An Invariant Formulation of Special Relativity

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This article presents an invariant formulation of special relativity which can be applied in any inertial reference frame. In addition, a new universal force is proposed.

Introduction

The intrinsic mass $(\, m \,)$ and the frequency factor $(\, f \,)$ of a massive particle are given by:

$$m \doteq m_o$$

$$f \doteq \left(1 - \frac{\mathbf{v} \cdot \mathbf{v}}{c^2}\right)^{-1/2}$$

where $(\, m_o \,)$ is the rest mass of the massive particle, $(\, \mathbf{v} \,)$ is the relational velocity of the massive particle and $(\, c \,)$ is the speed of light in vacuum.

The intrinsic mass $(\, m \,)$ and the frequency factor $(\, f \,)$ of a non-massive particle are given by:

$$m \doteq \frac{h \, \kappa}{c^2}$$

$$f \doteq \frac{\nu}{\kappa}$$

where $(\, h \,)$ is the Planck constant, $(\, \nu \,)$ is the relational frequency of the non-massive particle, $(\, \kappa \,)$ is a positive universal constant with dimension of frequency and $(\, c \,)$ is the speed of light in vacuum.

In this article, a massive particle is a particle with non-zero rest mass and a non-massive particle is a particle with zero rest mass.

The Invariant Kinematics

The special position ($\bar{\mathbf{r}}$), the special velocity ($\bar{\mathbf{v}}$) and the special acceleration ($\bar{\mathbf{a}}$) of a (massive or non-massive) particle are given by:

$$\bar{\mathbf{r}} \doteq \int f\,\mathbf{v}\,dt$$

$$\bar{\mathbf{v}} \doteq \frac{d\bar{\mathbf{r}}}{dt} = f\,\mathbf{v}$$

$$\bar{\mathbf{a}} \doteq \frac{d\bar{\mathbf{v}}}{dt} = f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}$$

where (f) is the frequency factor of the particle, ($\mathbf{v}$) is the relational velocity of the particle and (t) is the relational time of the particle.

The Invariant Dynamics

If we consider a (massive or non-massive) particle with intrinsic mass (m) then the linear momentum ($\mathbf{P}$) of the particle, the angular momentum ($\mathbf{L}$) of the particle, the net force ($\mathbf{F}$) acting on the particle, the work (W) done by the net force acting on the particle, and the kinetic energy (K) of the particle are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}} = m\,f\,\mathbf{v}$$

$$\mathbf{L} \doteq \mathbf{P} \mathbin{\dot{\times}} \mathbf{r} = m\,\bar{\mathbf{v}} \mathbin{\dot{\times}} \mathbf{r} = m\,f\,\mathbf{v} \mathbin{\dot{\times}} \mathbf{r}$$

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = m\,\bar{\mathbf{a}} = m\left[f\,\frac{d\mathbf{v}}{dt} + \frac{df}{dt}\,\mathbf{v}\right]$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{d\mathbf{P}}{dt}\cdot d\mathbf{r} = \Delta\,\mathrm{K}$$

$$\mathrm{K} \doteq m\,f\,c^2$$

where (f, $\mathbf{r}$, $\mathbf{v}$, t, $\bar{\mathbf{v}}$, $\bar{\mathbf{a}}$) are the frequency factor, the relational position, the relational velocity, the relational time, the special velocity and the special acceleration of the particle and (c) is the speed of light in vacuum. The kinetic energy (K_o) of a massive particle at relational rest is ($m_o\,c^2$)

Relational Quantities

From an auxiliary massive particle (called auxiliary-point) some kinematic quantities (called relational quantities) can be obtained. These are invariant under transformations between inertial reference frames.

An auxiliary-point is an arbitrary massive particle free of external forces (or that the net force acting on it is zero)

The relational time $(\,t\,)$, the relational position $(\,\mathbf{r}\,)$, the relational velocity $(\,\mathbf{v}\,)$ and the relational acceleration $(\,\mathbf{a}\,)$ of a (massive or non-massive) particle relative to an inertial reference frame S are given by:

$$t \doteq \gamma \left(\mathtt{t} - \frac{\vec{r}\cdot\vec{\varphi}}{c^2} \right)$$

$$\mathbf{r} \doteq \left[\vec{r} + \frac{\gamma^2}{\gamma+1} \frac{(\vec{r}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} - \gamma\,\vec{\varphi}\,\mathtt{t} \right]$$

$$\mathbf{v} \doteq \left[\vec{v} + \frac{\gamma^2}{\gamma+1} \frac{(\vec{v}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} - \gamma\,\vec{\varphi} \right] \frac{1}{\gamma\,(1 - \frac{\vec{v}\cdot\vec{\varphi}}{c^2})}$$

$$\mathbf{a} \doteq \left[\vec{a} - \frac{\gamma}{\gamma+1} \frac{(\vec{a}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} + \frac{(\vec{a}\times\vec{v})\times\vec{\varphi}}{c^2} \right] \frac{1}{\gamma^2\,(1 - \frac{\vec{v}\cdot\vec{\varphi}}{c^2})^3}$$

where $(\,\mathtt{t},\ \vec{r},\ \vec{v},\ \vec{a}\,)$ are the time, the position, the velocity and the acceleration of the particle relative to the inertial reference frame S, $(\,\vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the inertial reference frame S and $(\,c\,)$ is the speed of light in vacuum. $(\,\vec{\varphi}\,)$ is a constant and $\gamma = (1 - \vec{\varphi}\cdot\vec{\varphi}/c^2)^{-1/2}$

The relational frequency $(\,\nu\,)$ of a non-massive particle relative to an inertial reference frame S is given by:

$$\nu \doteq \mathtt{v}\ \frac{\left(1 - \dfrac{\vec{c}\cdot\vec{\varphi}}{c^2}\right)}{\sqrt{1 - \dfrac{\vec{\varphi}\cdot\vec{\varphi}}{c^2}}}$$

where $(\,\mathtt{v}\,)$ is the frequency of the non-massive particle relative to the inertial reference frame S, $(\,\vec{c}\,)$ is the velocity of the non-massive particle relative to the inertial reference frame S, $(\,\vec{\varphi}\,)$ is the velocity of the auxiliary-point relative to the inertial reference frame S and $(\,c\,)$ is the speed of light in vacuum.

§ In arbitrary inertial reference frames ($t_\alpha \neq \tau_\alpha$ or $\mathbf{r}_\alpha \neq 0$) (α = auxiliary-point) a constant must be add in the definition of relational time such that the relational time and the proper time of the auxiliary-point are the same ($t_\alpha = \tau_\alpha$) and another constant must be add in the definition of relational position such that the relational position of the auxiliary-point is zero ($\mathbf{r}_\alpha = 0$)

§ In the particular case of an isolated system of (massive or non-massive) particles, inertial observers should preferably use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ($\sum_z m_z \bar{\mathbf{v}}_z = 0$)

General Observations

§ Forces and fields must be expressed with relational quantities (the Lorentz force must be expressed with the relational velocity $\mathbf{v}$, the electric field must be expressed with the relational position $\mathbf{r}$, etc.)

§ The operator ($\dot{\times}$) must be replaced by the operator ($\times$) or the operator ($\wedge$) as follows: ($\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \times \mathbf{a}$) or ($\mathbf{a} \dot{\times} \mathbf{b} = \mathbf{b} \wedge \mathbf{a}$)

§ The intrinsic mass quantity (m) is invariant under transformations between inertial and non-inertial reference frames.

§ The relational quantities ($\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) are invariant under transformations between inertial reference frames.

§ Therefore, the kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) are invariant under transformations between inertial reference frames.

§ However, it is natural to consider the following generalization:

• It would also be possible to obtain relational quantities ($\nu, t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) that would be invariant under transformations between inertial and non-inertial reference frames.

• The kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) would also be given by the equations of this article.

• Therefore, the kinematic and dynamic quantities ($f, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}$) would be invariant under transformations between inertial and non-inertial reference frames.

Vector Lorentz Transformations

If we consider two inertial reference frames (S and S') whose origins coincide at time zero (in both frames) then the time ($\mathtt{t}'$), the position ($\vec{r}\,'$), the velocity ($\vec{v}\,'$) and the acceleration ($\vec{a}\,'$) of a (massive or non-massive) particle relative to the inertial reference frame S' are given by:

$$\mathtt{t}' = \gamma\left(\mathtt{t} - \frac{\vec{r}\cdot\vec{\varphi}}{c^2}\right)$$

$$\vec{r}\,' = \left[\,\vec{r} + \frac{\gamma^2}{\gamma+1}\frac{(\vec{r}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} - \gamma\,\vec{\varphi}\,\mathtt{t}\,\right]$$

$$\vec{v}\,' = \left[\,\vec{v} + \frac{\gamma^2}{\gamma+1}\frac{(\vec{v}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} - \gamma\,\vec{\varphi}\,\right]\frac{1}{\gamma\,(1-\frac{\vec{v}\cdot\vec{\varphi}}{c^2})}$$

$$\vec{a}\,' = \left[\,\vec{a} - \frac{\gamma}{\gamma+1}\frac{(\vec{a}\cdot\vec{\varphi})\,\vec{\varphi}}{c^2} + \frac{(\vec{a}\times\vec{v})\times\vec{\varphi}}{c^2}\,\right]\frac{1}{\gamma^2\,(1-\frac{\vec{v}\cdot\vec{\varphi}}{c^2})^3}$$

where ($\mathtt{t}$, $\vec{r}$, $\vec{v}$, $\vec{a}$) are the time, the position, the velocity and the acceleration of the particle relative to the inertial reference frame S, ($\vec{\varphi}$) is the velocity of the inertial reference frame S' relative to the inertial reference frame S and (c) is the speed of light in vacuum. ($\vec{\varphi}$) is a constant and $\gamma = (1-\vec{\varphi}\cdot\vec{\varphi}/c^2)^{-1/2}$

Transformation of Frequency

The frequency ($\mathtt{v}'$) of a non-massive particle relative to an inertial reference frame S' is given by:

$$\mathtt{v}' = \mathtt{v}\,\frac{\left(1-\dfrac{\vec{c}\cdot\vec{\varphi}}{c^2}\right)}{\sqrt{1-\dfrac{\vec{\varphi}\cdot\vec{\varphi}}{c^2}}}$$

where ($\mathtt{v}$) is the frequency of the non-massive particle relative to an inertial reference frame S, ($\vec{c}$) is the velocity of the non-massive particle relative to the inertial reference frame S, ($\vec{\varphi}$) is the velocity of the inertial reference frame S' relative to the inertial reference frame S and (c) is the speed of light in vacuum.

The Kinetic Force

The kinetic force $\mathbf{K}^a_{ij}$ exerted on a particle i with intrinsic mass m_i by another particle j with intrinsic mass m_j is given by:

$$\mathbf{K}^a_{ij} \;=\; - \left[\frac{m_i\, m_j}{\mathbb{M}} \,(\, \bar{\mathbf{a}}_i - \bar{\mathbf{a}}_j \,) \right]$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i, $\bar{\mathbf{a}}_j$ is the special acceleration of particle j and $\mathbb{M}$ ($= \sum_z m_z$) is the sum of the intrinsic masses of all the particles of the Universe.

The kinetic force $\mathbf{K}^u_i$ exerted on a particle i with intrinsic mass m_i by the Universe is given by:

$$\mathbf{K}^u_i \;=\; -\, m_i\, \frac{\sum_z m_z\, \bar{\mathbf{a}}_z}{\sum_z m_z}$$

where m_z and $\bar{\mathbf{a}}_z$ are the intrinsic mass and the special acceleration of the *z*-th particle of the Universe.

From the above equations it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_j \mathbf{K}^a_{ij}$ $+ \mathbf{K}^u_i$) acting on a particle i with intrinsic mass m_i is given by:

$$\mathbf{K}_i \;=\; -\, m_i\, \bar{\mathbf{a}}_i$$

where $\bar{\mathbf{a}}_i$ is the special acceleration of particle i.

Now, substituting ($\mathbf{F}_i = m_i\, \bar{\mathbf{a}}_i$) and rearranging, we obtain:

$$\mathbf{T}_i \;\doteq\; \mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Therefore, the total force $\mathbf{T}_i$ acting on a particle i is always zero.

Bibliography

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Appendix I

System of Equations I

$$
\begin{tikzcd}
 & \boxed{[1]} \arrow[d, "\downarrow dt \downarrow"] & \\
\boxed{[4]} \arrow[d, "\downarrow dt \downarrow"] & \boxed{[2]} \arrow[l, "\dot{\times}\, \mathbf{r}"'] \arrow[d, "\downarrow dt \downarrow"] & \\
\boxed{[5]} & \boxed{[3]} \arrow[l, "\dot{\times}\, \mathbf{r}"'] \arrow[r, "\int d\mathbf{r}"] & \boxed{[6]}
\end{tikzcd}
$$

$$[\,1\,] \qquad \frac{1}{\mu}\left[\int \mathbf{P}\, dt - \iint \mathbf{F}\, dt\, dt\right] = 0$$

$$[\,2\,] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\, dt\right] = 0$$

$$[\,3\,] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right] = 0$$

$$[\,4\,] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\, dt\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,5\,] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{dt} - \mathbf{F}\right] \dot{\times}\, \mathbf{r} = 0$$

$$[\,6\,] \qquad \frac{1}{\mu}\left[\int \frac{d\mathbf{P}}{dt} \cdot d\mathbf{r} - \int \mathbf{F} \cdot d\mathbf{r}\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass $(\,\mathrm{M}\,)$

Appendix II

System of Equations II

$$
\begin{tikzcd}
 & \boxed{[1]} \arrow[d, "dt"] & \\
\boxed{[4]} \arrow[d, "dt"] & \boxed{[2]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[d, "dt"] & \\
\boxed{[5]} & \boxed{[3]} \arrow[l, "\dot{\times}\,\mathbf{r}"'] \arrow[r, "\int d\mathbf{r}"] & \boxed{[6]}
\end{tikzcd}
$$

$$[\,1\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{r}} - \int\!\!\int \mathbf{F}\; dt\, dt \,\right] = 0$$

$$[\,2\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\; dt \,\right] = 0$$

$$[\,3\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] = 0$$

$$[\,4\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{v}} - \int \mathbf{F}\; dt \,\right] \dot{\times}\;\mathbf{r} = 0$$

$$[\,5\,] \qquad \frac{1}{\mu}\left[\, m\,\bar{\mathbf{a}} - \mathbf{F} \,\right] \dot{\times}\;\mathbf{r} = 0$$

$$[\,6\,] \qquad \frac{1}{\mu}\left[\, m\, f\, c^2 - \int \mathbf{F} \cdot d\mathbf{r} \,\right] = 0$$

$[\,\mu\,]$ is an arbitrary constant with dimension of mass $(\,\mathrm{M}\,)$