of parameter t, then we can write:

$$\sum_k \dot{v}_k \frac{\partial F}{\partial \dot{v}_k} - F = const. \tag{42}$$

Eq. (42) is an analogue of the definition of energy in classical mechanics in generalized coordinates if we replace F by the Lagrange function of the massive point.

It follows from eq. (39):

$$\frac{\partial F}{\partial \dot{v}_k} = \frac{\dot{v}_k}{1 - v^2} + \frac{v_k(\mathbf{v} \cdot \dot{\mathbf{v}})}{(1 - v^2)^2}; \quad k = 1, 2, 3 \tag{43}$$

$$\frac{\partial F}{\partial v_k} = v_k \left(\frac{\dot{\mathbf{v}}^2}{(1 - v^2)^2} + 2\frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{(1 - v^2)^3}\right) + \frac{\dot{v}_k(\mathbf{v} \cdot \dot{\mathbf{v}})}{(1 - v^2)^2}; \quad k = 1, 2, 3. \tag{44}$$

Let us introduce the vector $\mathbf{w}$ by the definition

$$w_k = \frac{\dot{v}_k}{(1 - v^2)}. \tag{45}$$

Then eqs. (43), (44) can be evidently written in the form

$$\frac{\partial F}{\partial \dot{v}_k} = w_k + \frac{v_k(\mathbf{v} \cdot \mathbf{w})}{(1 - v^2)}; \quad k = 1, 2, 3 \tag{46}$$

$$\frac{\partial F}{\partial v_k} = v_k \left(w^2 + \frac{2(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{(1 - v^2)}\right) + v_k(\mathbf{v} \cdot \mathbf{w}); \quad k = 1, 2, 3. \tag{47}$$

After t-derivation of eq. (47) and expressing $\dot{\mathbf{v}}$ as a function of $\mathbf{w}$, we get

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{v}_k} = \dot{w}_k + \frac{v_k(\mathbf{v} \cdot \dot{\mathbf{w}})}{(1 - v^2)} \quad +$$

$$v_k \left(w^2 + \frac{2(\mathbf{v} \cdot \mathbf{w})^2}{(1 - v^2)}\right) + v_k(\mathbf{v} \cdot \mathbf{w}); \quad k = 1, 2, 3. \tag{48}$$

After insertion of $\partial F/\partial v_k$ (47) and equation (48) into Lagrange equation (41), we get

$$\dot{w}_k + \frac{v_k(\mathbf{v}\cdot\dot{\mathbf{w}})}{(1-v^2)} = 0. \tag{49}$$

After multiplication of the last equation by v_k we get

$$\mathbf{v}\cdot\dot{\mathbf{w}} = 0, \tag{50}$$

from which equation follows $\dot{\mathbf{w}} = 0$, or, $\mathbf{w} = const$. However, $\mathbf{w}$ is collinear with $\dot{\mathbf{v}}$, then $\mathbf{w}\cdot\dot{\mathbf{v}} = 0$, Or,

$$\mathbf{w}\cdot\mathbf{v} = const. \tag{51}$$

It gives still two linearly independent integrals of the Lagrange equations.

1.7 The length of the straight segment in the Lobachevsky-Beltrami-Fok space

We consider the length AB as the shortest line segment from A to B in the Lobachevsky space. Let us introduce two vectors $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_2$. Then the parametric form of the segment is

$$\mathbf{v} = \mathbf{v}_1 + \mu(\mathbf{v}_2 - \mathbf{v}_1); \quad 0 < \mu < 1. \tag{52}$$

After insertion of (52) into F in (35), we get

$$2F = \frac{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1-v^2)^2}\dot{\mu}^2. \tag{53}$$

Putting

$$a = \sqrt{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}, \tag{54}$$

we get from eq. (36) the time integral from t_1 to t_2

$$dl_v = \int_0^1 \frac{ad\mu}{1-v^2} \tag{55}$$

Using substitution

$$\mu = \frac{(1-v_1^2)\xi}{1 - \mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{v}_1\cdot\mathbf{v}_2 - v_1^2)\xi}, \tag{56}$$

we get

$$l_v = \int_0^1 \frac{abd\xi}{b^2 - a^2\xi^2} = \frac{1}{2}\ln\frac{b+a}{b-a}, \tag{57}$$

where $b = 1 - \mathbf{v}_1 \cdot \mathbf{v}_2$ and

$$\frac{a}{b} = \tanh l_v; \quad \left(\frac{a}{b}\right)^2 = \tanh^2 l_v. \tag{58}$$

Or,

$$\frac{(\mathbf{v}_2 - \mathbf{v}_1)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)} = \tanh l_v. \tag{59}$$

The left side of (59) is the relative velocity, So

$$|v'| = \tanh l_v. \tag{60}$$

Putting

$$v_1 = \tanh l_{1v}, \quad v_2 = \tanh l_{2v} \tag{61}$$

we get for $\mathbf{v}_1 \parallel \mathbf{v}_2$

$$V' = \tanh(l_{2v} - l_{1v}) = \frac{\tanh l_{2v} - \tanh l_{1v}}{1 - \tanh l_{2v} \cdot \tanh l_{1v}}, \tag{62}$$

Or,

$$V' = \frac{v_2 - v_1}{1 - v_1.v_2}, \tag{63}$$

which is the famous Einstein formula for the addition of velocities.

1.8 The Lobachevsky-Beltrami-Fok triangle in particle physics

Let us investigate the angle between vectors of the velocities velocities of the two bodies. Let the vectors are taken with regard to the point which is in the state of rest. The vectors are $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the obligate formula for cosine of the angle of the two vectors is:

$$\cos(\mathbf{v}_1, \mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1| \cdot |\mathbf{v}_2|}. \tag{64}$$

However if the reference point of vectors is moving with the velocity $\mathbf{u}$, then the angle between vectors are given by the relativistic formula:

$$\cos\alpha = \cos(\mathbf{v}'_1, \mathbf{v}'_2) = \frac{\mathbf{v}'_1 \cdot \mathbf{v}'_2}{|\mathbf{v}'_1| \cdot |\mathbf{v}'_2|}, \tag{65}$$

where the prime symbols are vectors after the Lorentz transformations of the velocities. The Lorentz transformation of the velocities is as follows:

$$\mathbf{v}' = \frac{\mathbf{v} - \mathbf{u} + (a_{00} - 1)\frac{\mathbf{u}}{u^2}\left((\mathbf{u}\cdot\mathbf{v} - u^2)\right)}{a_{00}\,(1 + \mathbf{u}\cdot\mathbf{v})}. \tag{66}$$

If we express $\mathbf{v}_1'$ and $\mathbf{v}_2'$ and by $\mathbf{v}_1$ and by $\mathbf{v}_2$, we then get cosine of the angle α as it follows:

$$\cos\alpha = \frac{(\mathbf{v}_1 - \mathbf{u})\cdot(\mathbf{v}_2 - \mathbf{u}) - (\mathbf{v}_1 \times \mathbf{u})\cdot(\mathbf{v}_2 \times \mathbf{u})}{\sqrt{(\mathbf{v}_1 - \mathbf{u})^2 - (\mathbf{v}_1 \times \mathbf{u})^2}\cdot\sqrt{(\mathbf{v}_2 - \mathbf{u})^2 - (\mathbf{v}_2 \times \mathbf{u})^2}}. \tag{67}$$

This is the expression for the cosine of the angle of the triangle in the space of Lobachevsky. In other words, this is cosine of the angle in the vertex $\mathbf{u}$ in the triangle with the vertexes at points $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ where the relative velocities $\mathbf{v}_1 - \mathbf{u}$ and $\mathbf{v}_2 - \mathbf{u}$ are sides of the triangle and form the angle α.

It can be explained by the different way. The length element in the Lobachevsky space corresponding to $d\mathbf{v}$ is

$$dl_v^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1 - v^2)^2}, \tag{68}$$

And the length element in the Lobachevsky space corresponding to $\delta\mathbf{v}$ is

$$\delta l_v^2 = \frac{(\delta\mathbf{v})^2 - (\mathbf{v} \times \delta\mathbf{v})^2}{(1 - v^2)^2}. \tag{69}$$

Then we can define the relation for the cosine between $d\mathbf{v}$ and $\delta\mathbf{v}$ by the relation:

$$dl_v \delta l_v \cos\alpha = \frac{d\mathbf{v}\delta\mathbf{v} - (\mathbf{v} \times d\mathbf{v})(\mathbf{v} \times \delta\mathbf{v})}{(1 - v^2)^2}. \tag{70}$$

The angle between the relative velocities can be considered as the angle of the Lobachevsky triangle. If we have three bodies moving with velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then the corresponding triangle will have the vertexes in points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and the relative velocities are the sides of the triangle. This construction is the analogue of the non-relativistic case, but we have here the Lobachevsky triangle on the Lobachevsky 2D manifold. The generalization of the Lobachevsky triangle to the Euler-Lobachevsky tetrahedron is evident.

Lobachevsky, in his pangeometry, presents the idea (many years before Einstein) that his geometry is probably realized in the near vicinity of atoms and molecules and also in the cosmical space (Norden, 1956). Now, we see that his geometry is realized in particle physics of LHC in CERN.

2 The nonlinear Lorentz group for accelerated systems

We determine here the nonlinear transformations between coordinate systems which are mutually in a constant symmetrical accelerated motion. The maximal acceleration limit follows from the kinematical origin. Maximal acceleration is an analogue of the maximal velocity in special relativity. We derive the dependence of mass, length, time, Doppler effect on acceleration as an analogue phenomena in special theory of relativity. The derived addition theorem for acceleration can play crucial role in modern particle physics and cosmology.

In case of noninertial systems the principle of the constant light velocity cannot be accepted. An outstanding physicist Mandelstam wrote in his book (Mandelstam, 1972): "... special relativity theory cannot answer the question, how a clock behaves when moving with acceleration and why it slows down, because it does not deal with reference systems moving with acceleration ".

The problem of acceleration of charged particles or systems of particles is the permanent and the most prestige problem in the accelerator physics. Particles can be accelerated by different ways. Usually by the classical electromagnetic fields, or, by light pressure of the laser fields (Baranova et al., 1994; Pardy, 1998, 2001a, 2002a). The latter method is the permanent problem of the laser physics for many years.

Here, we determine transformations between coordinate systems which moves mutually with the same acceleration. We determine transformations between non relativistic and relativistic uniformly accelerated systems.

Let us remind the special theory of relativity velocity and acceleration The Lorentz transformation between two inertial coordinate systems $S(0, x, y, z)$ and $S'(0, x', y', z')$ (where system S' moves in such a way that x-axes converge, while y and z-axes run parallel and at time $t = t' = 0$ for the origin of the systems O and O' it is $O \equiv O'$) is as follows:

$$x' = \gamma(v)(x - vt), \quad y' = y, \quad z' = z', \quad t' = \gamma(v)\left(t - \frac{v}{c^2}x\right), \tag{1}$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2}$$

The infinitesimal form of this transformation is evidently given by differentiation of the every equation. Or,

$$dx' = \gamma(v)(dx - vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(v)\left(dt - \frac{v}{c^2}dx\right). \tag{3}$$

It follows from eqs. (3) that if v_1 is velocity of the inertial system 1 with regard to S and v_2 is the velocity of the inertial systems 2 with regard to 1, then the relativistic sum of the two velocities is

$$u_2 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \tag{4}$$

The mathematic object called four-velocity follows from the Lorentz transformation after some additional operations. From the ordinary three-dimensional velocity vector one can form a four-vector. This four-dimensional velocity (four-velocity) of a particle is the vector

$$u^\mu = \frac{dx^\mu}{ds}, \tag{5}$$

where, according to Landau et al. (1987)

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}} \tag{6}$$

with v being the ordinary three-dimensional velocity of the particle and c being the velocity of light. Thus

$$u^1 = \frac{dx^1}{ds} = \frac{dx}{cdt\sqrt{1 - \frac{v^2}{c^2}}} = \frac{v_x}{c\sqrt{1 - \frac{v^2}{c^2}}}. \tag{7}$$

Then,

$$u^\mu = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\mathbf{v}}{c\sqrt{1 - \frac{v^2}{c^2}}}\right). \tag{8}$$

Note, that the four-velocity is a dimensionless quantity. The components of the four-velocity are not independent. Noting that $dx^\mu dx_\mu = ds^2$, we have

$$u^\mu u_\mu = 1. \tag{9}$$

Geometrically, u^μ is a unit four-vector tangent to the world line of the particle.

Similarly to the definition of the four-velocity, the second derivative

$$a^\mu = \frac{d^2x^\mu}{ds^2} = \frac{du^\mu}{ds} \tag{10}$$

may be called the four-acceleration. Differentiating formula (9), we find:

$$u^\mu a_\mu = 0, \tag{11}$$

i.e. the four-vectors of velocity and acceleration are "mutually perpendicular".

Now, let us determine the relativistic uniformly accelerated motion, i.e. the rectilinear motion for which the acceleration a^μ in the proper reference frame (at each instant of time) remains constant. We proceed as follows.

In the reference frame in which the particle velocity is $v = 0$, the components of the four-acceleration $a^\mu = (0, a/c^2, 0, 0)$ (where $\mathbf{a}$ is the ordinary three-dimensional acceleration, which is directed along the x axis). The relativistically invariant condition for uniform acceleration must be expressed by the constancy of the four-scalar which coincides with a^2 in the proper reference frame:

$$a^\mu a_\mu = const = -\frac{a^2}{c^4}. \tag{12}$$

In the "fixed" frame, with respect to which the motion is observed, writing out the expression for $a^\mu a_\mu$ gives the equation:

$$\frac{d}{dt}\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = a, \tag{13}$$

or,

$$\frac{v}{c\sqrt{1-\frac{v^2}{c^2}}} = at + const. \tag{14}$$

Setting $v = 0$ for $t = 0$, we find that $const = 0$, so that

$$v = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}, \tag{15}$$

Integrating once more and setting $x = 0$ for $t = 0$, we find:

$$x = \frac{c^2}{a}\left(\sqrt{1+\frac{a^2t^2}{c^2}}-1\right). \tag{16}$$

For $at \ll c$, these formulas go over the classical expressions $v = at, x = \frac{a}{2}t^2$. For $at \to \infty$, the velocity tends toward the constant value c.

The proper time of a uniformly accelerated particle is given by the integral (Landau et al., 1987)

$$\int_0^t \sqrt{1+\frac{v^2(t)}{c^2}}dt = \frac{c}{a}\text{arcsinh}\frac{at}{c}. \tag{17}$$

At the limit $t \to \infty$ it increases much more slowly than t, according to the law

$$\frac{c}{a}\ln\frac{2at}{c}. \tag{18}$$

The infinitesimal form of Lorentz transformation (3) evidently gives the Lorentz length contraction and time dilation. Namely, if we put $dt = 0$ in the first equation of system (3), then the Lorentz length contraction follows in the infinitesimal form $dx' = \gamma(v)dx$. Or, in other words, if in the system S' the infinitesimal length is dx', then the relative length with regard to the system S is $\gamma^{-1}dx'$. Similarly, from the last equation of (3) it follows the time dilatation for $dx = 0$. Historical view on this effect is in the Selleri article (Selleri, 1997).

2.1 Uniformly accelerated frames with space-time symmetry

Let us take two systems $S(0, x, y, z)$ and $S'(0, x', y', z')$, where system S' moves in such a way that x-axes converge, while y and z-axes run parallel and at time $t = t' = 0$ for the beginning of the systems O and O' it is $O \equiv O'$. Let us suppose that system S' moves relative to some basic system B with acceleration $a/2$ and system S' moves relative to system B with acceleration $-a/2$. It means that both systems moves one another with acceleration a and are equivalent because in every system it is possibly to observe the force caused by the acceleration $a/2$. In other words no system is inertial.

Now, let us consider the formal transformation equations between two systems. At this moment we give to this transform only formal meaning because at this time, the physical meaning of such transformation is not known. On the other hand, the consequences of the transformation will be shown very interesting. The first published derivation of such transformation by the standard way was given by author (Pardy, 2003; 2004; 2005), and the same transformations were submitted some decades ago (Pardy, 1974). The old results can be obtained if we perform transformation

$$t \to t^2, \quad t' \to t'^2, \quad v \to \frac{1}{2}a, \quad c \to \frac{1}{2}\alpha \tag{19}$$

in the original Lorentz transformation (1). We get:

$$x' = \Gamma(a)(x - \frac{1}{2}at^2), \quad y' = y, \quad z' = z, \quad t'^2 = \Gamma(a)\left(t^2 - \frac{2a}{\alpha^2}x\right) \tag{20}$$

with

$$\Gamma(a) = \frac{1}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{21}$$

We used practically new denotation of variables in order to get the transformation (20) between accelerated systems.

The transformations (20) form the one-parametric group with the parameter a. The proof of this mathematical statement can be easy performed if we perform the transformation T_1 from S to S', transformation T_2 from S' to S'' and transformation T_3 from S to S''. Or,

$$x' = x'(x, t; a_1), \quad t' = t'(x, t; a_1), \tag{22}$$

$$x'' = x''(x', t'; a_2), \quad t'' = t''(x', t'; a_2), \tag{23}$$

After insertion of transformations (22) into (23), we get

$$x'' = x''(x, t; a_3), \quad t'' = t''(x, t; a_3), \tag{24}$$

where parameter a_3 is equal to

$$a_3 = \frac{a_1 + a_2}{1 + \frac{a_1 a_2}{\alpha^2}}. \tag{25}$$

The inverse parameter is $-a$ and parameter for identity is $a = 0$. It may be easy to verify that the final relation for the definition of the continuous group transformation is valid for our transformation. Namely (Eisenhart, 1943):

$$(T_3 T_2)\, T_1 = T_3\, (T_2 T_1)\,. \tag{26}$$

The physical interpretation of this nonlinear transformations is the same as in the case of the Lorentz transformation only the physical interpretation of the invariant function $x = (1/2)\alpha t^2$ is different. Namely it can be expressed by the statement. If there is a physical signal in the system S with the law $x = (1/2)\alpha t^2$, then in the system S' the law of the process is $x' = (1/2)\alpha t'^2$, where α is the constant of maximal acceleration. It is new constant and cannot be defined by the game with known physical constants.

Let us remark, that it follows from history of physics, that Lorentz transformation was taken first as physically meaningless mathematical object by Larmor, Voigt and Lorentz and later only Einstein decided to put the physical meaning to this transformation and to the invariant function $x = ct$. We hope that the derived transformation will appear as physically meaningful.

Using relations $t \leftarrow t^2, \quad t' \leftarrow t'^2, \quad v \leftarrow \frac{1}{2}a, \quad c \leftarrow \frac{1}{2}\alpha$, the nonlinear transformation is expressed as the Lorentz transformation forming the one-parametric group. This proof is equivalent to the proof by the above direct calculation. The integral part of the group properties is the so called addition theorem for acceleration.

$$a_3 = \frac{a_1 + a_2}{1 + \frac{a_1 a_2}{\alpha^2}}. \tag{27}$$

where a_1 is the acceleration of the S' with regard to the system S, a_2 is the acceleration of the system S'' with regard to the system S' and a_3 is the acceleration of the system S'' with regard to the system S. The relation (27), expresses the law of acceleration addition theorem on the understanding that the events are marked according to the relation (20).

If $a_1 = a_2 = a_3 = ... + a_n = a$, for n accelerated carts which rolls in such a way that the first cart rolls on the basic cart, the second rolls on the first cart and so on, then we get for the sum of n accelerated carts the following formula

$$a_{sum} = \frac{1 - \left(\frac{1-a/\alpha}{1+a/\alpha}\right)^n}{1 + \left(\frac{1-a/\alpha}{1+a/\alpha}\right)^n}, \tag{28}$$

which is an analogue of the formula for the inertial systems (Lightman et al., 1975).

In this formula as well as in the transformation equation (20) appears constant α which cannot be calculated from the theoretical considerations, or, constructed from the known physical constants (in analogy with the velocity of light). What is its magnitude can be established only by experiments. The notion maximal acceleration was introduced some decades ago by author (Pardy, 1974). Caianiello (1981) introduced it as some consequence of quantum mechanics and Landau theory of fluctuations (Landau, et al., 1982). Revisiting view on the maximal acceleration was given by Papini (2003). At present time it was argued by Lambiase et al. (1999) that maximal acceleration determines the upper limit of the Higgs boson and that it gives also the relation which links the mass of W-boson with the mass of the Higgs boson. The LHC and HERA experiments presented different answer to this problem.

2.2 Transformation with constant acceleration in the fixed frame

In the "fixed" frame, with respect to which the motion is observed, we use the equation (16) to derive the adequate transformation: Or,

$$\xi(a,t) = \frac{c^2}{a}\left(\sqrt{1+\frac{a^2t^2}{c^2}} - 1\right). \tag{29}$$

For $at \ll c$, these formulas go over the classical expressions $v = at, x = \frac{1}{2}at^2$. For $at \to \infty$, the velocity tends toward the constant value c.

The transformation equations between S and S' can be easily derived. Let us give some instructions.

It may be easy to see, that

$$x' = \Gamma(a)(x - \xi(a,t)), \quad y' = y, \quad z' = z, \tag{30}$$

with

$$\Gamma(a) = \frac{1}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{31}$$

Then,

$$x = \Gamma(a)(x + \xi(a,t')) \tag{32}$$

and

$$\xi(a,t') = \Gamma^{-1}x - x' = x/\Gamma - \Gamma x + \Gamma\xi(a,t) \tag{33}$$

It follows from the last equation the variable t' and the identity $\xi(\alpha,t') = \xi(\alpha,t)$.

Let us remark, that if we use the infinitesimal transformation (3) with the velocity depending on time (15), then we obtain after integration the new original transformation for accelerated systems (Pardy, 2003, 2004, 2005) with the new physical meaning.

2.3 Mass shift by acceleration

If the maximal acceleration is the physical reality, then it should have the similar consequences in a dynamics as the maximal velocity of motion has consequences in the dependence of mass on velocity. We can suppose in analogy with the special relativity that mass depends on the acceleration for small velocities, in the similar way as it depends on velocity in case of uniform motion. Of course such assumption must be experimentally verified and in no case it follows from special theory of relativity, or, general theory of relativity (Fok, 1961). So, we postulate ad hoc, in analogy with special theory of relativity:

$$m(a) = \frac{m_0}{\sqrt{1-\frac{a^2}{\alpha^2}}}; \quad v \ll c, \quad a = \frac{dv}{dt}. \tag{34}$$

Let us derive as an example the law of motion when the constant force F acts on the body with the rest mass m_0. Then, the Newton law reads (Landau et al., 1987):

$$F = \frac{dp}{dt} = m_0 \frac{d}{dt}\frac{v}{\sqrt{1-\frac{a^2}{\alpha^2}}}. \tag{35}$$

The first integral of this equation can be written in the form:

$$\frac{Ft}{m_0} = \frac{v}{\sqrt{1-\frac{a^2}{\alpha^2}}}; \quad a = \frac{dv}{dt}, \quad F = const.. \tag{36}$$

Let us introduce quantities

$$v = y, \quad a = y', \quad A(t) = \frac{F^2t^2}{m_0^2\alpha^2}. \tag{37}$$

Then, the quadratic form of the equation (36) can be written as the following differential equation:

$$A(t)y'^2 + y^2 - A(t)\alpha^2 = 0, \tag{38}$$

which is nonlinear differential equation of the first order. The solution of it is of the form $y = Dt$, where D is some constant, which can be easily determined. Then, we have the solution in the form:

$$y = v = Dt = \frac{t}{\sqrt{\frac{m_0^2}{F^2} + \frac{1}{\alpha^2}}}. \tag{39}$$

For F $\to \infty$, we get $v = \alpha t$. This relation can play substantial role at the beginning of the big-bang, where the accelerating forces can be considered as infinite, however the law of acceleration has finite nonsingular form.

At this moment it is not clear if the dependence of the mass on acceleration can be related to the energy dependence on acceleration similarly to the Einstein relation uniting energy, mass and velocity (Okun, 2001, 2002; Sachs, 1973).

The infinitesimal form of author transformation (20) evidently gives the length contraction and time dilation. Namely, if we put $dt = 0$ in the first equation of system (20), then the length contraction follows in the infinitesimal form $dx' = \Gamma(a)dx$. Or, in other words, if in the system S' the infinitesimal length is dx', then the relative length with regard to the system S is $\Gamma^{-1}dx'$. Similarly, from the last equation of (20) it follows the time dilatation for $dx = 0$.

The relativistic Doppler effect is the change in frequency (and wavelength) of light, caused by the relative motion of the source and the observer (as in the classical Doppler effect), when taking into account effects described by the special theory of relativity.

2.4 Doppler effect due to an acceleration

The relativistic Doppler effect is different from the non-relativistic Doppler effect as the equations include the time dilation effect of special relativity and do not involve the medium of propagation as a reference point (Rohlf, 1994).

The Doppler shift caused by acceleration can be also derived immediately from the original relativistic equations for the Doppler shift. We only make the transformation $v \to a/2, c \to \alpha/2$ to get

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1 - a/\alpha}{1 + a/\alpha}} \tag{40}$$

when the photons of the wave length λ are measured toward photon source, and

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1 + a/\alpha}{1 - a/\alpha}} \tag{41}$$

when the photons of the wave length λ are measured in the frame that is moving away from the photon source. Different approach used Friedman et al. (2010).

2.5 The Cherenkov effect and the transition radiation due to an acceleration

Concerning the Cherenkov radiation, it is based on the fact that the speed of light in the medium with the index of refraction n is c/n. A charged particle moving in such medium can have the speed greater than it is the speed of light in this medium. When a charged particle moves faster than the speed of light in this medium, a portion of the electromagnetic radiation emitted by excited atom along the path of the particle is coherent. The coherent radiation is emitted at a fixed angle with respect to the particle trajectory. This radiation was observed by Cherenkov in 1935. The characteristic angle was derived by Tamm and Frank in the form (Rohlf, 1994)

$$\cos\theta = \frac{c}{vn}. \tag{42}$$

The Cherenkov angle caused by acceleration cannot be derived immediately from the original Frank-Tamm equations for this effect.

In case of the Ginzburg transition radiation the radiation in concentrated in the angle

$$1/\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{43}$$

The transition radiation angle caused by acceleration cannot be derived immediately from the original Ginzburg formula for this effect.

2.6 The rotating systems

It is defined by equations

$$x = r\cos(\varphi + \omega t), \quad y = r\sin(\varphi + \omega t). \tag{44}$$

The corresponding space-time element is as follows:

$$ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right)(cdt)^2 - \frac{2\omega r^2}{c} d\varphi(cdt) - dz^2 - dr^2 - r^2 d\varphi^2. \qquad (45)$$

Although the rotating system cannot be considered as equivalent to the linear accelerated system, nevertheless, the radial component of every part of this system is in the permanent acceleration. The application in the galactic space is evident. In other words, if the radial coordinate of Earth with regard to Sun is r_E and its radial acceleration w_E and the radial coordinate of Moon with regard to Earth is r_M and acceleration w_M, then the relative acceleration w_r of Moon with regard to Sun is not $w_E + w_M$, but it is given by the formula

$$w_r = \frac{w_E + w_M}{1 + \frac{w_E w_M}{\alpha^2}}. \qquad (46)$$

The last formula is an analogue of the formula which determines the relative velocities in case of the inertial motion in the special theory of relativity. The last formula is true only if the transverse effects do not influence the radial effects. It can be verified optically, because we know that the optical frequency of the emission source is influenced by acceleration.

Similarly, it is possible to verify the dependence of mass on acceleration, also by the ultracentrifuge, or immediately by physics in LHC, or ELI.

3 The Thomas precession by an uniform acceleration

Thomas precession, named after Llewellyn Thomas, is a relativistic motion of a particle following a curvilinear orbit. Algebraically, it is a result of the non-commutativity of Lorentz transformations. Thomas precession is a kinematic effect in the flat spacetime of special relativity. This rotation is called Thomas rotation, Thomas and Wigner rotation or Wigner rotation. The rotation was discovered by Thomas (Thomas, 1926) and derived by Wigner (Wigner, 1939). If a sequence of non-collinear boosts returns an object to its initial velocity, then the sequence of Wigner rotations can combine to produce a net rotation called the Thomas precession (Rhodes et al., 2005).

Thomas precession is always accompanied by dynamical effects (Malykin, 2006). We calculate here Thomas precession caused by accelerated motion of the systems. In other words we show that Thomas precession can be initiated by acceleration of a point particle. The problem of acceleration of charged particles or systems of particles is the permanent and the most prestige problem in the accelerator physics. Particles can be accelerated by different ways. Usually by the classical electromagnetic fields, or, by light pressure of the laser fields (Baranova et al., 1994; Pardy, 1998, 2001a, 2002a). The latter method is the permanent problem of the laser physics for many years.

We have determined transformations between coordinate systems which moved mutually with the uniform acceleration (Pardy, 2003, 1974, 2004, 2005). They involved so called maximal acceleration discussed also in journals (Caianiello, 1981; Lambiase et al., 1998; Papini, 2003).

The Lorentz transformation between two inertial coordinate systemssystems $S(0, x, y, z)$ and $S'(0, x', y', z')$ (where system S' moves in such a way that x-axes converge, while y and z-axes run parallel and at time $t = t' = 0$ for the origin of the systems O and O' it is $O \equiv O'$) is as follows:

$$x' = \gamma(v)(x - vt), \quad y' = y, \quad z' = z', \quad t' = \gamma(v)\left(t - \frac{v}{c^2}x\right), \tag{1}$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2}$$

The infinitesimal form of this transformation is evidently given by differentiation of the every equation. Or,

$$dx' = \gamma(v)(dx - vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(v)\left(dt - \frac{v}{c^2}dx\right). \tag{3}$$

It follows from eqs. (3) that if v_1 is velocity of the inertial system 1 with regard to S and v_2 is the velocity of the inertial systems 2 with regard to 1, then the relativistic sum of the two velocities is

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \tag{4}$$

The infinitesimal form of Lorentz transformation (3) evidently gives the Lorentz length contraction and time dilation. Historical view on this effect is in the Selleri article (Selleri, 1997).

With regard to the previous chapter, the results in uniformly accelerated systems can be obtained from the old relativistic form $f(v/c)$ if we use transformation $v \leftarrow \frac{1}{2}a, \quad c \leftarrow \frac{1}{2}\alpha$.

So, let us first remind the relativistic derivation of the Thomas angle, which has form $f(v/c)$. In other words, we consider the inverse transformations to T_1 from S to S' with velocity $\mathbf{v}||x$, inverse transformation to T_2 from S' to S'' with velocity $\mathbf{u}||y$ and inverse transformation T_3 from S to S^+ with velocity $\mathbf{v}\oplus\mathbf{u}$ where mathematical symbol $\oplus$ is the expression for the relativistic addition of the velocities $\mathbf{v},\mathbf{u}$. Then $S=S^+$, if S^+ is turned in the xy plane with angle φ, which is given by the formula (Tomonaga, 1997):

$$\varphi = \arctan\frac{uv(\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}-1)}{u^2\sqrt{1-\frac{v^2}{c^2}}+v^2\sqrt{1-\frac{u^2}{c^2}}}. \tag{5}$$

Now, let us perform the transformations gradually. Let be $S\to S'$, $\quad \mathbf{v}=(v,0,0)$, transformation. Or,

$$x=\gamma_v(x'+vt'),\quad y=y',\quad t=\gamma_v(t'+\frac{v}{c^2}x');\quad \gamma_v=\left(1-\frac{v^2}{c^2}\right)^{-1/2}. \tag{6}$$

Then, let be $S'\to S''$, $\quad \mathbf{u}=(0,u,0)$. Or,

$$x'=x'',\quad y'=\gamma_u(y''+ut''),\quad t'=\gamma_u(t''+\frac{u}{c^2}y'');\quad \gamma_u=\left(1-\frac{u^2}{c^2}\right)^{-1/2}. \tag{7}$$

The transformation from S to S'' is $S\to S''$. Or,

$$x=\gamma_v x''+\gamma_v\gamma_u\frac{vu}{c^2}y''+\gamma_v\gamma_u vt'',\quad y=\gamma_u y''+\gamma_u ut'', \tag{8}$$

$$t=\gamma_v\frac{v}{c^2}x''+\gamma_v\gamma_u\frac{u}{c^2}y''+\gamma_v\gamma_u t''. \tag{9}$$

Now, let us perform transformation from S to S^+, where S^+ moves with regard to S with velocity, which is the relativistic sum of $\mathbf{v}$ and $\mathbf{u}$, which is the velocity $\mathbf{v}\oplus\mathbf{u}$. Or, using formula (Batygin et al., 1970)

$$\mathbf{k}=\mathbf{v}\oplus\mathbf{u}=\frac{\mathbf{v}+\mathbf{u}+(\gamma_v-1)\frac{\mathbf{v}}{v^2}\left[\mathbf{v}\cdot\mathbf{u}+v^2\right]}{\gamma_v\left(1+\frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)}, \tag{10}$$

we get

$$\mathbf{k}=\mathbf{v}\oplus\mathbf{u}=\left(v,\frac{u}{\gamma_v},0\right);\quad \gamma_k=\left(1-\frac{k^2}{c^2}\right)^{-1/2}. \tag{11}$$

Then we have for radius vector $\mathbf{r}$ and time t we have transformations (Batygin et al., 1970):

$$\tilde{\mathbf{r}} = \mathbf{r}^+ + \mathbf{k}t^+ + (\gamma_k - 1)\frac{\mathbf{k}}{k^2}\left[\mathbf{k}\mathbf{r}^+ + k^2 t^+\right] \tag{12}$$

and

$$\tilde{t} = \gamma_k\left(t^+ + \frac{\mathbf{k}\mathbf{r}^+}{c^2}\right). \tag{13}$$

The t-transformation (13) can be expressed in variables t^+, x^+, y^+ as follows:

$$\tilde{t} = \gamma_k t^+ + \gamma_k \frac{v}{c^2}x^+ + \frac{\gamma_u}{\gamma_v}\frac{u}{c^2}y^+. \tag{14}$$

The last equation can be compared with the time transformation from S to S'', which is

$$t = \gamma_v\gamma_u t'' + \gamma_v\frac{v}{c^2}x'' + \gamma_v\gamma_u\frac{u}{c^2}y''. \tag{15}$$

Using $\gamma_k = \gamma_v\gamma_u$, we get two transformation of time following from eqs. (9) and (14):

$$S \to S'': t = \gamma_k t'' + \gamma_v\frac{v}{c^2}x'' + \gamma_k\frac{u}{c^2}y'', \tag{16}$$

$$S \to S^+: \tilde{t} = \gamma_k t^+ + \gamma_k\frac{v}{c^2}x^+ + \frac{\gamma_k}{\gamma_v}\frac{u}{c^2}y^+. \tag{17}$$

Now, let us perform rotation

$$x'' = x^+\cos\varphi + y^+\sin\varphi, \quad y'' = -x^+\sin\varphi + y^+\cos\varphi. \tag{18}$$

Then equation (17) is identical with eq. (18), if the angle φ is determined by equation

$$\varphi = \arctan\frac{(1 - \gamma_v\gamma_u)\,vu}{\gamma_v v^2 + \gamma_u u^2}. \tag{19}$$

The angle of rotation (19) is so called the Thomas angle of so called Thomas precession. With regard to the derived transformation of quantities $\mathbf{u}, \mathbf{v}, c$ to the uniformly accelerated system, or, $\mathbf{v} \to \mathbf{a}/2$, $\mathbf{u} \to \mathbf{w}/2$ $c \to \alpha/2$, we get immediately from the last formula (19) the Thomas precession angle:

$$\varphi = \arctan \frac{aw(\sqrt{1-\frac{a^2}{\alpha^2}}\sqrt{1-\frac{w^2}{\alpha^2}} - 1)}{a^2\sqrt{1-\frac{w^2}{\alpha^2}} + w^2\sqrt{1-\frac{a^2}{\alpha^2}}}, \tag{20}$$

which has the physical meaning of the Thomas precession caused by uniform acceleration. The last formula with uniform acceleration a and w can be used for the uniform equivalent gravity according to the principle of equivalence. It is not excluded that this formula will play the crucial role in modern physics with application for LHC in CERN.

4 The graviton energy loss of the binary with radiative corrections

4.1 Introduction

At present time the existence of the gravitational waves is confirmed thanks to the experimental proof of Taylor and Hulse who performed the systematic measurement of motion of the binary with pulsar PSR 1913+16. They found that the energy loss formula which follows from he Einstein general theory relativity is in accordance with their measurement.

The success was conditioned by the fact that the binary with the pulsar PSR 1913+16 is the gigantic system of two neutron stars emitting sufficient gravitational radiation for influencing the orbital motion of the binary at the observable scale.

Taylor and Hulse, working at the Arecibo radiotelescope, discovered the radiopulsar PSR 1913+16 in a binary in 1974 and it is now considered as the best general relativistic laboratory (Taylor, Jr., 1993).

Pulsar PSR 1913+16 is the massive body of the binary system where each of rotating pairs is of 1,4 times the mass of the Sun. These neutron stars rotate around each other in an orbit not much bigger than the Sun's diameter, with a period 7,8 hours. Every 59 ms the pulsar emits a short signal that is so clear that the arrival time of a 5-min string of the set of such signals can be resolved to within 15 μs.

A pulsar model based on strongly magnetized, rapidly spinning neutron stars was soon established as consistent with most of the known facts (Lyne, et al. 1968; Orsten et al., 1968) and the electrodynamical properties of it were studied (Gold, 1968) and shown to be plausibly capable of generating broadband radio noise detectable over interstelar distances. The binary pulsar PSR 1913+16 is now recognized as the harbinger of a new class of unusually short-period pulsars with numerous important applications.

Because the velocities and gravitational energies in a high-mass binary pulsar system can be significantly relativistic, strong-field and radiative effects come into play. The binary pulsar PSR 1913+16 provides significant tests of gravitation beyond the weak-field, slow-motion limit (Goldreich, et al., 1969; Damour, et al., 1992).

The goal of this article is not to repeat the derivation of the Einstein quadrupole formula, bacause it was performed many times in general relativity and also in source theory in the weak field limit at zero temperature (Manoukian, 1990). We will to show that in the framework of the source theory it is easy to determine the quantum energy loss formula of the binary system both in case with free graviton propagator and with graviton propagator with radiative corrections. It involves arbitrary strong gravity which overcomes all obstacles of the clasical gravity derivation.

Because the measurement of motion of the binaries goes on, we hope that the future experiments will verify the quantum version of the energy loss formula following from the source theory and that sooner or later the confirmation this formula will be established.

4.2 The source theory formulation of the problem

Source theory (Schwinger et al., 1976a; Dittrich, 1978; Schwinger, 1976b) was initially constructed for description of the particle physics situations occurring in the high-energy physics experiments. However, it was found that the original formulation simplifies the calculations in the electrodynamics and gravity where the interactions are mediated by photon or graviton respectively. The source theory of gravity forms the analogue of quantum electrodynamics because while in QED the interaction is mediated by the photon the gravitational interaction is mediated by the graviton (Schwinger, 1976b). The basic formula in the source theory is the vacuum-to-vacuum amplitude (Schwinger, et al., 1976a):

$$\langle 0_+|0_-\rangle = e^{\frac{i}{\hbar}W(S)}, \tag{1}$$

where the minus and plus tags on the vacuum symbol are causal labels, referring to any time before and after space-time region where sources are manipulated. The exponential form is introduced with regard to the existence of the physically independent experimental arrangements which has a simple consequence that the associated probability amplitudes multiply and corresponding W expressions add (Schwinger et al., 1976a; Dittrich, 1978).

In the flat space-time the field of gravitons is described by the amplitude (1) with the action ($c = 1$, in the following text) (Schwinger, 1976a):

$$W(T) =$$

$$4\pi G \int (dx)(dx')[T^{\mu\nu}(x)D_+(x-x')T_{\mu\nu}(x') - \frac{1}{2}T(x)D_+(x-x')T(x')], \quad (2)$$

where the dimensionality of $W(T)$ is the same as the dimensionality of the Planck constant $\hbar$. $T_{\mu\nu}$ is the tensor of momentum and energy, and for particle moving along the trajectory $\mathbf{x} = \mathbf{x}(t)$ it is defined by the equation (Weinberg, 1972)

$$T^{\mu\nu}(x) = \frac{p^\mu p^\nu}{E}\delta(\mathbf{x} - \mathbf{x}(t)), \quad (3)$$

where p^μ is the relativistic four-momentum of a particle with a rest mass m and

$$p^\mu = (E, \mathbf{p}) \quad (4)$$

$$p^\mu p_\mu = -m^2, \quad (5)$$

and the relativistic energy is defined by the known relation

$$E = \frac{m}{\sqrt{1 - \mathbf{v}^2}}, \quad (6)$$

where $\mathbf{v}$ is the three-velocity of the moving particle.

Symbol $T(x)$ in the formula (2) is defined as $T = g_{\mu\nu}T^{\mu\nu}$ and $D_+(x-x')$ is the graviton propagator whose explicit form will be determined later.

4.3 The power spectral formula in general

The probability of the persistence of vacuum is given by the following formula (Schwinger et al., 1976a):

$$|\langle 0_+|0_-\rangle|^2 = \exp\left\{-\frac{2}{\hbar}\mathrm{Im}W\right\} \stackrel{d}{=} \exp\left\{-\int dt d\omega \frac{1}{\hbar\omega}P(\omega, t)\right\}, \quad (7)$$

where the so called power spectral function $P(\omega, t)$ has been introduced (Schwinger et al., 1976a). In order to extract this spectral function from Im W, it is necessary to know the explicit form of the graviton propagator

$D_+(x-x')$. The physical content of this propagator is analogical to the photon propagator. It involves the property of spreading of the gravitons with velocity c. It means that its explicit form is just the same as of the photon propagator. With regard to Schwinger et al. (Schwinger et al., 1976a) the x-representation of $D(k)$ in eq. (2) is as follows:

$$D_+(x-x') = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} D(k), \tag{8}$$

where

$$D(k) = \frac{1}{|\mathbf{k}^2| - (k^0)^2 - i\epsilon}, \tag{9}$$

which gives

$$D_+(x-x') = \frac{i}{4\pi^2} \int_0^\infty d\omega \frac{\sin\omega|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|} e^{-i\omega|t-t'|}, \tag{10}$$

Now, using formulas (2), (7) and (10), we get the power spectral formula in the following form:

$$P(\omega,t) = 4\pi G\omega \int (d\mathbf{x})(d\mathbf{x}')dt' \frac{\sin\omega|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|} \cos\omega(t-t') \times$$

$$[T^{\mu\nu}(\mathbf{x},t)T_{\mu\nu}(\mathbf{x}',t') - \frac{1}{2} g_{\mu\nu}T^{\mu\nu}(\mathbf{x},t)g_{\alpha\beta}T^{\alpha\beta}(\mathbf{x}',t')]. \tag{11}$$

Because of definition (9), the general formula does not involve radiative corrections. In order to get the production of gravitons with the radiative corrections we must replace $D_+(x-x')$ by the more general propagator which involves radiative corrections.

4.4 The radiative corrections

We will investigate how the spectrum of the gravitational radiation is modified if we involve radiation correction corresponding to the virtual pair production and annihilation in the graviton propagator. Our analogue is the application of the photon propagator with radiative corrections for production of photons by the Cerenkov mechanism (Pardy, 1994). According to (Dittrich, 1978; Weinberg, 1972; Schwinger, 1973) the photon propagator in the Minkowski space-time with radiative correction is in the momentum representation of the form:

$$\tilde{D}(k) = D(k) + \delta D(k), \tag{12}$$

$$\tilde{D}(k) = \frac{1}{|\mathbf{k}|^2 - (k^0)^2 - i\epsilon} +$$

$$+ \int_{4m^2}^{\infty} dM^2 \frac{a(M^2)}{|\mathbf{k}|^2 - (k^0)^2 + \frac{M^2c^2}{\hbar^2} - i\epsilon}, \tag{13}$$

where m is mass of electron and the last term in equation (13) is derived on the virtual photon condition

$$|\mathbf{k}|^2 - (k^0)^2 = -\frac{M^2c^2}{\hbar^2}. \tag{14}$$

The weight function $a(M^2)$ has been derived in the following form (Schwinger et al., 1976a; Pardy, 1994):

$$a(M^2) = \frac{\alpha}{3\pi}\frac{1}{M^2}\left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2} \tag{15}$$

We suppose that the graviton propagator with the radiative correction forms the analogue of the photon propagator.

Now, with regard to the definition of the Fourier transform

$$D_+(x - x') = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} D(k), \tag{16}$$

we get for δD_+ the following relation ($c = \hbar = 1$):

$$\delta D_+(x - x') = \frac{i}{4\pi^2}\int_{4m^2}^{\infty} dM^2 a(M^2) \times$$

$$\times \int d\omega \, \frac{\sin[\omega^2 - M^2]^{1/2}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} e^{-i\omega|t-t'|}. \tag{17}$$

The function (17) differs from the the gravitational function "D_+" in (15) especially by the factor

$$\left(\omega^2 - M^2\right)^{1/2} \tag{18}$$

in the function 'sin' and by the additional mass-integral which involves the radiative corrections to the original power spectrum formula.

In order to determine the additional spectral function of produced gravitons, corresponding to the radiative corrections, we insert $D_+(x - x') + \delta D_+(x - x')$ into eq. (2), and using eq. (14) we obtain:

$$\delta P(\omega, t) = \frac{2G\omega}{\pi} \int (d\mathbf{x})(d\mathbf{x}')dt' \int_{4m^2}^{\infty} dM^2 a(M^2) \times$$

$$\times \frac{\sin[\omega^2 - M^2]^{1/2}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} \cos\omega(t - t') \times$$

$$[T^{\mu\nu}(\mathbf{x}, t) g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}(\mathbf{x}', t') - \frac{1}{2} g_{\mu\nu} T^{\mu\nu}(\mathbf{x}, t) g_{\alpha\beta} T^{\alpha\beta}(\mathbf{x}', t')]. \qquad (19)$$

4.5 The power spectral formula for the binary system

In case of the binary system with masses m_1 and m_2 we suppose that they move in a uniform circular motion around their centre of gravity in the xy plane with corresponding kinematical coordinates:

$$\mathbf{x}_1(t) = r_1(\mathbf{i}\cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t)) \qquad (20)$$

$$\mathbf{x}_2(t) = r_2(\mathbf{i}\cos(\omega_0 t + \pi) + \mathbf{j}\sin(\omega_0 t + \pi)) \qquad (21)$$

with

$$\mathbf{v}_i(t) = d\mathbf{x}_i/dt, \quad \omega_0 = v_i/r_i, \quad v_i = |\mathbf{v}_i|, \quad i = 1, 2 \qquad (22)$$

For the tensor of energy and momentum of the binary we have:

$$T^{\mu\nu}(x) = \frac{p_1^\mu p_1^\nu}{E_1} \delta(\mathbf{x} - \mathbf{x}_1(t)) + \frac{p_2^\mu p_2^\nu}{E_2} \delta(\mathbf{x} - \mathbf{x}_2(t)), \qquad (23)$$

where we have omit tensor $t_{\mu\nu}^G$ which is associated with the massles gravitational field distributed all over space and which is proportional to the gravitational constant G (Cho, et al., 1976):

After insertion of eq.(23) into eq. (11), we get (Pardy, 1983):

$$P_{total}(\omega, t) = P_1(\omega, t) + P_{12}(\omega, t) + P_2(\omega, t), \qquad (24)$$

where $(t' - t = \tau)$

$$P_1(\omega, t) = \frac{G\omega}{r_1\pi} \int_{-\infty}^{\infty} d\tau \frac{\sin[2\omega r_1 \sin(\omega_0\tau/2)]}{\sin(\omega_0\tau/2)} \cos\omega\tau \times$$

$$\left(E_1^2(\omega_0^2 r_1^2 \cos\omega_0\tau - 1)^2 - \frac{m_1^4}{2E_1^2} \right), \qquad (25)$$

$$P_2(\omega,t) = \frac{G\omega}{r_2\pi}\int_{-\infty}^{\infty} d\tau \frac{\sin[2\omega r_2 \sin(\omega_0\tau/2)]}{\sin(\omega_0\tau/2)} \cos\omega\tau \times$$

$$\left(E_2^2(\omega_0^2 r_2^2 \cos\omega_0\tau - 1)^2 - \frac{m_2^4}{2E_2^2}\right), \tag{26}$$

$$P_{12}(\omega,t) = \frac{4G\omega}{\pi}\int_{-\infty}^{\infty} d\tau \frac{\sin\omega[r_1^2 + r_2^2 + 2r_1r_2\cos(\omega_0\tau)]^{1/2}}{[r_1^2 + r_2^2 + 2r_1r_2\cos(\omega_0\tau)]^{1/2}} \cos\omega\tau \times$$

$$\left(E_1E_2(\omega_0^2 r_1 r_2 \cos\omega_0\tau + 1)^2 - \frac{m_1^2m_2^2}{2E_1E_2}\right). \tag{27}$$

Formulae (24)–(27) represent the power spectrum of gravitons without radiative corrections. The radiative correction contribution to those formulas can be expressed as follows:

$$\delta P_1(\omega,t) = \frac{G\omega}{r_1\pi}\int_{M_1^2}^{M_2^2} dM^2\, a(M^2) \times$$

$$\int_{-\infty}^{\infty} d\tau \frac{\sin[2r_1[\omega^2 - M^2]^{1/2} \sin(\omega_0\tau/2)]}{\sin(\omega_0\tau/2)} \cos\omega\tau \times$$

$$\left(E_1^2(\omega_0^2 r_1^2 \cos\omega_0\tau - 1)^2 - \frac{m_1^4}{2E_1^2}\right), \tag{28}$$

$$\delta P_2(\omega,t) = \frac{G\omega}{r_2\pi}\int_{M_1^2}^{M_2^2} dM^2\, a(M^2) \times$$

$$\int_{-\infty}^{\infty} d\tau \frac{\sin[2r_2[\omega^2 - M^2]^{1/2} \sin(\omega_0\tau/2)]}{\sin(\omega_0\tau/2)} \cos\omega\tau \times$$

$$\left(E_2^2(\omega_0^2 r_2^2 \cos\omega_0\tau - 1)^2 - \frac{m_2^4}{2E_2^2}\right), \tag{29}$$

$$\delta P_{12}(\omega,t) = \frac{4G\omega}{\pi}\int_{M_1^2}^{M_2^2} dM^2\, a(M^2)$$

$$\int_{-\infty}^{\infty} d\tau \frac{\sin[\omega^2 - M^2]^{1/2}[r_1^2 + r_2^2 + 2r_1r_2\cos(\omega_0\tau)]^{1/2}}{[r_1^2 + r_2^2 + 2r_1r_2\cos(\omega_0\tau)]^{1/2}} \cos\omega\tau \times$$

$$\left(E_1E_2(\omega_0^2 r_1 r_2 \cos\omega_0\tau + 1)^2 - \frac{m_1^2m_2^2}{2E_1E_2}\right). \tag{30}$$

4.6 The quantum energy loss of the binary

In this section we determine the quantum energy loss of the binary caused by its production of gravitons in case the graviton Green function does not involve radiative corrections and with radiative corrections. We follow here the derivation of Pardy (Pardy, 1983). We will show that while the quantum energy loss without radiative corrections can be solved exactly in the framework of the source theory, the presence of radiative corrections makes the problem more complex and the solution of it can be achieved with the arbitrary accuracy. First, let us approach the problem with the non-modified propagator of graviton.

Using the following relations

$$\omega_0\tau = \varphi + 2\pi l, \qquad \varphi \in (-\pi, \pi), \quad l = 0, \pm 1, \pm 2, ... \tag{31}$$

$$\sum_{l=-\infty}^{l=\infty} \cos 2\pi l \frac{\omega}{\omega_0} = \sum_{l=-\infty}^{\infty} \omega_0 \delta(\omega - \omega_0 l), \tag{32}$$

defining P_l by relation

$$P(\omega, t) = \sum_{l=1}^{\infty} \delta(\omega - \omega_0 l) P_l(\omega, t), \tag{33}$$

and using the definition of the Bessel function $J_{2l}(z)$

$$J_{2l}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \cos\left(z \sin\frac{\varphi}{2}\right) \cos l\varphi, \tag{34}$$

from which the derivatives and integral of it follow, we get for P_{1l} and P_{2l} the following formulas:

$$P_{il} = \frac{2G\omega}{r_i\pi}((E_i^2(v_i^2 - 1) - \frac{m_i^4}{2E_i^2})\int_0^{2v_il} dx\, J_{2l}(x) \quad +$$

$$4E_i^2(v_i^2-1)v_i^2 J'_{2l}(2v_il) + 4E_i^2 v_i^4 J'''_{2l}(2v_il)), \quad i = 1, 2. \tag{35}$$

Using $r_2 = r_1 + \epsilon$, where ϵ is supposed small in comparison with radii r_1 and r_2 we get

$$[r_1^2 + r_2^2 + 2r_1r_2\cos\varphi]^{1/2} \approx 2a\cos\left(\frac{\varphi}{2}\right), \tag{36}$$

$$a = r_1\left(1 + \frac{\epsilon}{2r_1}\right). \tag{37}$$

Using the alternative definition of the Bessel function $J_{2l}(z)$,

$$J_{2l}(z) = \frac{(-1)^l}{2\pi}\int_{-\pi}^{\pi} d\varphi \cos\left(z\cos\frac{\varphi}{2}\right)\cos l\varphi, \tag{38}$$

we easily derive the corresponding derivatives and integral which we can use for evaluation of interference term P_{12l}. We get the following approximative formula ($v = a\omega_0$):

$$P_{12l} = (-1)^l \frac{4G\omega}{a\pi}\left(AJ_{2l}'''(2vl) - BJ_{2l}'(2vl) + C\int_0^{2vl} dy\, J_{2l}(y)\right) \tag{39}$$

with

$$A = E_1E_2v_1^2v_2^2 \tag{40}$$

$$B = 4E_1E_2v_1v_2(1 - v_1v_2) \tag{41}$$

$$C = E_1E_2\left(1 - 2v_1v_2 + v_1^2v_2^2\right) - \frac{m_1^2m_2^2}{E_1E_2}. \tag{42}$$

Now, we can approach the evaluation of the energy loss formula for the binary from the power spectral formulas (35) and (39). The energy loss is defined by the relation

$$-\frac{dU}{dt} = \int P(\omega)d\omega =$$

$$\int d\omega \sum_l \delta(\omega - \omega_0 l)P_l = -\frac{d}{dt}(U_1 + U_2 + U_{12}). \tag{43}$$

According to [17] we have:

$$\sum_{l=1} \frac{J_{2l}(2lv)}{l^2} = \frac{v^2}{2}. \tag{44}$$

After derivation of the last relation with regard to v we have

$$\sum_{l=1} lJ_{2l}'''(2lv) = 0. \tag{45}$$

The second and the third relation which is necessary to know are

$$\sum_{l=1} 2lJ_{2l}'(2lv) = \frac{v}{(1 - v^2)^2} \tag{46}$$

and

$$\sum_{l=1} l \int_0^{2lv} J_{2l}(x)dx = \frac{v^2}{3(1-v^2)^3}. \tag{47}$$

So, after application of sums (45), (46) and (47) to equations (35) and (39), we get:

$$-\frac{dU_i}{dt} = \frac{Gm_i^2 v_i^2 \omega_0}{3\pi r_i (1-v_i^2)^3}[6v_i^3 + v_i^2 - 6v_i - 3] \tag{48}$$

and

$$-\frac{dU_{12}}{dt} = \sum_{l=1}(-1)^l \frac{4Gl\omega_0}{a\pi}\left(AJ_{2l}'''(2vl) - BJ_{2l}'(2vl) + C\int_0^{2vl} dy\, J_{2l}(y)\right), \tag{49}$$

where we did not evaluate here the corresponding Kapteyn's series. In ref. (Prudnikov, et al., 1983) is shown the second method of evaluation of the energy loss by the direct ω-integration.

Let us remark finally that derived formulas for the energy loss of the binary (48) and (49) are general and therefore their sum which forms the total produced gravitational energy need not to be in coincidence with the Einstein quadrupole formula because the Einstein derivation is based on the linearized version of classical gravity.

The evaluation of the additional energy loss corresponding to the extension of the gravitational propagator involving the radiative corrections consists in exact evaluation of eqs. (29)–(30). However because of the specific form of the integrals in these formulas the problem can be solved only approximately. The very simple method is to use Taylor expansion in the argument

$$z_i = 2[v_i^2 l^2 - M^2 r_i^2]^{1/2} \tag{50}$$

or

$$z_{12} = 2[v^2 l^2 - M^2 a^2]^{1/2} \tag{51}$$

or

$$z_i \approx 2v_i l + \delta z_i, \quad z_{12} \approx 2vl + \delta z_{12} \tag{52}$$

with

$$\delta z_i = -\frac{M^2 r_i^2}{v_i l}, \quad \delta z_{12} = -\frac{M^2 a^2}{vl} \tag{53}$$

From the inequality $[v^2l^2 - M^2r^2]^{1/2} \geq 0$ we get the minimal l is given by the condition $l = 2mr/v$ where m is the mass of electron and r and v are corresponding radii and velocities. It may be easy to show that $l \gg 1$ and therefore δz are sufficiently small quantities. At the same time the fine structure constant α which is involved in the mass integral is also very small. So we can let only the first term in the Taylor expansion. So in our approximation we obtain the following formulas for the additional power spectrum :

$$\delta P_{il} = \int_{M_1^2}^{M_2^2} a(M^2) dM^2 P_{il} \tag{54}$$

with

$$M_1^2 = 4m^2, \qquad M_2^2 = \frac{l^2 v_i^2}{r_i^2} \tag{55}$$

and

$$\delta P_{12l} = \int_{M_1^2}^{M_2^2} a(M^2) dM^2 P_{12l} \tag{56}$$

with

$$M_1^2 = 4m^2, \qquad M_2^2 = \frac{l^2 v^2}{a^2} \tag{57}$$

Using the substitution

$$t = \left(1 - 4m^2/M^2\right)^{1/2} \tag{58}$$

we get the mass integrals in the form (Pardy, 1994; Schwinger, 1973):

$$I_i = \int_{M_1^2}^{M_2^2} a(M^2) dM^2 = \frac{\alpha}{3\pi}\left\{\frac{s_i^2}{3} - 2s_i + \ln\left|\frac{1+s_i}{1-s_i}\right|\right\} \tag{59}$$

$$I_{12} = \int_{M_1^2}^{M_2^2} a(M^2) dM^2 = \frac{\alpha}{3\pi}\left\{\frac{s_{12}^2}{3} - 2s_{12} + \ln\left|\frac{1+s_{12}}{1-s_{12}}\right|\right\} \tag{60}$$

where

$$s_i = \left(1 - \frac{4m^2 r_i^2}{v_i^2 l^2}\right)^{1/2} \tag{61}$$

and

$$s_{12} = \left(1 - \frac{4m^2a^2}{v^2l^2}\right)^{1/2} \tag{62}$$

Then

$$\delta P_{il} \approx I_i(l)P_{il} \tag{63}$$

$$\delta P_{12l} \approx I_{12}(l)P_{12l} \tag{64}$$

and also

$$-\delta\frac{dU_i}{dt} \approx \sum_{l_i=l_{i(min)}}^{\infty} I_i(l)P_i, \tag{65}$$

$$-\delta\frac{dU_{12}}{dt} \approx \sum_{l_{12}=l_{12(min)}}^{\infty} I_{12}(l)P_{12l}, \tag{66}$$

where for the minimal values of quantities l we have the following relations:

$$l_{i(min)} = \frac{4m^2r_i^2}{v_i^2}, \qquad l_{12(min)} = \frac{4m^2a^2}{v^2}. \tag{67}$$

Because of the minimal values of l being very large in comparison with 1, we can use the asymptotical evaluation of the spectrum of gravitons. In other words we shall consider the so called high-energy gravitons, which can generate the one-loop radiative corrections.

The evaluation of the power spectral formulas for high-energy gravitons corresponds to evaluation of the Bessel functions, their derivatives and integrals for large l.

Using the formulas

$$J_{2l}'''(2lv) \sim -\left(1 - \frac{1}{v^2}\right) J_{2l}'(2lv), \quad l \gg 1 \tag{68}$$

$$J_{2l}'(2lv) \sim \frac{1}{\sqrt{3}}\frac{1}{\pi}\left(\frac{3}{2l_1}\right)^{2/3} K_{2/3}(l/l_1), \quad l \gg 1 \tag{69}$$

$$\int_0^{2lv} J_{2l}(y)dy \sim \frac{1}{\sqrt{3}}\frac{1}{\pi}\int_{l/l_1}^{\infty} K_{1/3}(y)dy, \quad l \gg 1 \tag{70}$$

$$l_1 = \frac{3}{2}(1 - v^2)^{-3/2} \tag{71}$$

$$K_{2/3}' = -\frac{1}{2}(K_{1/3} + K_{5/3}) \tag{72}$$

$$\int_{\xi}^{\infty} K_{5/3} dy \sim \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}, \quad \xi = l/l_1, \quad \xi \gg 1 \tag{73}$$

$$K_\nu \sim \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}, \quad \xi = l/l_1, \quad \xi \gg 1, \tag{74}$$

we get explicit asymptotic forms of the power spectral formulas:

$$P_{il}(\omega) \sim \frac{G\omega m^2}{r_i} \left(\frac{1}{6\pi\xi_i}\right)^{1/2} e^{-\xi_i}(v_i^2 - 3), \tag{75}$$

where

$$\xi_i = \frac{2}{3}\frac{\omega}{\omega_0}(1 - v_i^2)^{3/2}, \quad l = \omega/\omega_0. \tag{76}$$

Similarly,

$$P_{12l}(\omega, t) \sim \frac{4G\omega}{a}(-1)^l \left(A\frac{(1 - v^2)^2}{v^2} - B(1 - v^2) + C\right) \left(\frac{1}{6\pi\xi}\right)^{1/2} e^{-\xi}, \tag{77}$$

where

$$\xi = \frac{2}{3}\frac{\omega}{\omega_0}(1 - v^2)^{3/2}, \quad l = \omega/\omega_0, \quad a = r_1 + \epsilon/2, \quad v = a\omega_0, \tag{78}$$

and finally,

$$P_{(total)l} = P_{1l} + P_{2l} + P_{12l}. \tag{79}$$

The partial spectral formulas P_{il} and P_{12} have the following simple forms:

$$P_{il} \sim K_i L_i^{-l} l^{1/2}, \tag{80}$$

with

$$K_i = \frac{G\omega_0 m^2}{2r_i} \left(\frac{1}{\pi}\right)^{1/2} \left(1 - v_i^2\right)^{-3/4} \left(v_i^2 - 3\right) \tag{81}$$

and with

$$L_i = \exp\left\{\frac{2}{3}(1 - v_i^2)^{3/2}\right\}. \tag{82}$$

For interference term, we have:

$$P_{12l} \sim (-1)^l K_{12} L_{12}^{-l} l^{1/2} \tag{83}$$

with

$$K_{12} = \frac{G\omega_0 m^2}{a}\left[\frac{A}{v^2}(1-v^2)^2 - B(1-v^2) + C\right]\left(\frac{1}{4\pi}\right)^{3/2}\left(\frac{1}{1-v^2}\right)^{9/4} \tag{84}$$

and with

$$L_{12} = \exp\left\{\frac{2}{3}(1-v^2)^{3/2}\right\} \tag{85}$$

So

$$-\delta\frac{dE_i}{dt} \approx K_i \sum_{l=l_{i(min)}}^{\infty} I_i(l) L_i^{-l} l^{1/2}, \tag{86}$$

and

$$-\delta\frac{dE_{12}}{dt} \approx K_{12} \sum_{l=l_{12(min)}}^{\infty} (-1)^l I_{12}(l) L_{12}^{-l} l^{1/2} \tag{87}$$

and the additional quantum loss of energy caused by the radiative processes in the graviton propagator is in such a way given approximately by the last formulas.

The explicit evaluation of sums in the formulas (86) and (87) can be performed approximately using the so called Euler-McLaurin formula:

$$\sum_{l=a}^{l=b} f(l) = \int_a^b f(x)dx + \frac{1}{2}(f(b)+f(a)) + \frac{1}{12}(f'(b)-f'(a)) + ..., \tag{88}$$

where in our case it is $a = l_{i(min)}$ or $a = l_{12(min)}$ and $b = \infty$.

It is possible to show that $\delta P_i(l_{i(min)}) = 0$, $\delta P_i'(l_{i(min)}) \approx 0$, and At point $l = \infty$, we get $P_{il}(l = \infty) = 0$. So, we can transcribe eq. (86) in the following approximation:

$$-\delta\frac{dU_i}{dt} \approx K_i \int_{l=l_{i(min)}}^{\infty} dl I_i(l) L_i^{-l} l^{1/2}. \tag{89}$$

The sum in the right-side of equation (87) is evidently zero because of of the oscillating terms in the right-hand series. So The additional energy-loss of the binary, which is generated by the switching one loop radiative corrections in the graviton propagator is reduced to the formula (89). To our knowledge this formulas was never derived in any version of quantum gravity.

5 The Cherenkov radiation of gravitons in the Schwinger gravity

5.1 Introduction

The fast moving charged particle in a medium when its speed is faster than the speed of light in this medium produces electromagnetic radiation which is called the Vavilov-Cherenkov radiation. The prediction of the Cherenkov radiation came long ago. Heaviside in (1889) investigated the possibility of a charged object moving in a medium faster than electromagnetic waves in the same medium becomes a source of directed electromagnetic radiation. Kelvin (1901) presented an idea that the emission of particles is possible at a speed greater than that of light. Somewhat later, Sommerfeld (1904) proposed the hypothetical radiation with a sharp angular distribution. However, in fact, from experimental point of view, the electromagnetic Cherenkov radiation was first observed in the early 1900s by experiments developed by Marie and Pierre Curie when studying radioactivity emission. In essence they observed the emission of a bluish-white light from transparent substances in the neighborhood of strong radioactive source. But the first attempt to understand the origin of this was made by Mallet (Mallet, 1926; 1929a; 1929b) who observed that the light emitted by a variety of transparent bodies placed close to a radioactive source always had the same bluish-white quality, and that the spectrum was continuous, with no line or band structure characteristic of fluorescence. Unfortunately, these investigations were forgotten for many years. Cherenkov experiments (Cherenkov, 1934) were performed at the suggestion of Vavilov who opened a door to the true physical nature of this effect (Bolotovsky, 2009).

This radiation was first theoretically interpreted by Tamm and Frank (Tamm et al., 1937) in the framework of the classical electrodynamics. The source theoretical description of this effect was given by Schwinger et al. (1976a) at the zero temperature regime and the classical spectral formula was generalized to the finite temperature situation and for the massive photons by author Pardy (1989; 2002b). The Vavilov-Cherenkov effect was also used by author (1997) to possible measurement of the Lorentz contraction.

The gravitational Cherenkov radiation as the analogue of the electromagnetic effect is obviously conditioned by the existence of the gravitational index of refraction. There is a number of discussions concerning of the propagation of the gravitational waves in the bulk matter with the gravitational index of refraction. Szekeres (1971) has found the index of refraction of the gravitational waves propagating through matter which is composed of particles in which the incident wave induces quadrupole moments. Polnarev (1972) and Chesters (1973) have discussed the interaction of the gravitational waves in a hot gas and Peters (1974) has calculated the index of refraction of a cold gas of free particles.

In classical electrodynamics, the existence of the Cherenkov radiation is the natural consequence of the existence of the index of refraction of a medium. In the analogical gravitational situation the gravitational Cherenkov radiation is the natural consequence of the existence of the gravitational index of refraction.

In our article we do not consider the microscopical mechanism generating the gravitational index of refraction, however, we define the index of refraction by metric $g_{\mu\nu}$ which is involved in the equation for the Green function in the background gravitational field with metric $g_{\mu\nu}$:

$$\partial_\mu(\sqrt{-g}\partial^\mu)D_{+g}(x) = -\frac{1}{\sqrt{-g}}\delta(x) \tag{1}$$

where g is the determinant of the $g_{\mu\nu}$. Now, if we define the background metric by the following equations

$$g_{k0} = 0,\ g_{kl} = \delta_{kl},\ g_{00} = -n^2 \tag{2}$$

then, the left side of equation (1) is just the left side of the wave equation with the index of refraction n and obviously the Green function defined by eq. (1) is the Green function for propagation of massless particles in the background medium with velocity $c' = c/n$ and not c.

On such conditions we derive in this article the power spectral formula of gravitons in the framework of the Schwinger source theory (Schwinger et al., 1976a; Schwinger, 1970) at zero temperature and using the finite-temperature graviton propagator we generalize the result for the nonzero temperature situation.

First, we generalize the graviton action to the situation with the general metric $g_{\mu\nu}$ and then we specify the metric by relations (2). The derivation of the power spectral formula is analogical to the electromagnetic case. The obtained result is the gravitational analogue of the Frank-Tamm formula for the electromagnetic Cherenkov radiation. The finite-temperature gravitational Cherenkov radiation is derived here by the finite temperature procedure (Pardy, 1989).

5.2 The source theory formulation of the problem in the Riemann space-time

Source theory (Schwinger et al., 1976a; Schwinger, 1970) is the theoretical construction which uses quantum-mechanical particle language. Initially it was constructed for description of the particle physics situations occurring in the high-energy physics experiments. However, it was found that the original formulation simplifies the calculations in the electrodynamics and gravity where the interactions are mediated by photon or graviton respectively. The special values of mass and spin of photon or graviton combined with the general laws of quantum mechanics and special relativity are so restrictive that the essential frameworks of these fundamental theories are such analogical that it is possible to speak of the methodological unification of electromagnetism and gravity (Schwinger, 1976b). It means that the analogy can be expected also in case of the specific situation of production of gravitons by motion of particles in medium of the gravitational index of refraction n.

The basic formula of the source theory is the vacuum to vacuum amplitude (Schwinger, 1970):

$$< 0_+|0_- >= e^{\frac{i}{\hbar}W(S)} \tag{3}$$

where the minus and plus tags on the vacuum symbol are causal labels,referring to any time before and after space-time region where sources are manipulated. The exponential form is introduced with regard to the existence of the physically independent experimental arrangements which has a simple consequence that the associated probability amplitudes multiply and corresponding W expressions add (Schwinger et al., 1976a; Schwinger, 1970).

In the flat space-time the field of gravitons is described by the amplitude (3) with the action

$$W(T) =$$

$$\frac{4\pi G}{c^4}\int(dx)(dx')[T^{\mu\nu}(x)D_+(x-x')T_{\mu\nu}(x') - \frac{1}{2}T(x)D_+(x-x')T(x')] \tag{4}$$

where the dimensionality of $W(T)$ is the same as the dimensionality of the Planck constant $\hbar$. $T_{\mu\nu}$ is the tensor of momentum and energy and for particle moving along the trajectory $\mathbf{x} = \mathbf{x}(t)$ it is defined by the equation

$$T^{\mu\nu}(x) = c^2 \frac{p^\mu p^\nu}{E} \delta(\mathbf{x} - \mathbf{x}(t)) \tag{5}$$

where p^μ is the relativistic four-momentum of a particle with a rest mass m and

$$p^\mu = (E/c, \mathbf{p}) \tag{6}$$

$$p^\mu p_\mu = -m^2 c^2 \tag{7}$$

and the relativistic energy is defined by the known relation

$$E = \frac{mc^2}{\sqrt{1 - \mathbf{v}^2/c^2}} \tag{8}$$

where $\mathbf{v}$ being the three velocity of the moving particle.

Symbol $T(x)$ in the formula (4) is defined as $T = g_{\mu\nu} T^{\mu\nu}$ and symbol $D_+(x - x')$, is the graviton propagator and its explicit form will be determined later.

In case of the non-flat space-time with the general metric $g_{\mu\nu}$ there exists the system of rules how to transcribe the action $W(T)$. It follows from the general relativity theory (Weinberg, 1972) that all equations and formulas are influenced by gravity in the presence of the gravitational field expressed by the metrical tensor $g_{\mu\nu}$ .The general method how to involve the effect of gravity on mechanics and electrodynamics consists first in formulating the equations of motion from the viewpoint of the special theory of relativity and then in formulating them in the general covariant way which is equivalent to the situation with the gravitational field on condition that the system is sufficiently small in comparison with the scale of the fields. According to Weinberg (1972) the rules generating the general covariance are as follows:

$$(dx) \longrightarrow \sqrt{-g}(dx) \tag{9}$$

$$T_{\mu\nu} \longrightarrow \frac{1}{\sqrt{-g}} T_{\mu\nu} \tag{10}$$

$$T_{\mu\nu} \longrightarrow g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta} \tag{11}$$

$$D_+ \longrightarrow D_{+g}(x, x'), \tag{12}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. Function $D_{+g}(x,x')$ is the graviton propagator in the gravitational field and in our case it is the graviton propagator in the metric corresponding to the gravitational index of refraction n.

In such a way we get the action $W(T)$ embedded into the space-time with metric $g_{\mu\nu}$ has the following form:

$$W(T) = \frac{4\pi G}{c^4}\int (dx)(dx')[T^{\mu\nu}(x)g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}(x')D_{+g}(x,x')-$$

$$-\frac{1}{2}g_{\mu\nu}T^{\mu\nu}(x)D_{+g}(x,x')g_{\alpha\beta}T^{\alpha\beta}(x')] \tag{13}$$

The formula (13) describes the interaction of the particle with zero mass and spin 2 and spirality ± 2—(graviton)—with the metric field of the external gravity. The procedure of derivation of the general covariant action is in agreement with discussion in the Yilmaz (1975) article concerning gravity and source theory.

5.3 The power spectral formula

It may be easy to show that the probability of the persistence of vacuum is given by the following formula (Schwinger et al., 1976a):

$$|<0_+|0_->|^2 = exp\{-\frac{2}{\hbar}\mathrm{Im}W\} \stackrel{d}{=} exp\{-\int dt d\omega \frac{c}{\hbar\omega}P(\omega,t)\} \tag{14}$$

where we have introduced the so called power spectral function (Schwinger et al., 1976a) $P(\omega,t)$. In order to extract this spectral function from $\mathrm{Im}W$, it is necessary to know the explicit form of the graviton propagator $D_{+g}(x-x')$.The physical content of this propagator is analogical to the photon propagator . It involves the property of spreading of the gravitons with velocity c/n. It means that its explicit form is just the same as of the photon propagator. With regard to Schwinger et al. (1976a) and eq. (1) with metric (2), we can therefore write for our problem:

$$D_{+g}(x-x') = \frac{1}{n^2}\int_0^\infty \frac{(dk)}{(2\pi)^4}\frac{e^{ik(x-x'}}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} =$$

$$\frac{i}{4\pi^2 cn^2}\int_0^\infty d\omega \frac{\sin\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|}e^{-i\omega|t-t'|} \tag{15}$$

Now, using formulas (13), (14) and (15), we get the power spectral formula in the following form:

$$P(\omega,t) = \frac{4\pi G}{c^4 n^2}\int (d\mathbf{x})(d\mathbf{x}')dt'\frac{\sin\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|}{|\mathbf{x}-\mathbf{x}'|}\cos\omega(t-t')\times$$

$$[T^{\mu\nu}(\mathbf{x},t)g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}(\mathbf{x}',t') - \frac{1}{2}g_{\mu\nu}T^{\mu\nu}(\mathbf{x},t)g_{\alpha\beta}T^{\alpha\beta}(\mathbf{x}',t')] \qquad (16)$$

The Cherenkov radiation in electrodynamics is produced in the linear case by uniformly moving charge with the constant velocity $\mathbf{v} = (v,0,0)$. In the gravitational situation the gravitational Cherenkov radiation is generated by the energy-momentum tensor of the uniformly linearly moving particle with the rest mass m and with the constant velocity $\mathbf{v}$. If we insert the tensor of the energy-momentum of the particle moving along the trajectory $\mathbf{x} = \mathbf{v}t$ into eq. (16), then using the metric tensor (2) and $\tau = t - t'$, and $\beta = v/c$, we get instead of the formula (16) the following relation:

$$P(\omega,t) = \frac{G\omega}{\pi v n^2}\frac{m^2}{1-\beta^2}\beta^4[1+\frac{n^2}{\beta^2}]^2\int_{-\infty}^{\infty} d\tau\frac{\sin n\omega\beta\tau}{\tau}\cos\omega\tau \qquad (17)$$

The formula (17) contents the known integral:

$$\int_{-\infty}^{\infty} d\tau\frac{\sin n\omega\beta\tau}{\tau}\cos\omega\tau = \begin{cases} \pi & n\beta > 1 \\ 0 & n\beta < 1 \end{cases} \qquad (18)$$

Using the integral (18) we finally get the power spectral formula of the produced gravitons:

$$P(\omega,t) = \frac{G\omega}{v n^2}\frac{m^2}{1-\beta^2}\beta^4[1+\frac{n^2}{\beta^2}]^2;\ n\beta > 1 \qquad (19)$$

and $P(\omega,t) = 0$ for $n\beta < 1$.

The power spectral formula (19) is the gravitational analogue of the Frank-Tamm formula for the Cherenkov radiation in electrodynamics.

The dimensionality of $P(\omega,t)$ is erg because $[G] = cm^3g^{-1}s^{-2}, [\omega] = s^{-1}, [m^2] = g^2, [v^{-1}] = cm^{-1}s$. This is in agreement with the definition of the power spectral formula involved in the energy loss equation of the produced radiation:

$$-\frac{dE}{dt} = \int d\omega P(\omega,t) \qquad (20)$$

5.4 Finite-temperature contribution

The finite-temperature quantum field theory (QFT) was developed a decade ago and at present time is intensively studied. The first formulation of the finite-temperature QFT was presented by Dolan and Jackiw (1974), Weinberg (1974) and Bernard (1974) and the first application of it concerned the effective potential in Higgs theories.

The quantum chromodynamics (QCD) was also studied at finite temperature and densities using the temperature Green functions (Kalashnikov, 1984). The systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one loop order was elaborated by Donoghue et al. (1985) and by Johansson et al. (1986). The finite temperature Cherenkov electrodynamical power spectral formula in source theory was also derived (Pardy, 1989).

Here we use the Pardy procedure in order to generalize the formula (19) to the finite-temperature regime. It consists in the real-time formulation in the following transformation in the graviton propagator (15):

$$\frac{1}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} \longrightarrow \frac{1}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} + \frac{2\pi i}{e^{\frac{|E|}{k_B T}} - 1}\delta(|\mathbf{k}|^2 - n^2(k^0)^2) \tag{21}$$

where $E = \hbar\omega$ is the energy of graviton, k_B is the Boltzmann constant and T is temperature of the graviton gas in the gravitational medium with the index of refraction n. In such a way the considered situation is the analogue of the electrodynamic one.

The transformation (21) enables immediately separate the finite-temperature part of the Green function. After inserting eq. (21) into eq. (15) we get using some obvious matematical operations the temparature part of the D_{+g}-function in the following form:

$$D_{+gT}(x - x') = \frac{i}{2\pi^2 cn^2}\int_0^\infty d\omega \frac{\sin\frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|}\frac{\cos\omega(t - t')}{\exp(\hbar\omega/k_B T) - 1} \tag{22}$$

It is obvious that D_{+gT} is pure imaginary. Using definition (14), we get for the finite-temperature part of the spectral function the following formula:

$$P_T(\omega, t) = \frac{2}{\exp(\hbar\omega/k_B T) - 1}\times$$

$$\frac{4\pi G}{c^4 n^2}\int (d\mathbf{x})(d\mathbf{x}')dt' \frac{\sin\frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|}\cos\omega(t - t')\times$$

$$[T^{\mu\nu}(\mathbf{x},t)g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}(\mathbf{x}',t') - \frac{1}{2}g_{\mu\nu}T^{\mu\nu}(\mathbf{x},t)g_{\alpha\beta}T^{\alpha\beta}(\mathbf{x}',t')] \qquad (23)$$

The last formula differs from the zero-temperature formula only by the multiplicative factor $2/[\exp(\hbar\omega/ck_BT) - 1]$. The total spectral formula is given obviously by the relation

$$P_{\text{total}} = P_{T=0} + P_T = P_{T=0}\left(1 + \frac{2}{\exp(\hbar\omega/k_BT) - 1}\right) \qquad (24)$$

or, after some algebra and using formula (19):

$$P_{\text{total}} = \frac{G\omega}{vn^2}\frac{m^2}{1-\beta^2}\beta^4[1 + \frac{n^2}{\beta^2}]^2 \coth\left(\frac{\hbar\omega}{2k_BT}\right); \, n\beta > 1 \qquad (25)$$

and $P_{\text{total}} = 0$ for $n\beta < 1$.

The power spectral formula (25) is the finite-temperature generalization of the power spectral formula for the zero-temperature gravitational Cherenkov radiation (19). This formula was never derived in the conventional gravity and at the same time is original in the Schwinger source theory.

6 Energy shift of H-atom electrons due to Gibbons-Hawking thermal bath

6.1 Introduction

The Gibbons-Hawking effect is the statement that a temperature can be associated to each solution of the Einstein field equations that contains a causal horizon. It is named after Gary Gibbons and Stephen William Hawking.

Schwarzschild spacetime contains an event horizon and so can be associated with temperature. In the case of Schwarzschild spacetime this is the temperature T of a black hole of mass M, satisfying T/M .

De Sitter space which contains an event horizon has the temperature T proportional to the Hubble parameter H. We consider here the influence of the heat bath of the Gibbons-Hawking photons on the energy shift of H-atom electrons.

The considered problem is not in the scientific isolation, because some analogical problems are solved in the scientific respected journals. At present time it is a general conviction that there is an important analogy between black hole and the hydrogen atom. The similarity between black hole and the hydrogen atom was considered for instance by Corda (2015a), who discussed the precise model of Hawking radiation from the tunneling mechanism. In this article an elegant expression of the probability of emission is given in terms of the black hole quantum levels. So, the system composed of Hawking radiation and black hole quasi-normal modes introduced by Corda (2015b) is somewhat similar to the semiclassical Bohr model of the structure of a hydrogen atom.

The time dependent Schrödinger equation was derived for the system composed by Hawking radiation and black hole quasi-normal modes (Corda, 2015c). In this model, the physical state and the correspondent wave function are written in terms of an unitary evolution matrix instead of a density matrix. Thus, the final state is a pure quantum state instead of a mixed one and it means that there is no information loss. Black hole can be well defined as the quantum mechanical systems, having ordered, discrete quantum spectra, which respect 't Hooft's assumption that Schrödinger equations can be used universally for all dynamics in the universe.

Thermal photons by Gibbons and Hawking form so called blackbody, which has the distribution law of photons derived in 1900 by Planck (1900, 1901), (Schöpf, 1978). The derivation was based on the investigation of the statistics of the system of oscillators inside of the blackbody. Later Einstein (1917) derived the Planck formula from the Bohr model of atom where electrons have the discrete energies and the energy of the emitted photons are given by the Bohr formula $\hbar\omega = E_i - E_f$, E_i, E_f are the initial and final energies of electrons.

6.2 The modified Coulomb potential

Now, let us calculate the modified Coulomb potential due to blackbody. The starting point of the determination of the energy shift in the H-atom is the potential $V_0(\mathbf{x})$, which is generated by nucleus of the H-atom. The potential at point $V_0(\mathbf{x} + \delta\mathbf{x})$, evidently is (Akhiezer, et al., 1953; Welton, 1948):

$$V_0(\mathbf{x} + \delta\mathbf{x}) = \left\{1 + \delta\mathbf{x}\nabla + \frac{1}{2}(\delta\mathbf{x}\nabla)^2 + ...\right\} V_0(\mathbf{x}). \tag{1}$$

If we average the last equation in space, we can eliminate so called the effective potential in the form

$$V(\mathbf{x}) = \left\{1 + \frac{1}{6}(\delta\mathbf{x})^2_T\Delta + ...\right\} V_0(\mathbf{x}), \tag{2}$$

where $(\delta\mathbf{x})^2_T$ is the average value of te square coordinate shift caused by the thermal photon fluctuations. The potential shift follows from eq. (2):

$$\delta V(\mathbf{x}) = \frac{1}{6}(\delta\mathbf{x})^2_T \Delta V_0(\mathbf{x}). \tag{3}$$

The corresponding shift of the energy levels is given by the standard quantum mechanical formula (Akhiezer, et al., 1953)

$$\delta E_n = \frac{1}{6}(\delta\mathbf{x})^2_T (\psi_n \Delta V_0 \psi_n). \tag{4}$$

In case of the Coulomb potential, which is the case of the H-atom, we have

$$V_0 = -\frac{e^2}{4\pi|\mathbf{x}|}. \tag{5}$$

Then for the H-atom we can write

$$\delta E_n = \frac{2\pi}{3}(\delta\mathbf{x})^2_T \frac{e^2}{4\pi}|\psi_n(0)|^2, \tag{6}$$

where we used the following equation for the Coulomb potential

$$\Delta\frac{1}{|\mathbf{x}|} = -4\pi\delta(\mathbf{x}). \tag{7}$$

Motion of electron in electric field is evidently described by elementary equation

$$\delta\ddot{\mathbf{x}} = \frac{e}{m}\mathbf{E}_T, \tag{8}$$

which can be transformed by the Fourier transformation into the following equation

$$|\delta\mathbf{x}_{T\omega}|^2 = \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\mathbf{E}^2_{T\omega}, \tag{9}$$

where the index ω concerns the Fourier component of above functions.

On the basis of the Bethe idea of the influence of vacuum fluctuations on the energy shift of electron (Bethe, 1947), the following elementary relations was used by Welton (1948), Akhiezer et al. (1953) and Berestetzkii et al. (1999):

$$\frac{1}{2}\mathbf{E}^2_\omega = \frac{\hbar\omega}{2} \tag{10}$$

and in case of the thermal bath of the blackbody, the last equation is of the following form (Isihara, 1971):

$$\mathbf{E}^2_{T\omega} = \varrho(\omega) = \left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{11}$$

because the Planck law in (11) was written as

$$\varrho(\omega) = G(\omega) < E_\omega >= \left(\frac{\omega^2}{\pi^2 c^3}\right)\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{12}$$

where the term

$$< E_\omega >= \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \tag{13}$$

is the average energy of photons in the blackbody and

$$G(\omega) = \frac{\omega^2}{\pi^2 c^3} \tag{14}$$

is the number of electromagnetic modes in the interval $\omega, \omega + d\omega$.

Then,

$$(\delta \mathbf{x}_{T\omega})^2 = \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{15}$$

where $(\delta \mathbf{x}_{T\omega})^2$ involves the number of frequencies in the interval $(\omega, \omega + d\omega)$.

So, after some integration, we get

$$(\delta \mathbf{x})^2_T = \int_{\omega_1}^{\omega_2} \frac{1}{2}\left(\frac{e^2}{m^2\omega^4}\right)\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)\frac{d\omega}{e^{\frac{\hbar\omega}{kT}} - 1} = \frac{1}{2}\left(\frac{e^2}{m^2}\right)\left(\frac{\hbar}{\pi^2 c^3}\right)F(\omega_2 - \omega_1), \tag{16}$$

where $F(\omega)$ is the primitive function of the omega-integral

$$J = \frac{1}{\omega}\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}, \tag{17}$$

which cannot be calculated by the elementary integral methods and it is not involved in the tables of integrals.

Frequencies ω_1 and ω_2 will be determined with regard to the existence of the fluctuation field of thermal photons. It was determined in case of the Lamb shift (Bethe, 1947 ; Welton, 1947) by means of the physical analysis of the interaction of the Coulombic atom with the surrounding fluctuation field. We suppose here that the Bethe and Welton arguments are valid and so we take the frequencies in the Bethe-Welton form. In other words, electron cannot respond to the fluctuating field if the frequency which is much less than the atom binding energy given by the Rydberg constant (Rohlf, 1994) $E_{Rydberg} = \alpha^2 mc^2/2$. So, the lower frequency limit is

$$\omega_1 = E_{Rydberg}/\hbar = \frac{\alpha^2 mc^2}{2\hbar}, \tag{18}$$

where $\alpha \approx 1/137$ is so called the fine structure constant.

The specific form of the second frequency follows from the elementary argument, that we expect the effective cutoff, since we must neglect the relativistic effect in our non-relativistic theory. So, we write

$$\omega_2 = \frac{mc^2}{\hbar}. \tag{19}$$

If we take the thermal function of the form of the geometric series

$$\frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} = q(1 + q^2 + q^3 + ...); \quad q = e^{-\frac{\hbar\omega}{kT}}, \tag{20}$$

$$\int_{\omega_1}^{\omega_2} q(1 + q^2 + q^3 + ...)\frac{1}{\omega}d\omega = \ln|\omega| + \sum_{k=1}^{\infty}\frac{(-\frac{\hbar\omega}{kT})^k}{k!k} + ...; \quad q = e^{-\frac{\hbar\omega}{kT}} \tag{21}$$

and the first thermal contribution is

$$Thermal\ contribution = \ln\frac{\omega_2}{\omega_1} - \frac{\hbar}{kT}(\omega_2 - \omega_1), \tag{22}$$

Then, with eq. (6)

$$\delta E_n \approx \frac{2\pi}{3}\left(\frac{e^2}{m^2}\right)\left(\frac{\hbar}{\pi^2 c^3}\right)\left(\ln\frac{\omega_2}{\omega_1} - \frac{\hbar}{kT}(\omega_2 - \omega_1)\right)|\psi_n(0)|^2, \tag{23}$$

where (Sokolov et al., 1962)

$$|\psi_n(0)|^2 = \frac{1}{\pi n^2 a_0^2} \tag{24}$$

with

$$a_0 = \frac{\hbar^2}{me^2}. \tag{25}$$

Let us only remark that the numerical form of eq. (23) has deep experimental astrophysical meaning.

In article by author (Pardy, 1994), which is the continuation of author articles on the finite-temperature Cherenkov radiation and gravitational

Cherenkov radiation (Pardy, 1989a; ibid., 1989b), the temperature Green function in the framework of the Schwinger source theory was derived in order to determine the Coulomb and Yukawa potentials at finite-temperature using the Green functions of a photon with and without radiative corrections, and then by considering the processes expressed by the Feynman diagrams.

The determination of potential at finite temperature is one of the problems which form the basic ingredients of the quantum field theory (QFT) at finite temperature. This theory was formulated some years ago by Dolan and Jackiw (1974), Weinberg (1974) and Bernard (1974) and some of the first applications of this theory were the calculations of the temperature behavior of the effective potential in the Higgs sector of the standard model. Information on the systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one-loop order was given by Donoghue, Holstein and Robinett (1985). Partovi (1994) discussed the QED corrections to Plancks radiation law and photon thermodynamics.

A similar discussion of QED was published by Johansson, Peressutti and Skagerstam (1986) and Cox et al. (1984). Serge Haroche (2012) and his research group in the Paris microwave laboratory used a small cavity for the long life-time of photon quantum experiments performed with the Rydberg atoms. We considered here the thermal gas corresponding to the Gibbons- Hawking theory of space-time (at temperature T) as the preamble for new experiments for the determination of the energy shift of H-atom electrons interacting with the Gibbons- Hawking on thermal gas.

7 The space-time deformation origin of gravity

7.1 Introduction

Space-time is a medium which can be deformed in such a way that the deformation of space-time is equivalent to the existence of the Riemann metric being equivalent to gravity. The deformation of space-time by two massive bodies generates the Newton attractive force between them. This situation has an analogue in the attraction of two electrons (the Cooper pair) in the solid state medium where the lattice is deformed by electrons and leads to the superconductivity of the medium. In other words, the Einstein metric for the two body system can be transformed to the Newton gravitation formula .

There is a possibility that during the big bang, supernova explosion, gravitational collapse, collisions of the high-energy elementary particles and so on, the deformations called dislocations in space-time are created and the deflection of light caused by the dislocations in space-time is physically possible .

Einstein gives no explanation of the origin of the metric, or, metrical tensor. He only introduces the Riemann geometry as the basis for the general relativity (Kenyon, 1996). He "derived" the nonlinear equations for the metrical tensor (Chandrasekhar, 1972) and never explained what is microscopical origin of the metric of space-time. Einstein assumption was that metric follows from differential equations as their solutions. However, the metric has an microscopical origin similarly to the situation where the phenomenological thermodynamics has also the microscopical and statistical origin.

Let us remember the different origins of metric. First, let us show that metric is generated by the coordinate transformations. We demonstrate it using the spherical transformations:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{1}$$

The square of the infinitesimal element is as follows:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2. \tag{2}$$

We see that the nonzero components of the metrical tensor are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta. \tag{3}$$

For $r = const$, it is $dr = 0$ and the element of the length is the element of the 2-dimensional sphere in the 3-dimensional space. The resulting metric is not the 3-dimensional one but only 2-dimensional in the 3D space. So, in order to generate the 2D metric, it is necessary to use the 3D transformations in 3D space. If we want the generate the metric on the 3D sphere, then it is necessary to use the 4D transformations for x, y, z, ξ, where ξ is the extra-coordinate. So, the metric is generated by the curvilinear transformations. Einstein suggested the possibility that metric can be generated by gravitational field. He created the general theory of relativity and gravitation. Henri Poincaré never accepted the metric generated by the gravitational field. According to Poincaré, light interacting with gravity is not the geometrical problem but the optical one.

The Riemann element $(ds)^2$ is defined as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{4}$$

and it is composed from the four infinitesimal coordinate differentials. It means if we want to generate the metric of this 4-dimensional space-time by the coordinate transformations, then it necessary to use the coordinate transformations in 5D space-time. Or, in other words, it is necessary to introduce the extra-dimension. Einstein radically refused the extra-dimensions and he pedagogically explained the curvature of a space-time by introduction the metric which depends on the temperature of the surface (Einstein, 1919). Of course such explanation of the origin of metric is not generally accepted in the textbooks and monographs (Rindler, 2003). It was only pedagogical explanation. Some mathematicians (Natorp, 1901) tried to proof that our space is 3-dimensional and they automatically excluded the extra-dimensions. However, such proofs are misleading because we know from the Bertrand Russell philosophy of mathematics that the mathematical theorems are not existential. In other words, mathematics cannot say anything on the existence of electron, proton, quarks, strings and so on, because these things does not follow from the mathematical axiomatic system. They are only things of the external world and not of the world of mathematics. At the same time pure mathematics cannot predict any fundamental physical constant, because every physical constant including also the fundamental ones is of the physical origin. Mathematics gives only such informations which follow from its axiomatical systems.

Extra-dimensions can be introduced only by the definition and the existence of them cannot be mathematically proved. We know that the 3-dimensional space was confirmed by the most precise theory in the history of physics - QED, and it means that the extra-dimensions were not confirmed. Also the Planck law of the blackbody radiation in 4D space differs form the Planck law in 3D space. The formation of galaxies in the 3D space substantially differs from the formation of galaxies in the 4-dimensional space.

Einstein avoids the extra-dimensionality and compactification. He uses argumentation (Einstein, 1919) on the existence of the non-euclidean geometry using the 2D hot plane, where the magnitude of a ruler changes from point to point being dependent on the temperature at a given point. This method was also used by Feynman (Feynman, 1999). Rindler does not use this method of argumentation (Rindler, 2003).

7.2 The metric of space-time as deformation

So, the question we ask, is, what is the microscopical origin of the metric of space-time. We postulate that the origin of metric is the specific deformation of space-time continuum. We take the idea from the mechanics of continuum and we apply it to the space-time medium. The similar approach can be found in the Tartaglia article and his e-print (Tartaglia, 1995), where space-time is considered as a deformable medium.

The mathematical description of the 3-dimensional deformation is given for instance in (Landau, et al., 1995). The fundamental quantity is the tensor of deformation expressed by the relative displacements u^i as follows:

$$u_{ik} = \left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} + \frac{\partial u^l}{\partial x^i}\frac{\partial u_l}{\partial x^k}\right); \quad i, k = 1, 2, 3. \tag{5}$$

The last definition can be generalized to the 4-dimensional situation by the following relation:

$$u_{\mu\nu} = \left(\frac{\partial u_\mu}{\partial x^\nu} + \frac{\partial u_\nu}{\partial x^\mu} + \frac{\partial u_\alpha}{\partial x^\mu}\frac{\partial u^\alpha}{\partial x^\nu}\right); \quad \mu, \nu = 0, 1, 2, 3, \tag{6}$$

with $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$.

In order to establish the connection between metric $g_{\mu\nu}$ and deformation expressed by the tensor of deformation, we write for the metrical tensor $g_{\mu\nu}$ of the squared space-time element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{7}$$

the following relation

$$g_{\mu\nu} = (\eta_{\mu\nu} + u_{\mu\nu}), \tag{8}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

Instead of work with the metrical tensor $g_{\mu\nu}$, we can work with the tensor of deformation $u_{\mu\nu}$ and we can consider the general theory of relativity as the 4-dimensional theory of some real deformable medium as a corresponding form of the metrica theory. First, let us test the deformation approach to the space-time in case of the non-relativistic limit.

7.3 The non-relativistic test

The Lagrange function of a point particle with mass m moving in a potential φ is given by the following formula (Landau, et al., 1987):

$$L = -mc^2 + \frac{mv^2}{2} - m\varphi. \tag{10}$$

Then, for a corresponding action we have

$$S = \int L dt = -mc \int dt \left(c - \frac{v^2}{2c} + \frac{\varphi}{c} \right), \tag{11}$$

which ought to be compared with $S = -mc \int ds$. Then,

$$ds = \left(c - \frac{v^2}{2c} + \frac{\varphi}{c} \right) dt. \tag{12}$$

With $d\mathbf{x} = \mathbf{v}dt$ and neglecting higher derivative terms, we have

$$ds^2 = (c^2 + 2\varphi)dt^2 - d\mathbf{x}^2 = \left(1 + \frac{2\varphi}{c^2} \right) c^2 dt^2 - d\mathbf{x}^2. \tag{13}$$

The metric determined by this ds^2 can be be obviously related to the u_α as follows:

$$g_{00} = 1 + 2\partial_0 u_0 + \partial_0 u^\alpha \partial_0 u_\alpha = 1 + \frac{2\varphi}{c^2}. \tag{14}$$

We can suppose that the time shift caused by the potential is small and therefore we can neglect the nonlinear term in the last equation. Then we have

$$g_{00} = 1 + 2\partial_0 u_0 = 1 + \frac{2\varphi}{c^2}. \tag{15}$$

The elementary consequence of the last equation is

$$\partial_0 u_0 = \frac{\partial u_0}{\partial(ct)} = \frac{\varphi}{c^2}, \tag{16}$$

or,

$$u_0 = \frac{\varphi}{c} t + const. \tag{17}$$

Using $u_0 = g_{00} u^0$, or, $u^0 = g_{00}^{-1} u_0 = \frac{\varphi}{c} t$, we get with $const. = 0$ and

$$u^0 = ct' - ct, \tag{18}$$

the following result

$$t'(\varphi) = t(0)\left(1 + \frac{\varphi}{c^2}\right), \tag{19}$$

which is the Einstein formula relating time in the zero gravitational field to time in the gravitational potential φ. The time interval $t(0)$ measured remotely is so called the coordinate time and $t(\varphi)$ is local proper time. The remote observer measures time intervals to be deleted and light to be red shifted. The shift of light frequency corresponding to the gravitational potential is, as follows (Landau et al., 1987).

$$\omega = \omega_0\left(1 + \frac{\varphi}{c^2}\right). \tag{20}$$

The precise measurement of the gravitational spectral shift was made by Pound and Rebka in 1960. They predicted spectral shift $\Delta\nu/\nu = 2.46 \times 10^{-15}$ (Kenyon, 1996). The situation with the red shift is in fact the closed problem and no additional measurement is necessary.

While we have seen that the red shift follows from our approach immediately, without application of the Einstein equations, it is evident that the metric determined by the Einstein equations can be expressed by the tensor of deformation. And vice versa, to the every tensor of deformation the metrical tensor corresponds.

7.4 The deflection of light by the screw dislocation

The problem of the light deflection by the screw dislocation is the problem of the recent years (Katanaev et al., 1992), (Katanaev et al., 1998), (Moraes, 1996), (Andrade, 1998), and so on. The motivation was the old problem of the deflection of light by the gravitational field which according to Einstein causes the curvature of space-time.

We know from the history of physics that the deflection of light by the gravitational field of Sun was first calculated by Henri Cavendish in 1784 and it was never published. The first published calculation was almost 20 years later in 1981 by the Prussian astronomer Johann Soldner. Einstein's calculation in 1911 was 0.83 seconds of arc. Cavednish and Soldner predicted a deflection 0.875 seconds of arc. So, the prediction of Cavendish, Soldner (Brown, 2002) and Einstein in 1911 were approximately half of the correct value which was derived in 1919 by Einstein.

Einstein in 1911 used the principle of equivalence for the determination of the light deflection. As was shown by (Ferraro 2003) the Einstein application of this principle was incorrect. The correct application was given only by Ferraro in order to get the correct value. The deflection of light by the topological defects as dislocations, disclinations and so on was, to my knowledge, never calculated by Einstein. In the recent time such calculation was performed by (Katanaev et al., 1992; 1998; Moraes, 1996; Padua, 1998; Andrade, 1998) and so on. Here we use the different and more simple method and the definition of the screw dislocations which differs from the above authors.

According to (Landau et al., 1995), the screw deformation in the mechanics of continuum was defined by the tensor of deformation which is in the cylindrical coordinates as

$$u_{z\varphi} = \frac{b}{4\pi r}, \tag{21}$$

where b is the z-component of the Burgers vector. The Burgers vector of the screw dislocation has components $b_x = b_y = 0, b_z = b$. The Burgers vector is for the specific dislocation a constant geometrical parameter.

The postulation of the space-time as a medium enables to transfer the notions of the theory of elasticity into the relativistic theory of space-time and gravity. The considered transfer is of course the heuristical operation, nevertheless the consequences are interesting. To our knowledge, the problem, which we solve is new.

We know that the metric of the empty space-time is defined by the coefficients in the relation:

$$ds^2 = c^2dt^2 - dr^2 - r^2d\varphi^2 - dz^2. \tag{22}$$

If the screw deformation is present in space-time, then the generalized metric is of the form:

$$ds^2 = c^2dt^2 - dr^2 - r^2d\varphi^2 - 2u_{z\varphi}dzd\varphi - dz^2, \tag{23}$$

or,

$$ds^2 = c^2dt^2 - dr^2 - r^2d\varphi^2 - \frac{2b}{4\pi r}dzd\varphi - dz^2. \tag{24}$$

The motion of light in the Riemann space-time is described by the equation $ds = 0$. It means, that from the last equation the following differential equation for photon follows:

$$0 = c^2 - \dot{r}^2 - r^2\dot{\varphi}^2 - \frac{b}{2\pi r}\dot{z}\dot{\varphi} - \dot{z}^2. \tag{25}$$

Every parametric equations which obeys the last equation are equation of motion of photon in the space-time with the screw dislocation. Let us suppose that the motion of light is in the direction of the z-axis. Or, we write approximately:

$$r \approx a; \quad \dot{z} = v. \tag{26}$$

Then, we get equation of φ:

$$2\pi a^3 \dot{\varphi}^2 + bv\dot{\varphi} = 2\pi a(c^2 - v^2). \tag{27}$$

We suppose that the solution of the last equation is of the form

$$\varphi = At. \tag{28}$$

Then, we get for the constant A the quadratic equation

$$2\pi a^3 A^2 + bvA + 2\pi a(v^2 - c^2) = 0 \tag{29}$$

with the solution

$$A_{1/2} = \frac{-bv \pm \sqrt{b^2v^2 - 16\pi^2 a^4(v^2 - c^2)}}{4\pi a^3}. \tag{30}$$

Using approximation $v \approx c$, we get that first root is approximately zero and for the second root we get:

$$A \approx \frac{-bc}{2\pi a^3}, \tag{31}$$

which gives the function φ in the form:

$$\varphi \approx \frac{-bc}{2\pi a^3} t. \tag{32}$$

Then, if $z_2 - z_1 = l$ is a distance between two points on the straight line parallel with the axis of screw dislocation then, $\Delta t = l/c$, c being the velocity of light. For the deflection angle $\Delta\varphi$, we get:

$$\Delta\varphi \approx \frac{-bl}{2\pi a^3}. \tag{33}$$

So, we can say, that if we define the screw dislocation by the metric of eq. (24), then, the deflection angle of light caused by such dislocation is given by eq. (33). The result (33) is only approximative and we do not know what is the accuracy of such approximation. This problem can be solved using the approximation theory.

Let us remark that the exact trajectory of photon in the field of the screw dislocation can be determined from the trajectory equation

$$\frac{d^2x_\mu}{ds^2} + \Gamma_\mu^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0. \tag{34}$$

which was used in many textbooks. However, According to Landau et al. (1987), the equation is contradictory for photon, because in this case $ds = 0$, and it means that the last equation is not rigorously defined. Landau et al. derived the deflection of light from the Hamilton-Jacobi equation for particle with the rest mass $m = 0$, which moves with the light velocity. However, this approach is not also absolutely correct because in the classical field theory it is not possible to define photon. Photon is a quantum object. Rigorous derivation of the deflection of light was given by Fok (1961), who used the mathematical object "the front of wave" and his result is valid without any doubt.

Let us remark that equation (34) has two meanings: geometrical and physical. The geometrical meaning uses $g_{\mu\nu}$ which follows from the curvilinear transformations and the physical meaning of $g_{\mu\nu}$ is metric of the gravitational field calculated by means of the Einstein equation. The second meaning is the Einstein postulate and cannot be derived from so called pure mathematics. Only experiment can verify the physical meaning of equation (34).

The problem of interaction of light with the gravitational field is not exhausted by our example. We can define more difficult problems such as deflection of the coherent light, laser light, squeezed light, soliton light, massive light with massive photons, light of the entangled photons and so on. No of these problems was still solved because they are only for brilliant experts very well educated. And this is the pedagogical problem.

7.5 The physical generation of the screw dislocation

Now the question arises, how to determine the mechanical or electrodynamical or laser system which will generate the screw dislocation in space-time. We know, that for real crystals the generation of the screw dislocation is the elementary problem of the physics of crystals. If we use Einstein's equations, then the problem is mathematical one. Or, to see it, let us write the Einstein equations with the cosmological constant λ.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{35}$$

where $T_{\mu\nu}$ is the tensor of the energy and momentum.

In case of the perfect fluid and pressure, tensor of the energy and momentum is as follows:

$$T_{\mu\nu}(\text{mech}) = (\varrho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{36}$$

where ϱ is a density and p is a pressure of the fluid. The quantities u_μ are four velocities of the fluid.

In case that the tensor of the energy and momentum is created electromagnetically, then,

$$T_{\mu\nu}(\text{elmag}) = \frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu^\alpha - \frac{g_{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{37}$$

where $F_{\alpha\beta}$ is the tensor of the electromagnetic field.

So, because $g_{\mu\nu}$ is determined as the metrical tensor corresponding to the screw dislocation of the space-time, the left side of the Einstein equations is given and the problem is to find the mechanical and electrodynamical quantities, which determine corresponding tensors of energy and momentum.

The surprising thing is the fact that if there is no curvature of space time, than thanks to the existence of the cosmological constant, the solution corresponding to the mechanical or electrodynamical systems is not absolutely zero. The cosmological constant is in such a way very important quantity and evidently cannot be zero. The dislocations of space-time are in harmony with the cosmological constant.

Let us remark that Einstein equations were derived intuitively by Einstein (Chandrasekhar, 1972) and rigorously by Hilbert from the Lagrangian using the variational method (Kenyon, 1996). The Hilbert derivation is pure mathematical one and it means it is very simple. This variational method enables to start relativity physics from the general theory and to derive the special relativity as a classical limit of the general theory. This approach was presented by Rindler (Rindler, 1994). To our knowledge, such unconventional but very elegant approach was not presented in any textbook on relativity theory.

The tensor of the energy momentum in equation (36) is rigorously defined. The problem is, how to identify the distribution of the cosmological objects with this tensor. To our knowledge, there is no mathematical theorem for such rigorous identification.

7.6 Cosmological consequences

The verification of our theory and at the same time the existence of the nonzero cosmological constant can be performed during the measurement of the cosmical microwave background radiation. In case of the existence of some defects in space-time the distributions of this radiation will be inhomogeneous and it will depend on the density and orientations of the dislocations, screw dislocations, disclinations and other topological defects of the space-time.

It is evident that also in case that the curvature caused by the some topological defect is zero, then, thanks to the existence of the cosmological constant in the Einstein equations the defects can be generated mechanically or electrodynamically as it follows from the Einstein equations. So, investigation of the cosmical microwave background can inform us on the distribution of the topological defects in the space-time and on the possible origins of these defects (Rey et al., 1999).

7.7 The laboratory verification of a theory

We can consider the situation which is analogical to the space-time situation. In other words, we can consider the modified Planckian experiment with the black body radiation. The difference from the original Planck situation is that we consider inside of the black body some optical medium with dislocations. Then, in case that the optical properties depend also on the presence of dislocations, in other words, that the local index of refraction depend on presence of dislocations, then we can expect the modification of the Planck law of the blackbody radiation. We know that the most simple problem with the constant index of refraction was calculated (Kubo, 1965). To our knowledge, although this is only so called table experiment, it was never performed in some optical laboratory. It is possible to expect that during the experiments some surprises will appear. However, the practical situation can be realized, if we prepare some crystal with the screw dislocations with given orientation. Then, in case that the optical properties are expressed by means of metric in crystal, the metric will determine the optical path of light in the crystal and it means that the screw dislocations can be investigated by optical methods and not only by the electron microscope.

8 Summary and Perspectives

The book is a preamble of the unification of the Lobachevsky-Beltrami-Fok geometry with the physics of elementary particles and beyond. The starting point was the Fermat principle formulated by means of variational calculus. The Poincaré model of the Lobachevsky geometry was formulated as the optical model of the interaction of light with optical medium in space. Beltrami showed that the Lobachevsky geometry formulas follows from the spherical geometry by the elementary (Beltrami) operation $r \to ir$. The operation is not involved in the famous Euler monograph on spherical geometry (Euler, 1896). We have generalized the Beltrami operation to the operation $r \to ir + \varrho$, $r.r \to r.r^*$, in subsection 1.4, and by $r \to r + i\rho$, in section 1.5. Symbols ϱ and ρ are introduced as the new geometrical constants, which should not to be identified with the Einstein cosmological constant Λ.

Fok formulated the Lobachevsky geometry physically as the geometry of the relativistic velocity space. From this approach follows the adequate description of the decay of the neutral π-meson into two γ-photons. The angle between velocities of the gamma photons in the rest system of neutral meson is evidently π. However, according to the special theory of relativity the angle is transformed in the laboratory system innharmony with the Lorentz transformation and it is smaller than π. It is equivalent to the statement that the Lobachevsky angle Π is smaller than $\pi/2$, or, $\Pi < \pi/2$. Such experiment can be considered as the confirmation of the Lobachevsky geometry in the elementary particle physics. Similarly, the decay of the neutral η-meson $\eta^0 \to \gamma + \gamma$, axion $A^0 \to \gamma + \gamma$, or, the Higgs boson decay $H^0 \to \gamma + \gamma$, are the confirmation of the Lobachevsky geometry in the elementary particle physics and at present time can be tested in CERN.

Lobachevsky, in his pangeometry, presents the idea (many years before Einstein) that his geometry is probably realized in the near vicinity of atoms and molecules and also in the cosmical space (Norden, 1956). Now, we see that his geometry is realized in particle physics of LHC in CERN. On the other hand, Lobachevsky metric enables to formulate new problems such as the Lorentz-Dirac equation in the Lobachevsky geometry, the Bargaman-Michel-Telegdi (BMT) equation with the bremstrahlung term in the Lobachevsky geometry, the quantum hall physics in the Lobachevsky geometry, graphene physics in the Lobachevsky geometry, or, the Feynman integral in the Lobachevsky space-time and so on.

The Fermat principle enables to get the circular optical trajectories, or in other words the confinement of light by optical medium - so called optical black hole. It may be easy to prove it.

Let be the index of refraction $n(r)$ in the Euclidean plane with polar coordinates r, φ. The explicit form of the Fermat principle

$$\delta \int n(r) ds = 0 \tag{1}$$

is (Marklund et al., 2002)

$$\delta \int n(r) \sqrt{1 + r^2 \left(\frac{d\varphi}{dr}\right)^2} dr = 0. \tag{2}$$

The last equation is equivalent to the Euler-Lagrange variational equation for the functional $F(\varphi, \varphi')$

$$F_\varphi - \frac{d}{dr} F_{\varphi'} = 0. \tag{3}$$

Or,

$$\frac{d}{dr}\left[n(r) \frac{r^2 d\varphi/dr}{\sqrt{1 + r^2 \left(\frac{d\varphi}{dr}\right)^2}}\right] = 0. \tag{4}$$

It is evident that the elimination of $d\varphi/dr$ is as follows:

$$\frac{d\varphi}{dr} = \pm \frac{C}{\sqrt{r^4 n^2(r) - C^2 r^2}}. \tag{5}$$

The circular trajectory is defined by equation $dr/d\varphi = 0$, from which follows the index of refraction for the so called optical black hole

$$n(r) = \frac{const}{r}. \tag{6}$$

It is well known the Bose-Einstein condensate, where the optical light pulses travel with extremely small group velocity about 17 meters per second (Hau et al. 1999). This is the possible way for testing the optical black hole.

Without doubt, the monochromatic optical beam is composed from photons of energy $E = \hbar\omega$. While the rest mass of photon is zero, the relativistic mass follows from the Einstein relation $E = mc^2$. After identifying the relativity energy and quantum energy of photon we have

$$m = \frac{\hbar\omega}{c^2}. \tag{7}$$

The centrifugal force acting on photon moving with velocity v in optical medium along the circle with radius r is for the photon mass as follows:

$$F_{centrifugal} = \frac{\hbar\omega}{c^2}\frac{v^2}{r}. \tag{8}$$

The centrifugal force and the Kapitza effect (thermal fluctuations of the index of refraction) (Landau, et al., 1982) are the origin of the instability of the photon trajectory in the optical medium. So, the experimental investigation of the confinement of photon in the optical medium is meaningful at temperature $T \approx 0$. There is no doubt that the investigation of the photon trajectories is the crucial problem of the optical physics and it is interesting for all optical laboratories over the world.

The maximal acceleration constant was introduced in section 2 and it was derived as the kinematical constant and it differs from the Caianiello (1981) definition following from quantum mechanics. Our constant cannot be determined by the system of other physical constants. It is an analogue of the numeric velocity of light which cannot be composed from others physical constants, or, the Heisenberg fundamental length in particle physics. The nonlinear transformations (20) changes the Minkowski metric

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{9}$$

to the new metric with the Riemann form. Namely:

$$ds^2 = \alpha^2t^2dt^2 - dx^2 - dy^2 - dz^2 \tag{10}$$

and it can be investigated by the methods of differential geometry.

If some experiment will confirm the existence of kinematical maximal acceleration α, then it will have certainly crucial consequences for Einstein theory of gravity because this theory does not involve this factor. Also the cosmological theories constructed on the basis of the original Einstein equations will require modifications. The so called Hubble constant will be changed and the scenario of the accelerating universe modified.

Also the standard model of particle physics and supersymmetry theory will require generalization because they does not involve the maximal acceleration constant. It is not excluded that also the theory of parity nonconservation will be modified by the maximal acceleration constant. In such a way the particle laboratories have perspective programes involving the physics with maximal acceleration. Many new resuls can be obtained from the old relativistic results having the form of the mathematical objects involving function $f(v/c)$.

The prestige problem in the modern theoretical physics - the theory of the Unruh effect, or, the existence of thermal radiation detected by accelerated observer - is in the development (Fedotov et al., 2002) and the serious statement, or comment to the relation of this effect to the maximal acceleration must be elaborated.

It is not excluded that the maximal acceleration constant will be discovered by International Linear Collider (ILC). The unique feature of the ILC is the fact that its CM energy can be increased gradually simply by extending the main linac.

Let us remark that it is possible to extend and modify quantum field theory by maximal acceleration. It is not excluded that the kinematical maximal acceleration constant will enable to reformulate the theory of renormalization.

The derived formulas with uniform acceleration a and can be applied and verified in case of the uniform equivalent gravity according to the principle of equivalence.

The section 4 has dealed with the calculation of the Thomas precession caused by accelerated motion of the systems. In other words we have shown show that Thomas precession can be initiated by acceleration of a point particle. The problem of acceleration of charged particles or systems of particles is the permanent and the most prestige problem in the accelerator physics. Particles can be accelerated by different ways. Usually by the classical electromagnetic fields, or, by light pressure of the laser fields (Baranova et al., 1994; Pardy, 1998, 2002a). The latter method is the permanent problem of the laser physics for many years.

The section 4 deals with the quantum energy loss of binary involving radiative corrections. The general relativity necessarily does not contain the method how to express the quantum effects together with the radiative corrections by the geometrical language. So it cannot give the answer on the production of gravitons and on the graviton propagator with radiative corrections.

The section 4 is the extended version of the older article (Pardy, 1983), where only the spectral formulas without radiative corrections were derived. We have derived the quantum energy loss formulas with the arbitrary strength of the gravitational field.

At the present time the only classical radiation of gravity was confirmed. The production of gravitons is not involved in the Einstein theory. However, the idea, that radiative corrections can have macroscopical consequences is not new. Based on the Boulware quantum states (Boulware, 1975; Hiscock, 1988) has calculated the vacuum polarization induced by static star. He has found an energy density which is of course many orders of magnitude below what one could hope to detect by experiment.

In a similar way , Goldman et al. (1992), have pointed out the quantum gravity effects might play a role in the physics of cosmological scale, and Soleng (1992) has evaluated the post-Newtonian parameters, which follows from anisotropic vacuum energy. So, the situation in the gravity problems with radiative corrections is similar to the QED situation many years ago when the QED radiative corrections were theoretically predicted and then experimentally confirmed for instance in case of he Lamb shift or of the anomalous magnetic moment of electron.

Astrophysics is, therefore, in crucial position in prooving the influence of radiative corrections on the dynamics in the cosmic space. We hope that the further astrophysical observations will confirm the quantum version of the energy loss of the binary with graviton propagator with radiative corrections.

The power spectral formulas of the gravitational Cherenkov radiation at zero temperature (19) and at nonzero temperature (25) are derived in section 5 also in the framework of the Schwinger source theory.

These effects have been not discussed in the classical textbooks on gravity. Nevertheless these effects can be mathematically rigorously defined and described in the framework of the source theory embedded in the curved space time with the metric (2).

Formula (13) is valid for the general metric space-time and it enables to determine the power spectral formula of gravitons for the general metric space time if it is known the propagator $D_{+g}(x,x')$. It means it generates further problems of production of gravitons in the different metric space-times.

In electrodynamics, the Cherenkov effect occurs usually for the velocities comparable with velocities of light. However, if we consider the case with the cold gas (Peters, 1974), then the Cherenkov gravitational effect occurs practically for all velocities. In order to see the surprising result we write the gravitational index of refraction derived by Peters (1974) for the cold gas. Or,

$$n = 1 + \frac{2\pi\rho G}{\omega^2} \tag{11}$$

where ρ is the gas density. Then from condition $n\beta > 1$ we get using the eq. (21) the following inequality:

$$\omega < \left(\frac{2\pi\beta\rho G}{1-\beta}\right)^{1/2} \tag{12}$$

and it means that the interval of frequencies is limited and because ρ is for the cold gas very small , the gravitational Cherenkov radiation occurs only for very small frequencies. On the other hand the interval of allowed frequencies is greater for sufficiently fast moving bodies.

The amount of the produced gravitons by the Cherenkov mechanism depends on the square of the relativistic mass m, and it is obvious that for elementary particles as electrons, protons an so on the production of gravitons will be small.

It is obvious that such small energy cannot be observed by the any experimental equipment. On the other hand the big production of gravitons by the Cherenkov mechanism can occur for cosmological bodies with the sufficiently big masses and with energies exceeding the Cherenkov threshold. Of course, if such effect occured during the explosion of supernova SN 1987a, during the big bang, or during the collisions of galaxies is open question because of the nonexistence of the cold gas. On the other hand Polnarev (1972) has shown that in the ultrarelativistic case at the anisotropical situation there exists the possibility of the $n > 1$ or $n < 1$ and the effect is probable.

The investigation of the gravitational Cherenkov effect is analogical to the history of the Cherenkov effect. Heaviside (1889) investigated the possibility of a charged object moving in a medium faster than electromagnetic waves in the same medium becomes a source of directed electromagnetic radiation. Kelvin (1901) presented an idea that the emission of particles is possible at a speed greater than that of light. Somewhat later, Sommerfeld (1904; 1905) proposed the hypothetical radiation with a sharp angular distribution. His theory was never accepted in his time because of the priority of the special theory relativity, where the maximal velocity is the velocity of light. However, in fact, from experimental point of view, the electromagnetic Cherenkov radiation was first observed in the early 1900's by experiments developed by Marie and Pierre Curie when studying radioactivity emission. In essence they observed the emission of a bluish-white light from transparent substances in the neighborhood of strong radioactive source. But the first attempt to understand the origin of this was made by Mallet (1926; 1929a; 1929b) who observed that the light emitted by a variety of transparent bodies placed close to a radioactive source always had the same bluish white quality, and that the spectrum was continuous, with no line or band structure characteristic of fluorescence. Unfortunately, these investigations were forgotten for many years. Cherenkov (1934) experiments was performed at the suggestion of Vavilov who opened a door to the true physical nature of this effect (Bolotovskii, 2009)[1].

This radiation was first theoretically interpreted by Tamm and Frank (1937) in the framework of the classical electrodynamics. The source theoretical description of this effect was given by Schwinger et al. (1976a) at the zero temperature regime and the classical spectral formula was generalized to the finite temperature situation and for the massive photons by author (Pardy, 1989, 2002b). The Vavilov-Cherenkov effect was also used by author (Pardy, 1997) to possible measurement of the Lorentz contraction.

We hope in this book that the sympathy to the existence of the gravitational Cherenkov radiation will be sufficiently strong in order to have some followers.

Section 6 concerns the influence of the potential at finite temperature on the energy shift of the H-atom. The determination of potential at finite temperature is one of the problems which form the basic ingredients of the quantum field theory (QFT) at finite temperature. This theory was formulated some years ago by Dolan and Jackiw (1974), Weinberg (1974) and Bernard (1974) and some of the first applications of this theory were the calculations of the temperature behavior of the effective potential in the Higgs sector of the standard model.

Information on the systematic examination of the finite temperature effects in quantum electrodynamics (QED) at one-loop order was given by Donoghue, Holstein and Robinett (1985). Partovi (1994) discussed the QED corrections to Planck's radiation law and photon thermodynamics,

A similar discussion of QED was published by Johansson, Peressutti and Skagerstam (1986) and Cox et al. (1984).

Serge Haroche (2012) and his research group in the Paris microwave laboratory used a small cavity for the long life-time of photon quantum experiments performed with the Rydberg atoms. We considered here the thermal gas corresponding to the Gibbons-Hawking theory of space-time (at temperature T) as the preamble for new experiments for the determination of the energy shift of H-atom electrons interacting with the Gibbons-Hawking on thermal gas.

We have defined in section 7 with regard to previous author articles (Pardy, 2001a, 2001b) gravitation as a deformation of a medium called space-time.

[1]So, the adequate name of this effect is the Vavilov-Cherenkov effect. In the English literature, however, it is usually called the Cherenkov effect.

Because of this idea, we can write the Newton gravitational law in the form (D = deformation)

$$D_1 + D_2 \to -\kappa \frac{M_1 M_1}{r^3}\mathbf{r}, \tag{13}$$

where the quantities in the last equation have the standard textbook physical meanings.

We have used equation which relates Riemann metrical tensor to the tensor of deformation of the space-time medium and applied it to the gravitating system, which we call screw dislocation in space-time. The term screw dislocation was used as an analogue with the situation in the continuous mechanics. We derived the angle of deflection of light passing along the screw dislocation axis at the distance a from it on the assumption that trajectory length was l. This problem was not considered for instance in the Will monograph (Will, 1983). The screw dislocation was still not observed in space-time and it is not clear what role play the dislocations in the development of universe after big bang. Our method can be applied to the other types of dislocations in space-time and there is no problem to solve the problem in general. We have used here the specific situation because of its simplicity. We have seen that the problem of dislocation in space-time is interesting and it means there is some scientific value of this problem.

It is well known from the quantum field theory and experiment, that every particle has its partner in the form of the antiparticle (Maiani et al., 1995). For instance the antiparticle to the electron is positron. It is well known that after annihilation of particle-antiparticle pair, photons are generated. For instance

$$e^+ + e^- \quad \to \quad 2\gamma. \tag{14}$$

Now, if we define that to every dislocation D exists antidislocation $\bar{D}$, (which can be considered also as an analogue of the antistring, (Srivastava, 1992)), then, we can write the following equation which is analogical to (14)

$$D + \bar{D} \quad \to \quad N\gamma \tag{15}$$

where N is natural number and gamma denotes photon. The next equation is also possible:

$$D + \bar{D} \quad \to \quad Ng \tag{16}$$

where g denotes graviton.

It is possible also to consider the high-energy process with the incident particles a and b as follows:

$$a + b \quad \rightarrow \quad c_1 + c_2 + c_3 + ...c_n, \tag{17}$$

where c_i are denotations of identical or different particles. It is well-known that the equation (17) is the fundamental equation of LHC.

In case of the existence of the dislocations in universe, equations (39) and (40) of the section 7 represents the burst of photons or gravitons in the cosmical space. So, we defined the further possible interpretation of the photonic and gravitational bursts in cosmical space.

Let us remark that the dislocation approach to the particle physics are in harmony with the Einstein dream and later Misner-Wheeler geometrodynamics where all existing elementary objects can be defined as some form of space-time. Misner and Wheeler (Misner et l., 1957; Wheeler, 1957) consider also that neutrino is the specific form of the space-time. Let us still remark that we know from the history of philosophy that long time before Christ, Anaximandros introduced (Apeiron) as a medium from which all particles, and therefore all visible universe was created. So, we can say that the famous trinity of men, Einstein-Misner-Wheeler, is the follower of Anaximandros.

The identification of the fundamental particles by the dislocations is in harmony with the relation for the energy of the dislocation (Cottrell, 1964)

$$E = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{18}$$

where E_0 is the energy of dislocation when its velocity v is zero and c is the velocity of sound in the crystal. In case of medium called space-time the velocity c is the velocity of light in vacuum. So, vacuum is in a certain sense medium which is similar to the crystal medium. The analogy with the dislocations in crystal is of course heuristical step which is the integral part of the methodology of discoveries and it cannot be rigorously algorithmically defined.

Equation (15) can be verified by the table experiment, where the dislocations and the anti-dislocations in some crystal are annihilated by some suitable laboratory mechanism. The flash of light is observed. The laboratory mechanism can be realized for instance by the ultrasound, pressure, strike, and so on. The application of the ultrasound is analogical to the case of the sonoluminescence where the mechanism of the generation of light involves application of ultrasound. We hope that the energy released by such annihilation is greater than the energy released by the mechanism of cold fusion.

Although equation (18) can be identified with the relativistic equation for dependence of energy on velocity of elementary particle, we cannot identify electron with the screw dislocation in space-time. Why? Because we know that the attractive or repulsive force between two screw dislocations differs from the force between two electrons, two positrons or electron and positron. The second reason is that the anomalous magnetic moment of electron is of the dynamical origin as it follows from the Feynman diagram technique while the classical dislocation does not involve such dynamics. On the other hand, we do not know how other dislocations such as circular dislocations, cylindrical dislocations, helix, double helix, triple helix dislocations and so on are related to elementary particles, especially to neutrinos. We know that all physical constant in the standard model are of the dynamical origin, but at present time it is not clear what is their derivation from the more fundamental theory (subquark theory, string theory and so on), or, from the dislocation theory of elementary particles. We think that dislocation theory of elementary particles is not at present time prepared to give the answer to these difficult questions.

The 3D screw dislocation can be extended mathematically to the N-dimensional space, or, space-time. However the interpretation of the N-dimensional theory needs introducing of the compactification.

So, we can say that the Einstein dream of the unification of all objects and forces in nature in the framework of geometrodynamics is far from the successful realization because the identification of ultimate blocks of nature with dislocations and with the topological defects is not elaborated at this time.

In particle physics and in the string theory (Antoniadis, 2003) the confinement of quarks is mysterious and the solution of his problem in the dislocation theory of fundamental constituents is open. We think, that appropriate understanding of the string-like dislocations (Lund, 1985) and definition of the ultimate building blocks of nature can solve all problems of particle physics together with removing all mysteries.

While the verification of the existence of the optical bursts caused by the annihilations of the giant screw dislocations and anti-dislocations can be detected by the Hubble telescope, the gravitational bursts can be probably detected by LIGO (Barish, 1997), VIRGO (Vinet, 1997), GEO (Hough, 1997), TAMA (Tsubono, 1997), and so on.

Let us remark that the observations of recent gamma-ray bursts

GRB021004 and GRB990123 are interpreted as arising from the ultra-relativistic particles thrown out by a cataclysmic event as the collapse of massive star in the so called "cannonball" model. Such models apply to gamma-ray bursts lasting several seconds. Bursts shorter than about 2 seconds are thought to be due to coalescence of two neutron stars to form a black hole. So, our interpretation of light bursts differs from the traditional one because we explain the light bursts as an annihilation of dislocations and anti-dislocations existing in space-time. We hope that our interpretation of gamma-bursts based on particle physics, topology and general relativity is progressive.

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