The Divergence Myth in Gauss-Bonnet Gravity

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Abstract

In Riemannian geometry there is a unique combination of the Riemann-Christoffel curvature tensor, Ricci tensor and Ricci scalar that defines a fourth-order Lagrangian for conformal gravity theory. This Lagrangian can be greatly simplified by eliminating the curvature tensor term, leaving a unique combination of just the Ricci tensor and scalar. The resulting formalism and the associated equations of motion provide a tantalizing alternative to Einstein-Hilbert gravity that may have application to the problems of dark matter and dark energy without the imposition of the cosmological constant or extraneous scalar, vector and spinor terms typically employed in attempts to generalize the Einstein-Hilbert formalism.

Gauss-Bonnet gravity specifies that the full Lagrangian hides an ordinary divergence (or surface term) that can be used to eliminate the curvature tensor term. In this paper we show that the overall formalism, outside of surface terms necessary for integration by parts, does not involve any such divergence. Instead, it is the Bianchi identities that are hidden in the formalism, and it is this fact that allows for the simplification of the conformal Lagrangian.

1. Overview of Gauss-Bonnet Gravity

The free-space Einstein-Hilbert action in conventional gravity theory is

$$I_{EH} = \int \sqrt{-g}\, R\, d^4x \tag{1.1}$$

where $R = g^{\mu\nu} R_{\mu\nu} = R^\mu_\mu$ is the Ricci scalar. Upon variation with respect to the metric tensor $g^{\mu\nu}$, this goes to

$$\delta I_{EH} = \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}\, d^4x$$

Einstein gravity has proven to be a highly successful theory; its planetary and cosmological predictions typically agree with observation to a very high degree. However, the theory has nothing to say regarding the phenomena known as dark matter and dark energy. Indeed, the addition of the cosmological constant -2Λ to the action yields the revised Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

which may or may not relate to the dark energy question.

In accordance with the requirements of Lorentz/coordinate invariance, the Einstein-Hilbert Lagrangian is a pure scalar density and as such represents the simplest quantity that can be used to build up a consistent gravity theory. There is, however, another symmetry that the theory lacks, that of *conformal invariance*, a suspected invariance of Nature involving *scale* or distance. The notion of scale is normally associated with the metric tensor $g_{\mu\nu}$, which defines the magnitude or length L of an arbitary vector ξ^μ via the invariant form

$$L^2 = g_{\mu\nu} \xi^\mu \xi^\nu$$

One option for generalizing general relativity is to relax the Riemannian requirement that vector length should be necessarily fixed upon parallel transport, but such an option takes us into various non-Riemannian schemes that may or may not have any relevance in the real world. A different approach to scale symmetry is to require that the Lagrangian be invariant with respect to a change in the metric tensor given by $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, where $\Omega(x,t)$ is any arbitrary local function of space and time. A Lagrangian invariant with respect to such a rescaling of the metric thus arguably represents the simplest approach to generalizing Einstein gravity within the constraint of overall vector length invariance demanded by Riemannian geometry.

Clearly, the Einstein-Hilbert action in (1.1) is not scale invariant. It was recognized long ago by the German mathematical physicist Hermann Weyl and others that only a certain combination of the quadratic forms $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, R_{\mu\nu}R^{\mu\nu}$ and R^2 can be used to provide a scale-invariant action in four dimensions. That combination is

$$I = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) d^4x \tag{1.2}$$

Might not this action be suitable for deriving scale-invariant equations of motion? Perhaps, but the resulting equations are highly complicated, not to mention that the fact that they are necessarily of fourth order with respect to $g_{\mu\nu}$ and its derivatives, with the consequent appearance of quantities that appear to be nonphysical.

Much of the complication involves the presence of the $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term. However, it can be effectively eliminated using a clever approach first suggested by Cornelius Lanczos in 1938 (and later by Bryce DeWitt using a different argument). These approaches are rather complicated, but they both assert that the quantity

$$\int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) d^4x \tag{1.3}$$

is an ordinary divergence under conformal variation and can thus be set to zero. This integral can then be subtracted from (1.2), yielding the vastly simplified form

$$\int \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) d^4x \tag{1.4}$$

which is likewise fully conformally invariant. The integral in (1.3) is referred to as the Gauss-Bonnet gravity term (which has nothing to do with the Gauss-Bonnet theorem of differential surface theory), while (1.4) is now recognized as the formal action of modern conformal gravity theory.

We will now demonstrate a different approach in the derivation of (1.4) that is far simpler than those of Lanczos and DeWitt. This approach will also show that the Gauss-Bonnet Lagrangian in (1.3) is not a divergence at all, but vanishes only because it reduces to the Bianchi identities under conformal variation.

2. The Weyl Conformal Tensor

In 1921 Weyl discovered that in ordinary Riemannian geometry there is a unique tensor quantity that is conformally invariant. Now called the *Weyl conformal tensor* $C^\lambda_{\ \nu\alpha\beta}$, its definition in four dimensions is

$$C^\lambda_{\ \nu\alpha\beta} = R^\lambda_{\ \nu\alpha\beta} + \frac{1}{2}\left(\delta^\lambda_\beta R_{\nu\alpha} - \delta^\lambda_\alpha R_{\nu\beta} + g_{\nu\alpha}R^\lambda_\beta - g_{\nu\beta}R^\lambda_\alpha\right) + \frac{1}{6}\left(\delta^\lambda_\alpha g_{\beta\nu} - \delta^\lambda_\beta g_{\alpha\nu}\right)R \tag{2.1}$$

where $R^\lambda_{\ \nu\alpha\beta}$ is the Riemann-Christoffel curvature tensor

$$R^\lambda_{\ \nu\alpha\beta} = \begin{Bmatrix}\lambda \\ \nu\alpha\end{Bmatrix}_{|\beta} - \begin{Bmatrix}\lambda \\ \nu\beta\end{Bmatrix}_{|\alpha} + \begin{Bmatrix}\lambda \\ \sigma\beta\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\lambda \\ \sigma\alpha\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\beta\end{Bmatrix}$$

and where $R_{\nu\beta} = R^\lambda_{\ \nu\lambda\beta}$ and $R = g^{\mu\nu}R_{\mu\nu}$ are its contracted variants. As can be easily verified, the quantity $C^\lambda_{\ \nu\alpha\beta}$ remains unchanged when the metric tensor is rescaled via $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. Consequently, this tensor was considered early on as a candidate for a generalized version of Einstein's 1915 gravity theory based on this kind of scale or conformal symmetry. The Weyl tensor leads to a unique conformal Lagrangian that can be used to build an alternative gravity theory. That Lagrangian is $\sqrt{-g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$, which laboriously works out to be

$$\sqrt{-g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) \tag{2.2}$$

This quadratic quantity is of fourth order with respect to the metric tensor and its derivatives, an undesirable property that greatly complicates the solution of the associated equations of motion. But worse is its mixing of the Riemann-Christoffel curvature tensor with its Ricci cousins, which complicates consideration of spaces that are

Riemann-curved but Ricci-flat (such as the Schwarzschild metric). Nevertheless, if simple conformal invariance is to be demanded, then the Weyl Lagrangian is the only viable candidate.

3. Approach

In the following we will adopt the notation of using a single-bar subscript to denote ordinary partial differentiation, while covariant differentiation will be denoted by a double-bar subscript. Note also that the covariant divergence of a contravariant vector density is equal to the ordinary divergence, so that $(\sqrt{-g}\,\xi^\mu)_{||\mu} = \sqrt{-g}\,\xi^\mu)_{|\mu}$.

We can eliminate the Riemann-Christoffel term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ if we can find coefficients for the $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms that are different from the ones in (1.3). We therefore assume that the more general conformally invariant action

$$I = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right)d^4x \tag{3.1}$$

exists, where $A(\neq -2)$ and $B(\neq 1/3)$ are constants. We can then subtract (3.1) from (1.2) to eliminate the RC term, leaving an invariant Lagrangian consisting of just two terms. As we will see, this approach in fact produces two solutions, with one involving the Bianchi identities.

Consider the infinitesimal change of scale $\delta g^{\mu\nu} = -\pi(x,t)g^{\mu\nu}$, where $\pi \ll 1$. In four dimensions, the variation of $\sqrt{-g}$ is simple:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu} = 2\pi\sqrt{-g}$$

Of course, the variations of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$ and R^2 are much more involved, but the calculations are greatly simplified by using the Palatini identity

$$\delta R^\lambda{}_{\nu\alpha\beta} = \left(\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\alpha}$$

where, for the infinitesimal change of scale $\delta g^{\mu\nu} = -\pi g^{\mu\nu}$,

$$\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix} = \frac{1}{2}\delta^\lambda_\nu\pi_{|\alpha} + \frac{1}{2}\epsilon\delta^\lambda_\alpha\pi_{|\nu} - \frac{1}{2}g_{\nu\alpha}g^{\lambda\beta}\pi_{|\beta} \tag{3.2}$$

We can use integration by parts to transfer the covariant derivatives in the Palatini terms over to their respective coefficients, resulting in surface terms that can be neglected. This will leave a Lagrangian in which all the terms are coefficients of $\pi_{|\mu}$. Setting these terms to zero, we can then determine a unique combination of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms that provides a conformally invariant Lagrangian.

For brevity, we will simply write down the variations we'll need, all of which are easily confirmed. For example,

$$\delta\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 2\sqrt{-g}\,g_{\mu\lambda}R^{\lambda\nu\alpha\beta}\left(\delta\begin{Bmatrix}\mu\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - 2\sqrt{-g}\,g_{\mu\lambda}R^{\lambda\nu\alpha\beta}\left(\delta\begin{Bmatrix}\mu\\ \nu\beta\end{Bmatrix}\right)_{||\alpha}$$

Because of the antisymmetry of the α,β indices, this is just

$$\delta\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 4\sqrt{-g}\,g_{\mu\lambda}R^{\lambda\nu\alpha\beta}\left(\delta\begin{Bmatrix}\mu\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} \tag{3.3}$$

Similarly,

$$\delta\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu} = 2\sqrt{-g}\,R^{\mu\nu}\left(\delta\begin{Bmatrix}\alpha\\ \mu\alpha\end{Bmatrix}\right)_{||\nu} - 2\sqrt{-g}\,R^{\mu\nu}\left(\delta\begin{Bmatrix}\alpha\\ \nu\mu\end{Bmatrix}\right)_{||\alpha} \tag{3.4}$$

and

$$\delta\sqrt{-g}\,R^2 = 2\sqrt{-g}\,g^{\mu\nu}R\left(\delta\begin{Bmatrix}\alpha\\ \mu\alpha\end{Bmatrix}\right)_{||\nu} - 2\sqrt{-g}\,g^{\mu\nu}R\left(\delta\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\right)_{||\alpha} \tag{3.5}$$

Placing these expressions under their respective integrals allows us to integrate by parts over the covariant derivatives. Setting the surface terms to zero, we have

$$\delta \int \sqrt{-g}\, R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}\, d^4x = -4 \int \sqrt{-g}\, R^{\mu\nu}{}_{||\nu}\, \pi_{|\mu}\, d^4x \tag{3.6}$$

$$\delta \int \sqrt{-g}\, R_{\mu\nu} R^{\mu\nu}\, d^4x = -\int \sqrt{-g}\, \left(R^{\mu\nu}{}_{||\nu} + g^{\mu\nu} R_{|\nu} \right) \pi_{|\mu}\, d^4x \tag{3.7}$$

$$\delta \int \sqrt{-g}\, R^2\, d^4x = -6 \int \sqrt{-g}\, g^{\mu\nu} R_{|\nu}\, \pi_{|\mu}\, d^4x \tag{3.8}$$

Putting all this together, we arrive at

$$\delta \int \sqrt{-g}\, \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + A R_{\mu\nu} R^{\mu\nu} + B R^2 \right) d^4x = -\int \sqrt{-g}\, \left[2\,(A+2) R^{\mu\nu}{}_{||\nu} + (A+6B)\, g^{\mu\nu} R_{|\nu} \right] \pi_{|\mu}\, d^4x = 0 \tag{3.9}$$

The obvious solution is $A = -2$, $B = 1/3$. This reproduces (2.2), which was obtained by direct calculation.

However, if $A \neq -2$ then we can divide this term out from (3.9), giving

$$2 \int \sqrt{-g}\, \left[R^{\mu\nu}{}_{||\nu} + \frac{A+6B}{2(A+2)}\, g^{\mu\nu} R_{|\nu} \right] \pi_{|\mu}\, d^4x = 0 \tag{3.10}$$

If we now set

$$\frac{A+6B}{A+2} = -1 \tag{3.11}$$

we can write (3.10) as

$$\int \sqrt{-g}\, \left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R \right)_{||\nu} \pi_{|\mu}\, d^4x = 0 \tag{3.12}$$

We thus arrive at the Bianchi identities under the integral, which vanish identically. Notice that (3.12) is not a divergence/surface term!

We are still free to select constants A, B that satisfy (3.11), the sole constraint being $A \neq -2$. For reasons the writer cannot comprehend, the conventional choice is $A = -4$, $B = 1$, which is consistent with (1.3). But a better choice would be $A = -1$, $B = 0$, in which case the Gauss-Bonnet gravity term is simplified to

$$\int \sqrt{-g}\, \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu} \right) d^4x \qquad ((3.13)$$

Finally, to eliminate the obstreperous curvature term, we subtract (3.13) from (1.2), arriving at the standard conformal gravity action

$$\int \sqrt{-g}\, \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x$$

4. Conclusions/Comments

We have shown that the Gauss-Bonnet gravity term is not a divergence; in view of (3.11), it is not even unique. It vanishes solely because it embeds the Bianchi indenties. However the constants are chosen in (3.11), the standard conformal gravity action (1.4) still results, which is unique.

Notice that this entire formalism is valid only in four dimensions. For spacetimes with $n < 4$, the Weyl conformal tensor is identically zero, and there is no candidate for a conformally invariant action.

It should be noted that the equations of motion for conformal gravity were calculated exactly by Mannheim and Kazanas in 1989, who found that the Schwarzschild-like metric

$$ds^2 = e^{\nu} \left(dx^0 \right)^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

holds, where

$$e^{\nu} = 1 + \frac{\alpha}{r} + \beta r + \gamma r^2 + \delta, \quad e^{\lambda} = e^{-\nu}$$

where $\alpha, \beta, \gamma, \delta$ are constants. The γr^2 term represents an acceleration term, which may have something to do with dark matter and dark energy (although one might legitimately ask "acceleration with respect to *what?*") Nevertheless, the notion of a conformally invariant gravity theory remains an intriguing alternative to standard general relativity.

References

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