

# Kaotic Algebras $Ka^n(p, q)$

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**Abstract.** Manifolds of any dimension and signature have an associated Clifford algebra. Division algebras can be generated by repeated application of the Cayley-Dickson construction, which can be further extended to power associative algebras such as the sedenions. It is possible to apply the Cayley-Dickson construction to Clifford algebras. The series of algebras generated this procedure are referred to in this paper as kaotic algebras, and the notation  $Ka^n(p, q)$  is proposed to designate particular kaotic algebra obtained by  $n$  applications of the Cayley-Dickson construction to a matrix group isomorphic to a Clifford algebra  $Cl(p, q)$ . The Cayley table for  $Ka^3(1, 4)$  is generated. This table has aspects suggesting that this algebra may have applications in physics.

## 1. Introduction

### 2. The kaotic algebra $Ka^3(1, 4)$

Using  $Ka^n(p, q)$  to designate the kaotic algebra obtained by  $n$  applications of the Cayley-Dickson construction to a matrix group isomorphic to a Clifford algebra  $Cl(p, q)$ , the kaotic algebra obtained by three applications of the Cayley-Dickson construction to a Clifford algebra  $Cl(1, 4)$  is designated  $Ka^3(1, 4)$ . This kaotic algebra is not a Moufang loop, but can be constructed in a similar manner.

#### 2.1. Moufang Loop construction for octonions

Rules which construct Moufang Loops from groups can be used to construct octonions from quaternion pairs. A dis-association operator,  $\mu$ , is assigned to the second quaternion pair, and the product rule is, for  $(p, \mu p)$  times  $(q, \mu q)$ :

$$\begin{aligned} p \cdot q &= (pq) \\ p \cdot \mu q &= \mu(qp) \\ \mu p \cdot q &= \mu(pq^{-1}) \\ \mu p \cdot \mu q &= -(q^{-1}p) \\ \text{or, equivalently:} \\ p \cdot q &= (pq) \\ p \cdot \omega q &= \omega(p^{-1}q) \\ \omega p \cdot q &= \omega(qp) \\ \omega p \cdot \omega q &= -(qp^{-1}) \end{aligned}$$

#### 2.2. A Modified Moufang Loop construction for sedenions

A similar construction can be used for sedenions, using three dis-association operators,  $\mu$ ,  $\nu$  and  $\lambda$ , with rules for  $(p, \mu p, \nu p, \lambda p)$  times  $(q, \mu q, \nu q, \lambda q)$  as follows:

$$\begin{aligned} p \cdot q &= pq \\ p \cdot \mu q &= \mu qp \\ p \cdot \nu q &= \nu qp \\ p \cdot \lambda q &= \lambda qp^{-1} \\ \mu p \cdot q &= \mu pq^{-1} \\ \mu p \cdot \mu q &= -q^{-1}p \\ \mu p \cdot \nu q &= \lambda pq \\ \mu p \cdot \lambda q &= -\nu p^{-1}q \\ \nu p \cdot q &= \nu pq^{-1} \\ \nu p \cdot \mu q &= -\lambda qp \\ \nu p \cdot \nu q &= -q^{-1}p \\ \nu p \cdot \lambda q &= \mu qp^{-1} \\ \lambda p \cdot q &= \lambda pq \\ \lambda p \cdot \mu q &= \nu q^{-1}p \\ \lambda p \cdot \nu q &= -\mu pq^{-1} \\ \lambda p \cdot \lambda q &= -p^{-1}q \end{aligned}$$

### 2.3. A Modified Moufang Loop construction for trigtaduonions

A similar construction for trigtaduonions can be assembled, using seven dis-association operators,  $\mu, \nu, \lambda, \alpha, \beta, \gamma$  and  $\delta$ . Rules are imposed for the product:  $(p, \mu p, \nu p, \lambda p, \alpha p, \beta p, \gamma p \delta p)$  times  $(q, \mu q, \nu q, \lambda q, \alpha q, \beta q, \gamma q \delta q)$ .

As there are 64 products, it is easier to describe the construction using a table where each entry defines the product as a procedure, as follows:

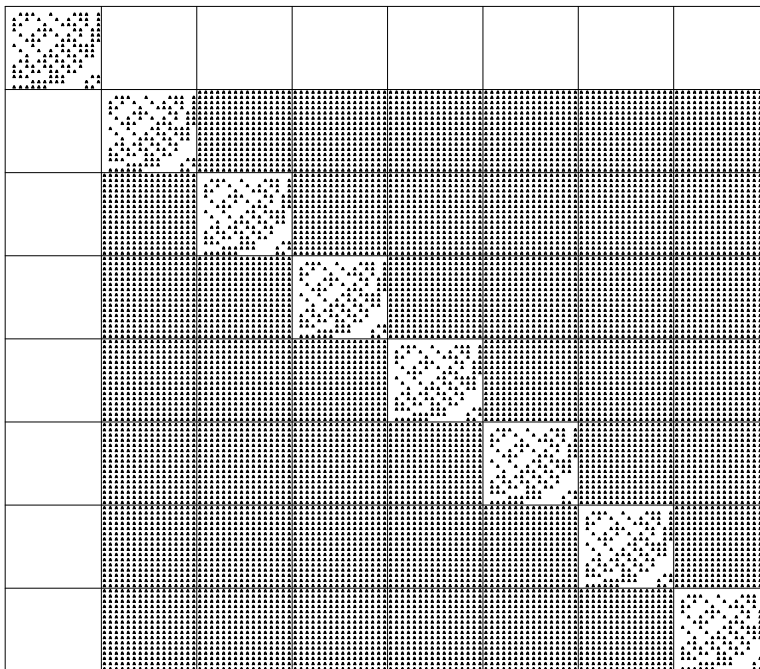
	$q$	$\mu q$	$\nu q$	$\lambda q$	$\alpha q$	$\beta q$	$\gamma q$	$\delta q$
$p$	$+pq$	$+\mu qp$	$+\nu qp$	$+\lambda qp^{-1}$	$+\alpha qp$	$+\beta qp^{-1}$	$+\gamma qp^{-1}$	$+\delta qp$
$\mu p$	$+\mu pq^{-1}$	$-q^{-1}p$	$+\lambda pq$	$-\nu p^{-1}q$	$+\beta pq$	$-\alpha p^{-1}q$	$-\delta pq$	$+\gamma p^{-1}q$
$\nu p$	$+\nu pq^{-1}$	$-\lambda qp$	$-q^{-1}p$	$+\mu qp^{-1}$	$+\gamma pq$	$+\delta qp$	$-\alpha p^{-1}q$	$-\beta qp^{-1}$
$\lambda p$	$+\lambda pq$	$+\nu q^{-1}p$	$-\mu pq^{-1}$	$-p^{-1}q$	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$-\alpha q^{-1}p$
$\alpha p$	$+\alpha pq^{-1}$	$-\beta qp$	$-\gamma qp$	$-\delta qp^{-1}$	$-q^{-1}p$	$+\mu qp^{-1}$	$+\nu qp^{-1}$	$+\lambda qp$
$\beta p$	$+\beta pq$	$+\alpha q^{-1}p$	$-\delta pq$	$+\gamma p^{-1}q$	$-\mu pq^{-1}$	$-p^{-1}q$	$-\lambda pq$	$+\nu p^{-1}q$
$\gamma p$	$+\gamma pq$	$+\delta qp$	$+\alpha q^{-1}p$	$-\beta qp^{-1}$	$-\nu pq^{-1}$	$+\lambda qp$	$-p^{-1}q$	$-\mu qp^{-1}$
$\delta p$	$+\delta pq^{-1}$	$-\gamma q^{-1}p$	$+\beta pq^{-1}$	$+\alpha p^{-1}q$	$-\lambda pq$	$-\nu q^{-1}p$	$+\mu pq^{-1}$	$-q^{-1}p$

If applied to the reals, this construction generates an octonion algebra. If applied to the complex numbers, it generates a sedenion algebra. If applied to a quaternion algebra, it generates a trigtaduonion algebra. When applied to unit multivector matrices for  $Cl(1, 4)$ , it generates the kaotic algebra  $Ka^3(1, 4)$ . The multiplication table for basis elements of  $Ka^3(1, 4)$  is shown in tables 1 to 32, in which hexadecimal numbering has been used to reduce column width.

### 3. Commutation properties

An overview of the commutation properties of products of  $Ka^3(1,4)$  is shown in condensed form in figure 1, where products that anti-commute have an A in their entry.

FIGURE 1. Pattern of anticommuting products for the kaotic algebra



#### 4. Discussion

$Ka^3(1, 4)$  has 16 real basis elements that square to +1 and 240 imaginary basis elements that square to  $-1$ . The imaginary elements can be sorted into two sets, 112 of them are based on real matrices and 128 of them are based on imaginary matrices. This suggests a relationship with the root vectors of the group  $E8$  and the octo-octonionic projective plane.  $E8$  and its subgroups have been used in many attempts to generate a grand unification algebra, by, amongst others, J.D.Smith and G.Lisi.  $Ka^3(1, 4)$  includes the division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . These algebras have been used in a model by G.Dixon.

A difficulty encountered when trying to generate the phenomenology of the standard model from an algebra is in finding a natural way to break the symmetry of the algebra that matches the asymmetry of the standard model. For  $Ka^3(1, 4)$ , as the algebra is based on matrices isomorphic to unit multivectors of  $Cl(1, 4)$ , which are also isomorphic to the complexification of unit multivectors of  $Cl(3, 1)$  or  $Cl(1, 3)$ , a natural way to break the symmetry is to assign basis elements to unit vectors and phase for a space-time manifold. Assigning one basis element to time and three basis elements of opposite signature to three spatial dimensions creates distinctions between subsets of the imaginary basis elements, and thus creates distinctions between the root vectors of  $E8$ , breaking the symmetry of  $E8$ .

An additional method of introducing symmetry breakage is a result of a feature of the trigtaduonions. In an analysis of the subalgebra structure of the trigtaduonions, R. Cawagas noted that there are four non-isotopic sedenion subalgebras within the trigtaduonions[1]. J.Baez has noted that there is a correspondence between Feynman vertices which feature two spinors and a vector component, and the way in which a higher division algebra can be assembled from two copies of a division algebra and a third component[2]. This suggests that the four non-isotopic sedenion subalgebras could result in four different types of Feynman vertices, resulting in four fundamental forces, possible comprising the forces of the standard model together with a force associated with dark matter.

Another difficulty lies in the possibility that some elements of the algebra may be associated with dark matter particles, whose phenomenology is unknown. However, this does offer the opportunity to make testable predictions. A possible approach to this assigns basis elements featuring  $\alpha, \beta, \gamma$  and  $\delta$  to subalgebras associated with dark matter. This is suggested by the observation that the other basis elements, which are the basis elements of  $Ka^2(1, 4)$ , generate copies of the subalgebra  $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$  (including 3 copies of  $\mathbb{O}$ ), which has been used by G. Dixon to describe one family of the fermions of the standard model[3].

A relationship with string and M-theory can be established for  $Ka^2(1, 4)$ .

## 5. Analysis of relationship of $Ka^2(1, 4)$ with string/M-theory

To relate  $Ka^2(1, 4)$  with string/M-theory, which feature one time-like dimension and 9 or 10 space-like dimensions, it is useful to assign orientation to basis elements for  $Ka^2(1, 4)$  using a notation that highlights relationships between basis elements and elements assigned to vectors for a  $Cl(1, 10)$  background. The multiplication table for  $Ka^2(1, 4)$  using this notation is shown in tables 33 to 40.

### 5.1. Notation

$32 \times 32$  matrices can be used to represent unit elements of a  $Cl(1, 10)$  multivector. To represent  $32 \times 32$  in compact form, letters are used to label unit  $2 \times 2$  and  $4 \times 4$  matrices as follows:

$$s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Note that the positive and negative forms of matrices  $R, Q, L, X, Y, Z$  are opposite to those used in a previous paper by this author, The Pattern of Reality[4], for consistency with a convention that matrices with a +1 entry in the first row are the positive form.

This notation allows a real  $32 \times 32$  matrix to be labelled using a three letter combination such as:  ${}^g Y X$ , and an imaginary  $32 \times 32$  matrix to be labelled using a four letter combination such as:  ${}^h D i F$ . Using this notation, products for two matrices can be obtained by multiplying the nested matrices separately. For instance:  
 ${}^g Y X \times {}^h D F = {}^{g,h} Y.D X.F$

To label basis elements for  $Ka^2(1, 4)$  it is only necessary to use  $4 \times 4$  complex matrices together with a dis-association operator which can be  $\nu$ ,  $\mu$  or  $\lambda$ , resulting in labels such as:  ${}^\nu i X$ . When referring to several unit matrices with a common dis-association operator, and/or common real/imaginary status, matrices are enclosed in square brackets, e.g.:  ${}^\nu [i X i Y i Z T]$  and  ${}^\lambda [P Q F D]$ .

### 5.2. Matrices chosen to represent unit vectors for $Cl(1, 10)$

${}^{sS} [i V T X Y Z]$  all anticommute with each other. If  ${}^{sS} [T X Y Z]$  are assigned to represent unit vectors for space-time dimensions,  $[i V]$  could be assigned to the fifth dimension used in the original Kaluza-Klein theory. Then, to expand that theory to eleven dimensions for use in M-theory, it would be logical to select eleven matrices to represent unit vectors such as:

$${}^{sS} [T X Y Z] \quad {}^g [T U V] V \quad {}^h [L M N] V \quad {}^j S i V$$

Then  ${}^{sS} [T X Y Z]$  can be assigned to space-time dimensions, and  ${}^g [T U V] V \quad {}^h [L M N] V \quad {}^j S i V$  can be assigned to compactified dimensions.

### 5.3. Mapping between $Cl(1, 10)$ and $Ka^2(1, 4)$

Space-time dimensions can be regarded as being generated using  $\mathbb{H} \otimes \mathbb{H}$ , the product of two independent quaternions. This suggests using a similar product for octonions to generate compactified dimensions. Consider octonionic elements from  $Ka^2(1, 4)$  such as:  $[STUV]{}^\mu [STUV]$  and  $[SLMN]{}^\nu [SLMN]$ , together with  ${}^\lambda i S$ . Their non-associative components,  ${}^\mu [STUV]{}^\nu [SLMN]{}^\lambda i S$  correspond to  ${}^g [T U V] V \quad {}^h [L M N] V \quad {}^j S i V$ . If sign is ignored, they generate the same product table.

This suggests that an equivalent of the Clifford algebra multivector for the compactified dimensions of M-theory can be identified for  $Ka^2(1, 4)$  elements.

The algebra multivector components for  ${}^g [L V^M V^N V] \quad {}^h [T V^U V^V V] \quad {}^j S V$  are:

Scalar:  ${}^S S$

Vector:  ${}^g [T U V] V, \quad {}^h [L M N] V, \quad {}^j S i V$

Bivector:  ${}^{[T U V L M N]} V, \quad {}^{j [X Y Z P Q R D E F]} V, \quad {}^h [T U V] i V, \quad {}^g [L M N] i V$

3-vector:  ${}^j [T U V L M N] i V, \quad {}^g [S X Y Z P Q R D E F] V, \quad {}^h [S X Y Z P Q R D E F] V, \quad [X Y Z P Q R D E F] i V$

4-vector:  ${}^j [T U V L M N] V, \quad {}^g [S X Y Z P Q R D E F] i V, \quad {}^h [S X Y Z P Q R D E F] i V, \quad [X Y Z P Q R D E F] V$

5-vector:  ${}^{[T U V L M N]} i V, \quad {}^{j [X Y Z P Q R D E F]} i V, \quad {}^h [T U V] V, \quad {}^g [L M N] V$

6-Vector:  ${}^g [T U V] i V, \quad {}^h [L M N] i V, \quad {}^j S V$

Pseudoscalar:  ${}^S i V$

Equivalents of  $Cl(0, 7)$  unit multivector components for  $Ka^2(1, 4)$  elements can be identified as:

Scalar:  $S$

Vector:  ${}^\mu[TUV], {}^\nu[LMN], {}^\lambda iS$

Bivector:  $[TUVLMN], {}^\lambda[XYZPQRDEF], {}^\nu i[TUV], {}^\mu i[LMN]$

3-vector:  ${}^\lambda i[TUVLMN], {}^\mu[XYZPQRDEF], {}^\nu[XYZPQRDEF], i[XYZPQRDEF]$

4-vector:  ${}^\lambda[TUVLMN], {}^\mu i[XYZPQRDEF], {}^\nu i[XYZPQRDEF], [XYZPQRDEF]$

5-vector:  $i[TUVLMN], {}^\lambda i[XYZPQRDEF], {}^\nu[TUV], {}^\mu[LMN]$

6-Vector:  ${}^\mu i[TUV], {}^\mu i[LMN], {}^\lambda S$

Pseudoscalar:  $iS$

This mapping may allow much of the mathematics of string/M-theory to be applied to  $Ka^2(1, 4)$  and  $Ka^3(1, 4)$ .

## 6. Acknowledgements

Many papers available on arXiv and elsewhere on the internet contributed to the development of this paper, but most use higher mathematical methods, which make them less accessible for a synthesist and making it hard to acknowledge particular contributions. I thank Wikipedia and its contributors and John Baez for information presented on the internet on a more accessible level. I thank F.D.Smith, G.Dixon and J.Koeplinger for their comments on earlier versions of this paper.

## References

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TABLE 5.  $Ka^3(1, 4)$  multiplication table, part B1

Table with columns labeled e20 to e3F and rows labeled e0 to e3F. Each cell contains a complex algebraic expression involving variables A, B, C, D and their products.

















TABLE 13.  $Ka^3(1, 4)$  multiplication table, part D1

Table with 30 columns (e00 to e7F) and 30 rows (e0 to e7F). The table contains a dense grid of algebraic expressions representing multiplication results for each combination of indices.





TABLE 16.  $K\alpha^3(1, 4)$  multiplication table, part D4

Table with columns labeled e60 through e7F and rows labeled eC0 through e7F. Each cell contains a linear combination of terms like -A0, +A1, -B2, etc.















TABLE 23.  $Ka^3(1, 4)$  multiplication table, part F3

	$\epsilon_{20}$	$\epsilon_{41}$	$\epsilon_{42}$	$\epsilon_{43}$	$\epsilon_{44}$	$\epsilon_{45}$	$\epsilon_{46}$	$\epsilon_{47}$	$\epsilon_{48}$	$\epsilon_{49}$	$\epsilon_{AA}$	$\epsilon_{AB}$	$\epsilon_{AC}$	$\epsilon_{AD}$	$\epsilon_{AE}$	$\epsilon_{AF}$	$\epsilon_{20}$	$\epsilon_{21}$	$\epsilon_{22}$	$\epsilon_{23}$	$\epsilon_{24}$	$\epsilon_{25}$	$\epsilon_{26}$	$\epsilon_{27}$	$\epsilon_{28}$	$\epsilon_{29}$	$\epsilon_{30}$	$\epsilon_{31}$	$\epsilon_{32}$	$\epsilon_{33}$	$\epsilon_{34}$	$\epsilon_{35}$	$\epsilon_{36}$	$\epsilon_{37}$	$\epsilon_{38}$	$\epsilon_{39}$	$\epsilon_{BA}$	$\epsilon_{BB}$	$\epsilon_{BC}$	$\epsilon_{BD}$	$\epsilon_{BE}$	$\epsilon_{BF}$																																																																																																																															
$\epsilon_{80}$	-20	-21	-22	-23	-24	-25	-26	-27	-28	-29	+2A	+2B	+2C	+2D	+2E	+2F	+30	+31	+32	+33	+34	+35	+36	+37	+38	+39	-3A	-3B	-3C	-3D	-3E	-3F	-30	-31	-32	-33	-34	-35	-36	-37	-38	-39	-3A	-3B	-3C	-3D	-3E	-3F																																																																																																																									
$\epsilon_{81}$	-21	-20	+AC	-AB	-AF	+29	-28	+AD	-26	+25	-2E	+A3	+A2	-A7	-2A	+AA	+31	+30	-BC	+BB	+BF	-39	+38	+36	-35	+3E	-B3	+B2	+B7	+3A	-B4	-22	-21	+AC	-AB	-AF	+29	-28	+AD	-26	+25	-2E	+A3	+A2	-A7	-2A	+AA	+31	+30	-BC	+BB	+BF	-39	+38	+36	-35	+3E	-B3	+B2	+B7	+3A	-B4																																																																																																											
$\epsilon_{82}$	-22	-AC	-20	-AA	-29	-AF	-27	-26	-24	+AD	-23	+A3	+2E	+A1	+A8	+2D	+A5	+32	+BC	+30	+BA	+39	+BF	+37	+36	+BD	+34	-B3	-3E	-B1	-B8	-3B	-B5	-23	-22	+AC	-20	-AA	-29	-AF	-27	-26	-24	+AD	-23	+A3	+2E	+A1	+A8	+2D	+A5	+32	+BC	+30	+BA	+39	+BF	+37	+36	+BD	+34	-B3	-3E	-B1	-B8	-3B	-B5																																																																																																						
$\epsilon_{83}$	-23	+AB	-AA	-20	-28	-27	+AF	-25	-24	+AD	-A2	-A1	+2E	-A9	+2C	-A6	+33	-BB	-BA	+30	+38	+37	-BF	+35	+34	-BD	+B2	+B1	-3E	+D9	-3C	+B6	-23	-22	+AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																											
$\epsilon_{84}$	-24	+AF	-29	-28	+AC	+AB	+AE	-23	-22	+2D	-A6	-A5	+2A	-A7	-A1	+A4	-BF	+39	+38	+30	-BC	-BB	-BE	+33	+32	-3D	+B6	+B5	-3A	+B7	+B1	-25	-24	+AF	-29	-28	+AC	+AB	+AE	-23	-22	+2D	-A6	-A5	+2A	-A7	-A1	+A4	-BF	+39	+38	+30	-BC	-BB	-BE	+33	+32	-3D	+B6	+B5	-3A	+B7	+B1																																																																																																										
$\epsilon_{85}$	-25	+29	+AF	-27	-AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2	-26	-25	+29	+AF	-27	-AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																								
$\epsilon_{86}$	-26	-28	-27	-AF	-AB	-AA	-20	-22	-21	-AE	+A5	+A4	+2D	+2C	+A9	+A3	+36	+38	+37	+BF	+BE	+BA	+30	+32	+31	-BE	-B5	-B4	-3D	-3C	-B9	-B3	-27	-26	-25	+AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																										
$\epsilon_{87}$	-27	-AD	-26	-25	-AE	-23	-22	-20	-AC	-AB	+2F	+A9	+A8	+A1	+A4	+2A	+37	+BD	+36	+35	+BE	+33	+32	+30	+BC	+BB	-3F	-B9	-B8	-B1	-B4	-3A	-27	-26	-25	+AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																										
$\epsilon_{88}$	-28	-26	+AD	-24	-23	+AE	-21	+AC	-20	+AA	-A9	+2F	-A7	-A2	-A5	+2B	+38	+36	-BD	+34	+33	-BE	+31	-BC	+30	-BA	+B9	-3F	+B7	+B2	+B5	-3B	-29	-28	-27	+AD	-24	-23	+AE	-21	+AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																					
$\epsilon_{89}$	-29	+25	-24	-AD	-22	+21	+AE	+AB	-AA	-20	+A8	-A7	-2F	+A3	-A6	-2C	+39	-35	+34	+BD	+32	-31	-BE	-BB	+BA	+30	-B8	+B7	+3F	-B3	+B6	+3C	-32	-31	-AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																											
$\epsilon_{8A}$	+2A	-2E	-A3	+A2	+2D	+A6	-A5	+2F	+A9	-A8	-20	-AC	+AB	-24	+21	-27	-3A	+3E	+B3	-B2	-3D	-B6	+B5	-3F	-B9	+B8	+30	+BC	-BB	+A4	-31	+37	-32	-31	-AC	-20	+AA	-23	-AE	+21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																											
$\epsilon_{8B}$	+2B	-A3	+2E	+A1	+A6	-2D	-A4	-A9	+2F	+A7	+AC	-20	-AA	+25	-22	-28	-B3	+B3	-3E	-B1	-B6	+3D	+B4	+B9	-3F	-B7	-BC	+30	+BA	-35	+32	+38	-29	-28	-27	+AD	-24	-23	+AE	-21	-A6	-2D	+A4	-2B	+A8	-A2	+35	-39	-BF	+37	+BC	+30	-BA	+33	+BE	-31	+D6	+3D	-B4	+3B	-B8	-B2																																																																																																											
$\epsilon_{8C}$	+2C	+A2	-A1	+2E	+A5	+A4	+2D	-A8	+2F	-AA	-20	-26	-23	+29	-3C	-B2	+B1	-3E	-B5	+B4	-3D	+B8	-B7	+3F	+BB	-BA	+30	+36	+33	-39	-38	-37	-36	-35	-34	-33	-32	-31	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100















TABLE 30.  $Ka^3(1, 4)$  multiplication table, part H2

Table with columns labeled εE0 through εFF and rows labeled εE0 through εFF. Each cell contains a linear combination of terms like +A, -A, +B, -B, etc.





















TABLE 40.  $Ka^2(1, 4)$  multiplication table, part D2

	$+s$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+s$	$+s$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+x$	$+x$	$+s$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+y$	$+y$	$+x$	$+s$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+z$	$+z$	$+x$	$+y$	$+s$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+w$	$+w$	$+x$	$+y$	$+z$	$+s$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+v$	$+v$	$+x$	$+y$	$+z$	$+w$	$+s$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+u$	$+u$	$+x$	$+y$	$+z$	$+w$	$+v$	$+s$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+t$	$+t$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+s$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+r$	$+r$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+s$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+q$	$+q$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+s$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+p$	$+p$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+s$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+o$	$+o$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+s$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+n$	$+n$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+s$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+m$	$+m$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+s$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+l$	$+l$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+s$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+k$	$+k$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+s$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+j$	$+j$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+s$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+i$	$+i$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+s$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+h$	$+h$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+s$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+g$	$+g$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+s$	$+f$	$+e$	$+d$	$+c$	$+b$	$+a$
$+f$	$+f$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+s$	$+e$	$+d$	$+c$	$+b$	$+a$
$+e$	$+e$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+s$	$+d$	$+c$	$+b$	$+a$
$+d$	$+d$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+s$	$+c$	$+b$	$+a$
$+c$	$+c$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+s$	$+b$	$+a$
$+b$	$+b$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+s$	$+a$
$+a$	$+a$	$+x$	$+y$	$+z$	$+w$	$+v$	$+u$	$+t$	$+r$	$+q$	$+p$	$+o$	$+n$	$+m$	$+l$	$+k$	$+j$	$+i$	$+h$	$+g$	$+f$	$+e$	$+d$	$+c$	$+b$	$+s$