An Elementary Proof of BEAL Conjecture

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Abstract

In 1997, Andrew Beal [1] announced the following conjecture : *Let A, B, C, m, n, and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then A, B, and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l. We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers.

Keywords: Prime numbers, divisibility, roots of polynomials of third degree.

O my Lord! Increase me further in knowledge.
(Holy Quran, Surah Ta Ha, 20:114.)

To my Wife Wahida

1 Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

***Conjecture* 1.1.** *Let A, B, C, m, n, and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1}$$

then A, B, and C have a common factor.

In this paper, we give an elementary proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots A^m, B^n and $-C^l$ with the condition (1.1). In the next section, we do some preliminaries calculs to give the expressions of the three roots of $P(x) = 0$. The proof of the conjecture (1.1) is the subject of the section 3.

We begin with the trivial case when $A^m = B^n$. The equation (1.1) becomes:

$$2A^m = C^l \tag{1.2}$$

then $2|C^l \Longrightarrow 2|C \Longrightarrow \exists\, c \in \mathbb{N}^*/\ C = 2c$, it follows $2A^m = 2^l c^l \Longrightarrow A^m = 2^{l-1}c^l$. As $l > 2$, then $2|A^m \Longrightarrow 2|A \Longrightarrow 2|B^n \Longrightarrow 2|B$. The conjecture (1.1) is verified.

We suppose in the following that $A^m > B^n$.

2 Preliminaries Calculs

Let $m, n, l \in \mathbb{N}^* \ > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{2.1}$$

We call:

$$P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) + x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \tag{2.2}$$

Using the equation (2.1), $P(x)$ can be written:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n (A^m + B^n)} \tag{2.3}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n \tag{2.4}$$

$$q = A^m B^n (A^m + B^n) \tag{2.5}$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \tag{2.6}$$

Equation (2.3) becomes:

$$P(x) = x^3 - px + q \tag{2.7}$$

Using the equation (2.2), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{2.8}$$

To resolve (2.8) let:

$$x = u + v \tag{2.9}$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0 \tag{2.10}$$

To determine u and v, we obtain the conditions:

$$u^3 + v^3 = -q \tag{2.11}$$

$$uv = p/3 > 0 \tag{2.12}$$

Then u^3 and v^3 are solutions of the second ordre equation:

$$X^2 + qX + p^3/27 = 0 \tag{2.13}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\overline{\Delta}}{27} \tag{2.14}$$

Let:

$$\begin{aligned}\overline{\Delta} = 27q^2 - 4p^3 &= 27(A^mB^n(A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^mB^n]^3 \\ &= 27A^{2m}B^{2n}(A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^mB^n]^3 \end{aligned} \tag{2.15}$$

Noting :

$$\alpha = A^mB^n > 0 \tag{2.16}$$

$$\beta = (A^m + B^n)^2 \tag{2.17}$$

we can write (2.15) as:

$$\overline{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \tag{2.18}$$

As $\alpha \neq 0$, we can also rewrite (2.18) as :

$$\overline{\Delta} = \alpha^3 \left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3\right) \tag{2.19}$$

We call t the parameter :

$$t = \frac{\beta}{\alpha} \tag{2.20}$$

$\overline{\Delta}$ becomes :

$$\overline{\Delta} = \alpha^3(27t - 4(t-1)^3) \tag{2.21}$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \tag{2.22}$$

Since $\alpha > 0$, the signe of $\overline{\Delta}$ is also the signe of $y(t)$. Let us study the signe of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1 + 2t)(5 - 2t) \tag{2.23}$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

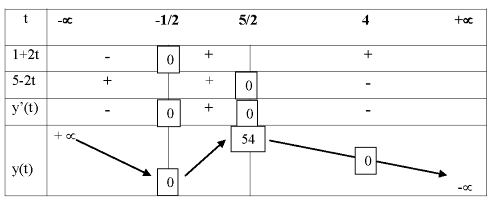

Fig. 1: The table of variation

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m + B^n)^2 > 4\alpha = 4A^mB^n \tag{2.24}$$

Then $y < 0 \Longrightarrow \overline{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (2.13) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positif or a negatif real and $u.v = p/3$.

2.1 Demonstration

Proof. The solutions of (2.13) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.25}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.26}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{2.27}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{2.28}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta} \tag{2.29}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \tag{2.30}$$

$$\text{and } sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \tag{2.31}$$

$$cos\theta = -\frac{q}{2\rho} < 0 \tag{2.32}$$

Then $\theta \in]+\frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{2.33}$$

and

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{2.34}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{2.35}$$

Let:

$$u = re^{i\psi} \tag{2.36}$$

$$\text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{2.37}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \overline{j} \tag{2.38}$$

j is a complex cubic root of the unity $\Longleftrightarrow j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{2.39}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{2.40}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{2.41}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{2.42}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{2.43}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{2.44}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{2.45}$$

$$v_2 = \overline{u_2} \tag{2.46}$$

$$v_3 = \overline{u_3} \tag{2.47}$$

We obtain as real solutions of the equation (2.10):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{2.48}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\frac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\frac{\theta}{3} + \sqrt{3}sin\frac{\theta}{3}\right) < 0 \tag{2.49}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\frac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\frac{\theta}{3} + \sqrt{3}sin\frac{\theta}{3}\right) > 0 \tag{2.50}$$

Using the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p}cos\frac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\frac{\theta}{3} + \sqrt{3}sin\frac{\theta}{3}\right)$$

$$3cos\frac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\frac{\theta}{3} \tag{2.51}$$

As $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$, then $sin\dfrac{\theta}{3}$ and $cos\dfrac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{2.52}$$

which is true since $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (2.8), we consider, as A^m is supposed great than B^n, the expressions:

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}} \tag{2.53}$$

□

3 Proof of the Main Theorem

Main Theorem: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{3.1}$$

then $A, B,$ and C have a common factor.

Proof. $A^m = 2\sqrt[3]{\rho}cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3}$ is an integer. But:

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{3.2}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{3.3}$$

As A^{2m} is an integer, and p is an integer then $cos^2\frac{\theta}{3}$ must be written in the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{3.4}$$

with $b \in \mathbb{N}^*$, for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

3.1 Case $cos^2\frac{\theta}{3} = \frac{1}{b}$

we obtain :

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{3.5}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

3.1.1 $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

3.1.2 $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2.p' \tag{3.6}$$

But :

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{1}{2}\right) = \frac{p}{3} = \frac{3p'}{3} = p' \tag{3.7}$$

On the one hand:

$$A^{2m} = (A^m)^2 = 2p' \Rightarrow 2|p' \Rightarrow p' = 2p"^2 \Rightarrow A^{2m} = 4p"^2$$
$$\Rightarrow A^m = 2p" \Rightarrow 2|A^m \Rightarrow 2|A$$

On the other hand:
$B^nC^l = p' = 2p"^2 \Rightarrow 2|B^n$ or $2|C^l$. If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A^m$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : A,B and C solutions of (2.1) have a common factor. Also if $2|C^l$, we obtain the same result : A,B and C solutions of (2.1) have a common factor.

3.1.3 $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{3} = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Longrightarrow p'$ is not a prime. Let μ a prime with $\mu|p' \Rightarrow \mu|A^{2m} \Rightarrow \mu|A$.

On the other hand:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 5p'$$

Then $\mu|B^n$ or $\mu|C^l$. If $\mu|B^n \Rightarrow \mu|B$. As $C^l = A^m + B^n$ and $\mu|A$ and $\mu|B$, it follows $\mu|A^m$ and $\mu|B^n$ then $\mu|(A^m + B^n) \Rightarrow \mu|C^l \Longrightarrow \mu|C$.

Then, we have : A,B and C solutions of (2.1) have a common factor. Also if $\mu|C^l$, we obtain the same result : A,B and C solutions of (2.1) have a common factor.

3.2 Case $a > 1,\ cos^2\frac{\theta}{3} = \frac{a}{b}$

That is to say:

$$cos^2\frac{\theta}{3} = \frac{a}{b} \tag{3.8}$$

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{3.9}$$

and a, b verify one of the two conditions:

$$\boxed{\{3|p \quad and \quad b|4p\}} \quad \text{or} \quad \boxed{\{3|a \quad and \quad b|4p\}} \tag{3.10}$$

and using the equation (2.34), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{3.11}$$

In these conditions, respectively, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (3.10).

3.2.1 Hypothesis: $\{3|p \quad and \quad b|4p\}$

3.2.1.1. Case $b = 2$ **and** $3|p$ **:** $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a \tag{3.12}$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \tag{3.13}$$

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

3.2.1.2. Case $b = 4$ **and** $3|p$ **:** We have $3|p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \tag{3.14}$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \tag{3.15}$$

But a, b are coprime. Then the case $b = 4$ and $3|p$ is impossible.

3.2.1.3. Case: $b \neq 2, b \neq 4$, $b|p$ **and** $3|p$ **:** As $3|p$ then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b} \tag{3.16}$$

We consider the case: $b|p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (if $p" = 1$, then $p = 3b$, see sub-paragraph 2^{sd} sous-case equation (3.36)). Hence :

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{3.17}$$

Let us calculate B^nC^l:

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = b.p".\frac{3b - 4a}{b} = p".(3b - 4a) \tag{3.18}$$

Finally, we have the two equations:

$$A^{2m} = \frac{4bp"a}{b} = 4ap" \tag{3.19}$$

$$B^nC^l = p".(3b - 4a) \tag{3.20}$$

Sous-case 1: p" is prime. From (3.19), $p"|A^{2m} \Rightarrow p"|A^m \Rightarrow p"|A$. From (3.20), $p"|B^n$ or $p"|C^l$. If $p"|B^n \Rightarrow p"|B$, as $C^l = A^m + B^n \Rightarrow p"|C^l \Rightarrow p"|C$. If $p"|C^l \Rightarrow p"|C$, as $B^n = C^l - A^m \Rightarrow p"|B^n \Rightarrow p"|B$.

Then A,B and C solutions of (2.1) have a common factor.

Sous-case 2: p" is not prime. Let λ one prime divisor of $p"$. From (3.19), we have :

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{3.21}$$

From (3.20), as $\lambda|p"$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \ \lambda|C^l \tag{3.22}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also :

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is prime, then} \quad \lambda|C \tag{3.23}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$.

Then: A, B and C solutions of (2.1) have a common factor.

Let us verify the condition (3.11) given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{3.24}$$

The $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ is verified.
If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^n Y - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

3.2.1.4. Case $b = 3$ and $3|p$: As $3|p \Longrightarrow p = 3p'$ and we write :

$$A^{2m} = \frac{4p}{3} cos^2 \frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3} \tag{3.25}$$

As A^{2m} is an integer and that a and b are coprime and $cos^2\frac{\theta}{3}$ can not be one in reference to the equation (2.33), then we have necessary $3|p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3 \times 3p" = 9$ but $p = A^{2m} + B^{2n} + A^m B^n > 9$, the hypothesis $p" = 1$ is impossible, then $p" > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a \tag{3.26}$$

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9 - 4a)}{3} = p".(9 - 4a) \tag{3.27}$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow a = 2$ as $a > 1$.
$a = 2$, we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a = 8p" \tag{3.28}$$

$$B^n C^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = p'\left(3 - 4\frac{a}{b}\right) = \frac{3p"(9 - 4a)}{3} = p" \tag{3.29}$$

The two last equations give that $p"$ is not prime. Then we use the same methodology describted above for the case 3.2.1.3., and we have : A,B and C solutions of (2.1) have a common factor.

3.2.1.5. Case $3|p$ and $b=p$: We have :

$$cos^2\frac{\theta}{3}=\frac{a}{b}=\frac{a}{p}$$

and :

$$A^{2m}=\frac{4p}{3}cos^2\frac{\theta}{3}=\frac{4p}{3}.\frac{a}{p}=\frac{4a}{3} \tag{3.30}$$

As A^{2m} is an integer, this implies that $3|a$, but $3|p \Longrightarrow\ 3|b$. As a and b are coprime, hence the contradiction. Then the case $3|p$ and $b=p$ is impossible.

3.2.1.6. Case $3|p$ and $b=4p$: $3|p \Longrightarrow p=3p'$, $p'\neq 1$ because $3 \ll p$, hence $b=4p=12p'$.

$$A^{2m}=\frac{4p}{3}cos^2\frac{\theta}{3}=\frac{4p}{3}\frac{a}{b}=\frac{a}{3}\Longrightarrow 3|a \tag{3.31}$$

because A^{2m} is an integer. But $3|p \Longrightarrow\ 3|\,[(4p)=b]$, that is in contradiction with the hypothesis a,b are coprime. Then the case $b=4p$ is impossible.

3.2.1.7. Case $3|p$ and $b=2p$: $3|p \Longrightarrow p=3p'$, $p'\neq 1$ because $3 \ll p$, hence $b=2p=6p'$.

$$A^{2m}=\frac{4p}{3}cos^2\frac{\theta}{3}=\frac{4p}{3}\frac{a}{b}=\frac{2a}{3}\Longrightarrow 3|a \tag{3.32}$$

because A^{2m} is an integer. But $3|p \Longrightarrow\ 3|(2p) \Longrightarrow\ 3|b$, that is in contradiction with the hypothesis a,b are coprime. Then the case $b=2p$ is impossible.

3.2.1.8. Case $3|p$ and $b\neq 3$ is a divisor of p : We have $b=p'\neq 3$, and p is written as:

$$p=kp' \quad with \quad 3|k\Longrightarrow k=3k' \tag{3.33}$$

and :

$$A^{2m}=\frac{4p}{3}cos^2\frac{\theta}{3}=\frac{4p}{3}.\frac{a}{b}=\frac{4\times 3.k'p'}{3}\frac{a}{p'}=4ak' \tag{3.34}$$

We calculate B^nC^l:

$$B^nC^l=\frac{p}{3}.\left(3-4cos^2\frac{\theta}{3}\right)=k'(3p'-4a) \tag{3.35}$$

<u>1^{st} Sous-case:</u> $k'\neq 1$, we use the same methodology described for the case 3.1.2.3., and we obtain: A,B and C solutions of (2.1) have a common factor.

<u>2^{nd} sous-case:</u>

$$k'=1\Longrightarrow p=3b \tag{3.36}$$

then we have:

$$A^{2m}=4a\Longrightarrow a \quad \text{is even} \tag{3.37}$$

and :

$$A^mB^n=2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3}-cos\frac{\theta}{3}\right)=\frac{p\sqrt{3}}{3}sin\frac{2\theta}{3}-2a \tag{3.38}$$

let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} sin\frac{2\theta}{3} = 2b\sqrt{3} sin\frac{2\theta}{3} \tag{3.39}$$

The left member of (3.39) is an integer and b also, then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written in the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.40}$$

where k_1, k_2 are two coprime integers and $k_2|b \Longrightarrow b = k_2.k_3$.

♢ - We suppose $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^m B^n = k_3.k_1 \tag{3.41}$$

Let μ is an prime integer such that $\mu|k_3$. If $\mu = 2 \Rightarrow 2|b$ but $2|a$ that is contradiction with a, b coprime. We suppose $\mu \neq 2$ and $\mu|k_3$, then $\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

*A-1- If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|4a \Longrightarrow \mu|a$. As $\mu|k_3 \Longrightarrow \mu|b$ and that a, b are coprime hence the contradiction.

*A-2- If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.42}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.43}$$

As $p = 3b = 3k_2.k_3$ and $\mu|k_3$ hence $\mu|p \Longrightarrow p = \mu\mu'$, so we have :

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{3.44}$$

and $\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n$ or $\mu|(B^n - A^m)$.

*A-2-1- If $\mu|B^n \Longrightarrow \mu|B$ which is in contradiction with *A-2.

*A-2-2- If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n \Longrightarrow \mu|B \text{ which is impossible} \\ or \\ \mu = 3 \end{cases} \tag{3.45}$$

*A-2-2-1- If $\mu = 3 \Longrightarrow 3|k_3 \Longrightarrow k_3 = 3k_3'$, and we have $b = k_2k_3 = 3k_2k_3'$, it follows $p = 3b = 9k_2k_3'$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then :

$$9k_2k_3' - 3A^mB^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{3.46}$$

hence $3|(3k_2k'_3 - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow 3|A^m$ or $3|B^n$.

*A-2-2-1-1- If $3|A^m \Longrightarrow 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \Longrightarrow\ 3|4a \Longrightarrow 3|a$. As $b = 3k_2k'_3$ then $3|b$, but a, b are coprime hence the contradiction. Then $3 \nmid A$.

*A-2-2-1-2- If $3|B^n \Longrightarrow 3|B$, but the (3.46) gives $3|(A^m-B^n)^2 \Longrightarrow 3|(A^m-B^n) \Longrightarrow 3|A^m \Longrightarrow 3|A$. But using the result of the last paragraph *A-2-2-1-1, we obtain $3 \nmid A$. Then the hypothesis $k_3 \neq 1$ is impossible.

♢- Now we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{3.47}$$

with k_1, b coprime. We write (3.47) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and remplacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \tag{3.48}$$

which implies that :

$$3|a \quad or \quad 3|(b-a)$$

*B-1- If $3|a$, as $A^{2m} = 4a \Longrightarrow 3|A^{2m} \Longrightarrow 3|A$. But $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Longrightarrow 3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n)$. But $3|A$ hence $3|B^n \Longrightarrow 3|B$, it follows $3|C^l \Longrightarrow 3|C$.

We obtain: A,B and C solutions of (2.1) have a common factor.

*B-2- Considering now that $3|(b-a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (3.41) and that $3|k_1 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow 3|A^m$ or $3|(A^m + 2B^n)$.

*B-2-1- If $3|A^m \Longrightarrow 3|A \Longrightarrow 3|A^{2m}$ then $3|4a \Longrightarrow 3|a$. But $3|(b-a) \Longrightarrow 3|b$ hence the contradiction with a, b are coprime.

*B-2-2- If:

$$3|(A^m + 2B^n) \Longrightarrow 3|(A^m - B^n) \tag{3.49}$$

But $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \Longrightarrow 9|(p - 3A^mB^n)$ or $9|(3b - 3A^mB^n)$, then $3|(b - A^mB^n)$ but $3|(b-a) \Longrightarrow 3|(a - A^mB^n)$. As $A^{2m} = 4a = (A^m)^2 \Longrightarrow \exists a' \in \mathbb{N}^*$ and $a = a'^2 \Longrightarrow A^m = 2a'$. We arrive to $3|(a'^2 - 2a'B^n) \Longrightarrow 3|a'(a' - 2B^n)$.

*B-2-2-1- If $3|a' \Longrightarrow 3|A^m \Longrightarrow 3|A$, but $3|(A^m+2B^n) \Longrightarrow 3|2B^n \Longrightarrow 3|B^n \Longrightarrow 3|B$, it follows $3|C$.

Hence A,B and C solutions of (2.1) have a common factor.

*B-2-2-2- Now if $3|(a'-2B^n) \Longrightarrow 3|(2a'-4B^n) \Longrightarrow 3|(A^m-4B^n) \Longrightarrow 3|(A^m-B^n)$, we refind the hypothesis (3.49) above.

The study of the case 3.2.1.8. is finished.

3.2.2 Hypothesis : $\{3|a \quad and \quad b|4p\}$

We have :

$$3|a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{3.50}$$

3.2.2.1. Case $b = 2$ and $3|a$: A^{2m} is written as :

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{3.51}$$

Using the equation (3.50), A^{2m} becomes:

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{3.52}$$

But $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

3.2.2.2. Case $b = 4$ and $3|a$: A^{2m} is written as :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{3.53}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{3.54}$$

which is impossible.

Then the case $b = 4$ is impossible.

3.2.2.3. Case $b = p$ and $3|a$: Then:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{3.55}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{3.56}$$

$$\exists a" \in \mathbb{N}^* \ / \ a' = a"^2 \tag{3.57}$$

We calculate A^mB^n, hence:

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{3.58}$$

The left member of (3.58) is an integer and p is also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.59}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = b = k_2.k_3,\ k_3 \in \mathbb{N}^*$.

$\diamondsuit$ - We suppose that $k_3 \neq 1$. We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{3.60}$$

Let us μ a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

* If $\mu|A^m \Longrightarrow \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu|4a' \Longrightarrow (\mu = 2$ but $2|a')$ or $(\mu|a')$. Then $\mu|a$ hence the contradiction with a, b coprime.

* If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.61}$$

It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.62}$$

Since $p = b = k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists\mu' \in \mathbb{N}^*$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{3.63}$$

From the last equation, we get $\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n$ or $\mu|(B^n - A^m)$. If $\mu|B^n$ which is contradiction with $\mu \nmid B^n$. If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, on arrive to:

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n \Longrightarrow & \text{which is contradiction} \\ or & \\ \mu = 3 & \end{cases} \tag{3.64}$$

Si $\mu = 3$, then $3|b$, but $3|a$ thus the contradiction with a, b coprime.

$\diamondsuit$ - We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1 \tag{3.65}$$

$$b = k_2 \tag{3.66}$$

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{3.67}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3}sin^2\frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3}sin^2\frac{\theta}{3}cos^2\frac{\theta}{3} = \frac{k_1^2}{b^2}$$
$$\frac{16}{3}sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2a'(p-a) = k_1^2 \tag{3.68}$$

but $a' = a"^2$ then $p-a$ is a square. Let us:

$$\lambda^2 = p-a \tag{3.69}$$

The equation (3.100) becomes:

$$4^2a"^2\lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{3.70}$$

taking the positif square root. Using (3.97), we get :

$$k_1 = 4a"\lambda \tag{3.71}$$

But $k_1 = A^m(A^m+2B^n) = 2a"(A^m+2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda \tag{3.72}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2 \Longrightarrow 2|(p-a)$ but a is even, then $2|p \Longrightarrow 2|b$ which is contradiction with a,b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|\lambda^2 \quad and \quad \lambda_1|(A^m+2B^n) \tag{3.73}$$
$$\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{3.74}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(p=b) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{3.75}$$

hence the contradiction with a,b coprime.

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(A^m+2B^n)^2$ that is $\lambda_1|(A^{2m}+4A^mB^n+4B^{2n})$, we write it as $\lambda_1|(p+3A^mB^n+3B^{2n}) \Longrightarrow \lambda_1|(p+3B^n(A^m+2B^n)-3B^{2n})$. But $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(p-3B^{2n})$, as $\lambda_1|(p-a)$ hence by difference, we obtain $\lambda_1|(a-3B^{2n})$ or $\lambda_1|(3a'-3B^{2n}) \Longrightarrow \lambda_1|3(a'-B^{2n}) \Longrightarrow \lambda_1 = 3$ or $\lambda_1|(a'-B^{2n})$.

*A-1- If $\lambda_1 = 3$ but $3|a \Longrightarrow 3|(p=b)$ hence the contradiction with a,b coprime.

*A-2- If $\lambda_1|(a'-B^{2n}) \Longrightarrow \lambda_1|(a"^2-B^{2n}) \Longrightarrow \lambda_1|(a"-B^n)(a"+B^n) \Longrightarrow \lambda_1|(a"+B^n)$ or $\lambda_1|(a"-B^n)$, because $(a"-B^n) \neq 1$ if not we obtain $a"^2-B^{2n} = a"+B^n \Longrightarrow a"^2-a" = B^n-B^{2n}$. The left member is positif and the right member is negatif, then the contradiction.

*A-2-1- If $\lambda_1|(a"-B^n) \Longrightarrow \lambda_1|2(a"-B^n) \Longrightarrow \lambda_1|(A^m-2B^n)$ but $\lambda_1|(A^m+2B^n)$ hence $\lambda_1|2A^m \Longrightarrow \lambda_1|A^m$, $\lambda_1 \neq 2$, it follows $\lambda_1|A^m$ hence the contradiction with

(3.106).

*A-2-2- If $\lambda_1|(a" + B^n) \Longrightarrow \lambda_1|2(a" + B^n) \Longleftrightarrow \lambda_1|(A^m + 2B^n)$. We refind the condition (3.105).

Then the case $k_3 = 1$ is impossible.

3.2.2.4. Case $b|p \Rightarrow\ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ **and** $3|a$ **:**

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{3.76}$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.77}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$ hence using $cos^2\frac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a') \tag{3.78}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = p'(b - 4a') \tag{3.79}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{3.80}$$

A - Let λ a prime divisor of p' (we suppose p' not prime). From (3.80), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is a prime, then} \quad \lambda|A \tag{3.81}$$

From (3.79), as $\lambda|p'$ we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \tag{3.82}$$

If $\lambda|B^n$, λ is a prime $\lambda|B$, but $C^l = A^m + B^n$, then we have also :

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is a prime, then } \lambda|C \tag{3.83}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that p' is prime, from the equations (3.79) and (3.80), we obtain then:

$$p'|A^{2m} \Rightarrow p'|A^m \Rightarrow p'|A \tag{3.84}$$

and:

$$p'|B^nC^l \Rightarrow p'|B^n \quad \text{or } p'|C^l \tag{3.85}$$

$$\text{If} \quad p'|B^n \Rightarrow p'|B \tag{3.86}$$

$$\text{As} \quad C^l = A^m + B^n \quad \text{and that} \ p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l \Rightarrow p'|C \tag{3.87}$$

By the same way, if $p'|C^l$, we arrive to $p'|B$.

Hence: A ,B and C solutions of (2.1) have a common factor.

3.2.2.5. Case $b = 2p$ and $3|a$: We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' \Longrightarrow 2|A^m \Longrightarrow 2|a \Longrightarrow 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with a, b coprime.

3.2.2.6. Case $b = 4p$ and $3|a$: We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a'$$

Calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$

$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} \tag{3.88}$$

let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{3.89}$$

The left member of (3.89) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{3.90}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = k_2.k_3$.

$\diamondsuit$ - Firstly, we suppose that $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^mB^n = k_3.k_1 \tag{3.91}$$

Let μ a prime integer and $\mu|k_3$, then $\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

* If $\mu|A^m \Longrightarrow \mu|A$. As $\mu|k_3 \Longrightarrow \mu|p$ and that $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow \mu|B^{2n}$ then $\mu|B$, it follows $\mu|C^l$, hence A, B and C solutions of (2.1) have a common factor.

* If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{3.92}$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \tag{3.93}$$

Then :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{3.94}$$

As $b = 4p = 4k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists\mu' \in \mathbb{N}^*$ that $b = \mu\mu'$, we obtain:

$$\mu'\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m) \tag{3.95}$$

The last equation implies $\mu|4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu|B^n$ or $\mu|(B^n - A^m)$. If $\mu|B^n \Longrightarrow$ it is contradiction with (3.92). If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we have:

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n & \text{it is contradiction with } 3.92 \\ or \\ \mu = 3 \end{cases} \tag{3.96}$$

If $\mu = 3$, then $3|b$, but $3|a$ which is contradiction with a, b coprime.

$\diamondsuit$ - We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^mB^n = k_1 \tag{3.97}$$

$$p = k_2 \tag{3.98}$$

$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{p} \tag{3.99}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2\frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2\frac{\theta}{3} cos^2\frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{3.100}$$

but $a' = a"^2$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \tag{3.101}$$

The equation (3.100) becomes:

$$a"^2\lambda^2 = k_1^2 \Longrightarrow k_1 = a"\lambda \tag{3.102}$$

taking the positif square root. Using (3.97), we get :

$$k_1 = a"\lambda \tag{3.103}$$

But $k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \tag{3.104}$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2|\lambda \Longrightarrow 2|\lambda^2$. As $2|(b = 4p) \Longrightarrow 2|(a = 3a')$ which is contradiction with a, b coprime).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Longrightarrow \lambda_1|(A^m + 2B^n) \tag{3.105}$$

$$\Longrightarrow \lambda_1 \nmid A^m \quad if\ not \quad \lambda_1|2B^n \tag{3.106}$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Longrightarrow \lambda_1|B$, it follows:

$$\lambda_1|(b = 4p) \quad and \quad \lambda_1|A^m \Longrightarrow \lambda_1|2a" \Longrightarrow \lambda_1|a \tag{3.107}$$

hence the contradiction with a, b coprime.

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(A^m+2B^n)^2$ that is $\lambda_1|(A^{2m}+4A^mB^n+4B^{2n})$, we write it as $\lambda_1|(p+3A^mB^n+3B^{2n}) \Longrightarrow \lambda_1|(p+3B^n(A^m+2B^n)-3B^{2n})$. But $\lambda_1|(A^m+2B^n) \Longrightarrow \lambda_1|(p-3B^{2n})$, as $\lambda_1|(4p-a)$ hence by difference, we obtain $\lambda_1|(a-3(B^{2n}+p))$ or $\lambda_1|(3a'-3(B^{2n}+p)) \Longrightarrow \lambda_1|3(a'-B^{2n}-p) \Longrightarrow \lambda_1 = 3$ or $\lambda_1|(a' - (B^{2n} + p))$.

*A-1- If $\lambda_1 = 3|\lambda \Rightarrow 3|\lambda^2 \Rightarrow 3|b - a$ but $3|a \Longrightarrow 3|(p = b)$ hence the contradiction with a, b coprime.

*A-2- If $\lambda_1 \neq 3$ and $\lambda_1|(a' - B^{2n} - p) \Longrightarrow \lambda_1|(A^mB^n + B^{2n}) \Longrightarrow \lambda_1|B^n(A^m + 2B^n) \Longrightarrow \lambda_1|B^n$ or $\lambda_1|(A^m + 2B^n)$. The case $\lambda_1|B^n$ was studied above.

*A-2-1- If $\lambda_1|(A^n + 2B^n)$. We refind the condition (3.105).

Then the case $k_3 = 1$ is impossible.

3.2.2.7. Case $3|a$ and $b = 2p'$ $b \neq 2$ with $p'|p$: $3|a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, hence:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \tag{3.108}$$

Calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.109}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$ hence en using $cos^2\frac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a') \tag{3.110}$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^nC^l = k(p' - 2a') \tag{3.111}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{3.112}$$

A - Soit λ a prime divisor of k (we suppose k not a prime). From (3.112), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{3.113}$$

From (3.111), as $\lambda|k$, we have:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l \tag{3.114}$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda|C^l \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|C \tag{3.115}$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that k is prime, from the equations (3.111) and (3.112), we obtain:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{3.116}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{3.117}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{3.118}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\, k|B^n \Rightarrow k|C^l$$

$$\Rightarrow k|C \tag{3.119}$$

By the same way, if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (2.1) have a common factor.

3.2.2.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$: $3|a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ a case has been studied (paragraph 3.2.2.6), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \tag{3.120}$$

Writing B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{3.121}$$

But $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, hence en using $cos^2\frac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a') \tag{3.122}$$

As $p = b.p'$, and $p' > 1$, we have:

$$B^nC^l = k(p' - 2a') \tag{3.123}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{3.124}$$

A - Let λ a prime divisor of k (we suppose k not a prime). From (3.124), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as}\,\lambda\,\text{is prime then} \quad \lambda|A \tag{3.125}$$

From (3.123), as $\lambda|k$ we obtain:

$$\lambda|B^nC^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \tag{3.126}$$

If $\lambda|B^n$, λ is a prime $\lambda|B$, and as $C^l = A^m + B^n$, then we have:

$$\lambda|C^l \quad \text{as } \lambda \text{ is prime, then } \lambda|C \tag{3.127}$$

By the same way if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that k is prime, from the equations (3.123) and (3.124), we have:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \tag{3.128}$$

and:

$$k|B^nC^l \Rightarrow k|B^n \quad \text{or } k|C^l \tag{3.129}$$

$$\text{if} \quad k|B^n \Rightarrow k|B \tag{3.130}$$

$$\text{as} \quad C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m,\ k|B^n \Rightarrow k|C^l$$
$$\Rightarrow k|C \tag{3.131}$$

By the same way if $k|C^l$, we arrive to $k|B$.

Hence: A ,B and C solutions of (2.1) have a common factor. □

The main theorem is proved.

Tunis, November 2013.

References

[1] R. Daniel Mauldin. *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem.* Notice of AMS, Vol 44, $n°11$, 1997, pp 1436-1437.