Quaternionic versus Maxwell based differential calculus

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Abstract

Two quite different forms of differential calculus exist that both have physical significance. The most simple version is quaternionic differential calculus. Maxwell based differential calculus is based on the equations that Maxwell and others have developed in order to describe electromagnetic phenomena. Both approaches can be represented by four-component “fields” and four-component differential operators. Both approaches result in a dedicated non-homogeneous second order partial differential equation. These equations differ and offer solutions that differ in details.

Maxwell based differential calculus uses coordinate time t, where quaternionic differential calculus uses proper time τ. The consequence is that also the interpretation of speed differs between the two approaches. A more intriguing fact is that these differences involve a different space-progression model and different charges and currents. The impacts of these differences are not treated in this paper.

By adding an extra Maxwell based differential equation the conformance between the two approaches increases significantly.

The formulation of physics in Maxwell based differential calculus differs significantly from the formulation of physics in quaternionic differential calculus. It results in a different space-progression model. The choice between the two approaches influences the description of physical reality. However, the selected formulation does not affect physical reality.

The conclusion of the paper is that depending on the type of investigated phenomena either the Maxwell based approach or the quaternionic approach fits better as a descriptor. The Maxwell based approach fits better for describing wave behavior. The quaternionic approach fits better for the description of the embedding process.

Quaternionic differential calculus also fits better with the application of Hilbert spaces in quantum physics than Maxwell based differential calculus does. However, Maxwell based differential calculus is the general trend in current physical theories.

Contents

1 Introduction

In this paper the quaternionic differential equations are compared to Maxwell based differential equations [1][2].

In order to ease the comparison of the two approaches, we apply four-component “fields” and four-component operators. The parameter space is represented in a similar way by a similar but flat four-component “field”.

We start with a four-component differentiable “field” φ and we also define the corresponding four-component differential operator ∇. This nabla operator is applicable in situations in which the continuity of the field is not too violently disrupted. We tolerate point-like artefacts that manifest as sources, drains, charges or transient embedding locations.

The four-component approach is sometimes implemented with the help of spinors and corresponding matrices. Here we could, but will not apply that methodology. The method confuses more than that it elucidates the situation. Instead, we consider the scalar part as a separate part and we apply base vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ rather than the corresponding Pauli matrices [3]. For the same reason we do not apply Clifford algebra, Jordan algebra or Grassmann algebra.

The investigated approaches both start with a basic “field” φ. Gravitation concerns applications where this “field” φ is always and everywhere present. This kind of field is suited as continuum for embedding discrete objects. It is also suited as long range transport medium for carriers of information and energy. Electromagnetic theory concerns applications where the existence of the “field” φ is determined by a set of charges in the form of nearby point-like artifacts. These two kinds of basic fields are related, but that is subject of another paper [4].

Double differentiation results in a non-homogeneous second order partial differential equation that reveals how the basic “field” φ can be deformed or vibrated and how the artifacts control the behavior and the existence of the field. This second order partial differential equation differs between the two approaches.

2 Notation

Italic font face without subscript indicates four-component “fields” or four-component operators. Bold italic font face indicates 3D vectors and vector functions or 3D operators.

The four-component “fields” consist of a combination of a scalar field and a field of 3D vectors.

$$\varphi = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\} = \{\varphi_0, \boldsymbol{\varphi}\} = \{\varphi_0, \boldsymbol{i}\, \varphi_1 + \boldsymbol{j}\, \varphi_2 + \boldsymbol{k}\, \varphi_3\} \tag{1}$$

Both approaches start with a basic “field” φ. A set of related “fields” is derived from this basic “field”.

Both approaches use the 3D nabla operator $\boldsymbol{\nabla}$.

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{2}$$

This vector operator can be applied when the deformations of the subjected fields are not too violent.

$$\nabla = \{\nabla_{0,} \nabla_{1,} \nabla_{2,} \nabla_3\} = \{\nabla_{0,} \boldsymbol{\nabla}\} \tag{3}$$

For example, the four-component "field" ϕ is defined as:

$$\phi = \{\varphi_{0,} \boldsymbol{\varphi}\} = \nabla \varphi = \{\nabla \varphi_{0,} \nabla \boldsymbol{\varphi}\} \tag{4}$$

The four-component differential operator differs between the two approaches. Quaternionic differential calculus uses proper time τ and partial derivative $\nabla_0 = \nabla_\tau = \frac{\partial}{\partial \tau}$ and Maxwell based differential calculus uses coordinate time t and partial derivative $\nabla_t = \frac{\partial}{\partial t}$.

We suppose that ∇_0 commutes with $\boldsymbol{\nabla}$.

3 Parameter spaces

The parameter space is represented by a four-component flat "field":

$$\{x_{0,}\, \boldsymbol{i}\, x_1 + \boldsymbol{j}\, x_2 + \boldsymbol{k}\, x_3\} = \{x_{0,}\, \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\};\ x_0 = \tau \text{ or } x_0 = t \tag{1}$$

Infinitesimal coordinate time steps Δt and infinitesimal proper time $\Delta\tau$ steps are related by:

$$\text{Coordinate time step vector} = \text{proper time step vector} + \text{spatial step vector} \tag{2}$$

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3}$$

In the quaternionic model, the formula indicates that the coordinate time step corresponds to the step of a full quaternion, which is a superposition of a proper time step and a perpendicular pure spatial step.

An infinitesimal spacetime step Δs is usually presented as an infinitesimal proper time step $\Delta\tau$.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{4}$$

The signs on the right side form the Minkowski signature $(+,-,-,-)$.

The quaternionic model offers a Euclidean signature $(+,+,+,+)$ as is shown in formula (3).

4 Definition of the differential

Locally, the deformation of the "field" φ is supposed to be sufficiently moderate, such that the nabla operator ∇ can be applied.

In the two approaches, the differentiations $\{\phi_0, \boldsymbol{\phi}\} = \{\nabla_0, \boldsymbol{\nabla}\}\{\varphi_0, \boldsymbol{\varphi}\}$ have different definitions.

We do not go further than double differentiation. This double differentiation results in a non-homogeneous second order partial differential equation. In the two approaches, the non-homogeneous second order partial differential equations have a different format. In the Maxwell based approach this equation it is known as wave equation.

5 Mathematical facts

Here α is a real or complex valued scalar function. $\boldsymbol{a}$ is a vector function.

The following formulas are just mathematical facts that generally hold for vector differential calculus:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla a} \rangle \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{a} \tag{1}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \alpha \rangle \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \alpha \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha = \boldsymbol{0} \tag{3}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \tag{4}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \boldsymbol{a} \rangle = \boldsymbol{0} \tag{5}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{a} \tag{6}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha) = \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \alpha \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\nabla} \alpha \tag{7}$$

6 Quaternionic differential calculus

Quaternions are a combination of a real scalar a_0 and a 3D vector $\boldsymbol{a}$, which forms the imaginary part. Quaternionic number systems are division rings. This means that every non-zero element has an inverse. Hilbert spaces can only cope with number systems that are division rings. Quaternionic number systems form the most elaborate division rings.

Continuous quaternionic functions represent skew fields. Quaternionic differential calculus uses proper time τ as progression parameter. For that reason all quaternionic differential equations are inherently Lorentz invariant.

Due to their four dimensions quaternionic number systems exist in 16 symmetry flavors that only differ in their discrete symmetry sets [1][4].

$$a \equiv a_0 + \boldsymbol{a} \tag{1}$$

The quaternionic conjugate is defined as:

$$a^* \equiv a_0 - \boldsymbol{a} \tag{2}$$

The norm is defined as:

$$|a| \equiv \sqrt{a^* a} \tag{3}$$

The norm of a quaternionic function φ is defined as

$$\|\varphi\| \equiv \sqrt{\int_V \varphi^* \varphi \, dV} \tag{4}$$

The quaternionic product is defined as:

$$c = ab = a_0 b_0 + a_0 \boldsymbol{b} + b_0 \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{5}$$

The $\pm$ sign indicates the freedom of choice between a left handed and a right handed external vector product. This indicates that quaternionic number systems exist in several versions.

The quaternionic nabla is defined by:

$$\nabla \equiv \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \equiv \frac{\partial}{\partial \tau} + \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{6}$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{7}$$

In quaternionic differential calculus ***the differential can be defined as a product***.

$$\phi = \nabla \varphi \equiv (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_0 \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \boldsymbol{\nabla} \varphi_0 + \nabla_0 \boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{8}$$

$$\phi_0 = \nabla_0 \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{9}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla} \varphi_0 + \nabla_0 \boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{10}$$

The second derivative delivers an non-homogeneous equation:

$$\zeta = \nabla^* \phi = \nabla^* \nabla \varphi = (\nabla_0 - \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) \tag{11}$$

$$= \{\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi = \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$\zeta_0 = \nabla_0 \phi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\phi} \rangle \tag{12}$$

$$\boldsymbol{\zeta} = \boldsymbol{\nabla} \phi_0 + \nabla_0 \boldsymbol{\phi} \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} \tag{13}$$

Notice that:

$$\nabla_0 \phi = \nabla_0 \nabla_0 \varphi_0 - \nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \nabla_0 \boldsymbol{\nabla} \varphi_0 + \nabla_0 \nabla_0 \boldsymbol{\varphi} \pm \nabla_0 \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{14}$$

$$-\boldsymbol{\nabla} \phi = -\boldsymbol{\nabla} \nabla_0 \varphi_0 + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \mp \nabla_0 \boldsymbol{\nabla} \times \boldsymbol{\varphi} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \tag{15}$$

$$= -\boldsymbol{\nabla} \nabla_0 \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_0 \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \mp \nabla_0 \boldsymbol{\nabla} \times \boldsymbol{\varphi} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi}$$

$$\rho_0 = \langle \boldsymbol{\nabla}, \boldsymbol{\phi} \rangle = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 \tag{16}$$

$$\boldsymbol{\rho} = \boldsymbol{\nabla} \phi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} \tag{17}$$

The gauge transformation $\varphi \to \varphi + \chi$, where $\nabla^* \chi = 0$, does not change ϕ in $\phi = \nabla \varphi$.

6.1 Solutions of the quaternionic second order partial differential equation

Solutions of the second order partial differential equation depend on start and boundary conditions. This second order partial differential equation only holds for moderate discontinuity conditions.

$$\nabla^* \nabla \varphi \equiv \{\nabla_0 \nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi = \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \zeta \tag{1}$$

Apart from more detailed conditions the equation can be reduced to special forms. Examples are the Poisson equation, the screened Poisson equation, the Helmholtz equation and the homogeneous second order partial differential equation.

6.1.1 Poisson Equations

In the screened Poisson equation the first term is reduced to multiplication with a real constant λ:

$$\nabla_0 \nabla_0 \varphi = -\lambda^2 \varphi \tag{1}$$

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi = \zeta \tag{2}$$

The corresponding solution is superposition of screened Green's functions.

Green functions represent solutions for point sources.

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{3}$$

$$\varphi = \iiint G(\boldsymbol{r} - \boldsymbol{r}')\, \zeta(\boldsymbol{r}')\, d^3\boldsymbol{r}' \tag{4}$$

In this case the Green's function for spherical symmetric conditions ($\boldsymbol{r} = \boldsymbol{0}$) is:

$$G(r) = \frac{\exp(-\lambda\, r)}{r} \tag{5}$$

A zero value of λ offers the normal Poisson equation.

If $\lambda \neq 0$ then equation (1) has a solution

$$\varphi = a(\boldsymbol{x}) \exp(\pm i\, \omega\, \tau);\ \lambda = \pm i\, \omega \tag{8}$$

ω represents a parameter space wide clock frequency.

6.1.2 Coherent swarm of charges

A coherent swarm of charges that can be described by a continuous quaternionic location density function represents a blurred Green's function. For example, in case of an isotropic Gaussian distribution ζ_0 the N contributions of the swarm elements to the integral φ will on average equal $\mathfrak{G}(r) = \mathrm{ERF}(r)/r$. $N\, \mathfrak{G}(r)$ represents the local potential.

6.1.3 The homogeneous quaternionic second order partial differential equation

Despite the fact that the equation is quite similar to a wave equation it does not support waves. Locally, this quaternionic second order partial differential equation is considered to act in a rather flat continuum φ.

$$\nabla^*\nabla\varphi = \nabla_0\nabla_0\varphi + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\varphi = 0 \tag{1}$$

First we look at:

$$\nabla^*\nabla\varphi_0 = 0 \tag{2}$$

φ_0 is a scalar function. For isotropic conditions in three participating dimensions equation (2) has three dimensional spherical wave fronts as one group of its solutions.

By changing to polar coordinates it can be deduced that a general solution is given by:

$$\varphi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} = \frac{f_0(\boldsymbol{r} - c\tau)}{|\boldsymbol{r}|} \tag{3}$$

where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. This solution describes a shape keeping front.

In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

We use

$$\langle \nabla, \nabla \rangle \varphi_0 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_0}{\partial r}\right)\right) = -\frac{f''(\boldsymbol{i}r - c\tau)}{r} = \frac{1}{c^2}\frac{\partial^2 \varphi_0}{\partial \tau^2} \tag{4}$$

Next we consider the vector function $\boldsymbol{\varphi}$

$$\nabla^* \nabla \boldsymbol{\varphi} = 0 \tag{5}$$

Equation (5) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\varphi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{6}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function. Again the solution describes a shape keeping front, but this time the traveling front also keeps its amplitude.

The imaginary $\boldsymbol{i}$ represents a normalized base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one dimensional wave front.

These solutions do not represent waves. Instead they represent moving fronts that keep their shape.

6.1.4 No waves

A solution based on

$$\varphi = a(\boldsymbol{x}) \exp(\pm i\, \omega\, \tau) \tag{1}$$

$$\nabla_0 \nabla_0 \varphi = -\omega^2 \varphi \tag{2}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi = \omega^2 \varphi \tag{3}$$

does not lead to spatial waves. Thus separation of variables does not work well for the quaternionic homogeneous second order partial differential equation.

In order to show waves, a change to another parameter space is required. Instead of parameters τ and x we might select parameter t which is a function of τ and $\boldsymbol{x}$.

Let us investigate function $f(t, r)$ where $r = |\boldsymbol{x}|$ and $t = t(\tau, r)$. Now

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial t} \frac{\partial t}{\partial r} \qquad (4$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial t^2} \left(\frac{\partial t}{\partial r} \right)^2 + \frac{\partial f}{\partial t} \frac{\partial^2 t}{\partial r^2} \tag{5}$$

A homogeneous (spherical) wave equation looks like:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r^2} = 0 \tag{6}$$

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial t^2} \left(\frac{\partial t}{\partial r} \right)^2 - \frac{\partial f}{\partial t} \frac{\partial^2 t}{\partial r^2} = 0 \qquad (7$$

This is achieved if:

$$\frac{\partial t}{\partial r} = \pm 1 \text{ and } \frac{\partial f}{\partial t} = 0 \tag{8}$$

Taking $t = |x| = |\tau + \boldsymbol{x}|$ will do the job.

In contrast to equation (6) the equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial r^2} = 0 \qquad (9$$

does not describe waves, but like equation (6) it can describe shape keeping fronts.

7 Maxwell based differential calculus

The majority of the Maxwell based differential equations are quite similar to the quaternionic differential equations. We know that Maxwell based differential calculus supports a wave equation. As has been indicated above, the quaternionic second order partial differential equation is not suitable as a wave equation. For that reason, we introduce a new variable t, which replaces parameter τ. In Maxwell based differential calculus, the partial differential $\frac{\partial}{\partial t}$ replaces $\frac{\partial}{\partial \tau}$.

Let us investigate function $f(t, r)$ where $r = |\boldsymbol{x}|$ and $t = t(\tau, r)$. Now

(1

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial r}$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial t^2}\left(\frac{\partial t}{\partial r}\right)^2 + \frac{\partial f}{\partial t}\frac{\partial^2 t}{\partial r^2} \tag{2}$$

A homogeneous (spherical) wave equation looks like:

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r^2} = 0 \tag{3}$$

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial t^2}\left(\frac{\partial t}{\partial r}\right)^2 - \frac{\partial f}{\partial t}\frac{\partial^2 t}{\partial r^2} = 0 \qquad (4$$

This is achieved if:

$$\frac{\partial t}{\partial r} = \pm 1 \text{ and } \frac{\partial f}{\partial t} = 0 \tag{5}$$

Taking $t = |x| = |\tau + \boldsymbol{x}|$ will do the job.

In contrast to equation (3) the equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial r^2} = 0 \qquad (6$$

does not describe waves, but like equation (3) it can describe shape keeping fronts.

7.1 Maxwell-like equations

We start from the quaternionic differential and use control switch $\alpha = -1$.

$$\phi = \nabla\varphi = (\nabla_t + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_\tau\varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_\tau\boldsymbol{\varphi} \pm \boldsymbol{\nabla}\times\boldsymbol{\varphi} \tag{1}$$

$$\phi_0 = -\alpha\, \nabla_\tau\varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{2}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla}\varphi_0 + \nabla_\tau\boldsymbol{\varphi} - \boldsymbol{\nabla}\times\boldsymbol{\varphi} \tag{3}$$

We define new symbols:

$$\mathfrak{E} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_\tau\boldsymbol{\varphi} \tag{4}$$

$$\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\nabla}\times\boldsymbol{\varphi} \tag{5}$$

$$\nabla_\tau\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla}\times\nabla_\tau\boldsymbol{\varphi} = -\boldsymbol{\nabla}\times\mathfrak{E} \tag{6}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\varphi}) = \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi} \tag{7}$$

$$\nabla_\tau \boldsymbol{\mathfrak{E}} \equiv -\nabla_\tau \boldsymbol{\nabla}\varphi_0 - \nabla_\tau \nabla_\tau \boldsymbol{\varphi} \tag{8}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\mathfrak{E}} \rangle = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{9}$$

$$\nabla_\tau \phi_0 = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{10}$$

$$\boldsymbol{\nabla}\phi_0 \equiv -\alpha\, \nabla_\tau \boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{11}$$

$$\begin{aligned} \nabla_\tau \phi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\mathfrak{E}} \rangle &= -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \\ &= (-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\varphi_0 = \xi_0 \end{aligned} \tag{12}$$

$$\begin{aligned} -\boldsymbol{\nabla}\phi_0 + \alpha\, \nabla_\tau \boldsymbol{\mathfrak{E}} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \\ &= +\alpha\, \nabla_\tau \boldsymbol{\nabla}\varphi_0 + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \alpha\, \nabla_\tau \boldsymbol{\nabla}\varphi_0 - \alpha\, \nabla_\tau \nabla_\tau \boldsymbol{\varphi} - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} \\ &= (-\,\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\boldsymbol{\varphi} = \boldsymbol{\xi} \end{aligned} \tag{13}$$

$$\boldsymbol{\phi} = -\,\boldsymbol{\mathfrak{E}} \mp \boldsymbol{\mathcal{B}} \tag{14}$$

$$\begin{aligned} (-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\varphi_0 &= -\alpha \frac{\partial^2 \varphi_0}{\partial \tau^2} + \frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = \zeta_0 \\ &= \nabla_\tau \phi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\mathfrak{E}} \rangle \end{aligned} \tag{15}$$

$$\begin{aligned} (-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle)\boldsymbol{\varphi} &= -\alpha \frac{\partial^2 \boldsymbol{\varphi}}{\partial \tau^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = \boldsymbol{\zeta} \\ &= -\boldsymbol{\nabla}\phi_0 + \alpha\, \nabla_\tau \boldsymbol{\mathfrak{E}} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \end{aligned} \tag{16}$$

$$-\alpha \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \zeta = \nabla^* \nabla \varphi \tag{17}$$

With $\alpha = -1$, this corresponds to a Euclidean signature.

7.2 Maxwell equations

The Maxwell based formulas that are used here are taken from Bo Thidé; “Electromagnetic field theory”; second edition.

We use these formulas without units. Thus $c = 1;\ \varepsilon_0 = 1; \mu = 1$.

The Maxwell equations use coordinate time t. Just changing parameter τ into variable t, which is a function of τ and $\boldsymbol{x}$, does not affect field φ. It only changes the parameter space and the formulas that describe φ. This means that φ still obeys all the quaternionic partial differential equations, including the second order partial differential equation! With other words, Maxwell equations just offer a different view on field φ. We will use a selector α that will distinguish pure quaternionic differential formulas ($\alpha = -1$) from nearly equivalent Maxwell based differential formulas ($\alpha = +1$).

∇_t stands for $\frac{\partial}{\partial t}$. In quaternionic parameter space, function t plays the role of quaternionic distance $|x|$, where:

$$t = |x| = |\tau + \boldsymbol{x}| \tag{1}$$

In Maxwell equations the symbol $\boldsymbol{E}$ is usually used for the electrical field and symbol $\boldsymbol{B}$ is usually used for the magnetic field. Here we use the special symbols $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ in order to indicate the more general usage.

$$\boldsymbol{\mathcal{E}} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_t\boldsymbol{\varphi} \tag{2}$$

$$\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

In order to support the comparison, we introduce $\varkappa$ as a new scalar field. *This field is not subject of a regular Maxwell equation.*

$$\varkappa \equiv \alpha\, \nabla_t\varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \Leftrightarrow -\phi_0 = \alpha\, \nabla_\tau\varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{4}$$

In EMFT the scalar field $\varkappa$ is taken as a gauge with

$\alpha = 1$; Lorentz gauge

$\alpha = 0$; Coulomb gauge

$\alpha = -1$; Kirchhoff gauge.

In Maxwell based differential calculus the scalar field $\varkappa$ is ignored or it is taken equal to zero. As will be shown, zeroing $\varkappa$ is not necessary for the derivation of the Maxwell based wave equation.

$$\nabla_t\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \nabla_t\boldsymbol{\varphi} = -\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \boldsymbol{\varphi} \tag{6}$$

$$\nabla_t \boldsymbol{\mathcal{E}} \equiv -\nabla_t \boldsymbol{\nabla} \varphi_0 - \nabla_t \nabla_t \boldsymbol{\varphi} \tag{7}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}}\rangle = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \varphi_0 - \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{8}$$

$$\nabla_t \varkappa = \alpha \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \Leftrightarrow \nabla_\tau \phi_0 = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{9}$$

$$\boldsymbol{\nabla} \varkappa \equiv \alpha \nabla_t \boldsymbol{\nabla} \varphi_0 + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \Leftrightarrow \boldsymbol{\nabla} \phi_0 = -\alpha\, \nabla_\tau \boldsymbol{\nabla} \varphi_0 - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{10}$$

$$\nabla_t \varkappa + \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}}\rangle = \alpha \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \varphi_0 - \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{11}$$

$$= \alpha \nabla_t \nabla_t \varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \varphi_0 = (\alpha \nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle) \varphi_0$$

$$-\boldsymbol{\nabla} \varkappa - \alpha\, \nabla_t \boldsymbol{\mathcal{E}} + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \tag{12}$$

$$= -\alpha \nabla_t \boldsymbol{\nabla} \varphi_0 - \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \alpha \nabla_t \boldsymbol{\nabla} \varphi_0 + \alpha \nabla_t \nabla_t \boldsymbol{\varphi} + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \boldsymbol{\varphi}$$

$$= \alpha \nabla_t \nabla_t \boldsymbol{\varphi} - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle \boldsymbol{\varphi}$$

In quaternionic differential calculus $\alpha = -1$ and $\phi_0 = -\varkappa$. If in Maxwell based differential calculus the Lorentz gauge $\alpha = 1$ is applied, then the Maxwell based wave functions result:

$$(\alpha\, \nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle) \varphi_0 = \rho_0 = \nabla_t \varkappa + \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}}\rangle \Leftrightarrow \nabla_\tau \phi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\mathfrak{E}}\rangle \tag{13}$$

$$\alpha \frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = \rho_0 \tag{14}$$

$$(\alpha\, \nabla_t \nabla_t - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle) \boldsymbol{\varphi} = \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \alpha\, \nabla_t \boldsymbol{\mathcal{E}} - \boldsymbol{\nabla} \varkappa \Leftrightarrow \boldsymbol{\nabla} \phi_0 - \alpha\, \nabla_\tau \boldsymbol{\mathfrak{E}} + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \tag{15}$$

(16)

$$\alpha \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} - \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} - \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = \boldsymbol{J}$$

This corresponds to the Minkowski signature.

$$\{\rho_0, \boldsymbol{J}\} \Leftrightarrow \{\nabla_t \varkappa - \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle, -\boldsymbol{\nabla} \varkappa + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \alpha\, \nabla_\tau \boldsymbol{\mathcal{E}}\} \tag{17}$$

$$= \{\nabla_t \varkappa, -\boldsymbol{\nabla} \varkappa\} + \{\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle, \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} - \alpha\, \nabla_\tau \boldsymbol{\mathcal{E}}\} \tag{18}$$

Notice that we did not need to take $\varkappa = 0$, which is used in the gauge. Adding equation (3) as an extra Maxwell equation would bring Maxwell equations more in conformance with the equations of quaternionic differential calculus.

Notice the difference of the Minkowski signature of these equations with the Euclidean signature of the wave function of quaternionic differential calculus. This difference is enforced by the selection of the value of α.

7.2.1 Poisson equations

The Poisson equations for the Maxwell based differential calculus are similar to the Poisson equations for the quaternionic differential calculus.

$$\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = -\rho_0 = -\nabla_t \varkappa - \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle \tag{1}$$

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = -\boldsymbol{J} = \boldsymbol{\nabla} \varkappa + \alpha \nabla_t \boldsymbol{\mathcal{E}} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \tag{2}$$

7.2.2 The screened Poisson equation

The screened Poisson equation runs:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \chi - \lambda^2 \chi = \rho \tag{3}$$

In Maxwell based differential calculus this corresponds to:

$$\nabla_t \nabla_t\, \chi = \lambda^2\, \chi \tag{4}$$

A solution of this equation is

$$\chi = a(\boldsymbol{x}) \exp(\pm \lambda t) \tag{5}$$

This differs significantly from the quaternionic differential calculus version.

7.2.3 The Maxwell-Huygens homogeneous wave equation

In Maxell format the homogeneous wave equation uses coordinate time t. It runs as:

$$\frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = 0 \tag{1}$$

Papers on Huygens principle work with the homogeneous version of this formula or it uses the version with polar coordinates.

For isotropic conditions in three participating dimensions the general solution runs:

$$\varphi_0 \ = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{2}$$

This follows from

$$\langle \nabla, \nabla \rangle \varphi_0 \equiv \frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi_0}{\partial r} \right) \right) = \frac{f''(r - ct)}{r} = \frac{1}{c^2} \partial^2 \varphi_0 / \partial t^2 \tag{3}$$

In a single participating dimension the general solution runs:

$$\varphi_0 \ = f(x - ct), \text{ where } c = \pm 1; f \text{ is real} \tag{4}$$

8 Phenomena

The two approaches are two different views of the same investigated field. Each view corresponds to a set of equations. These sets differ in the format of some of the equations and the equations differ in the selected scalar parameter. The actual behavior of the field and its features are not affected by the selected view. The two sets of equations might describe different aspects of the reaction of the field on discontinuities. Two fields might show different behavior if the type of artifacts that cause the discontinuities of the field differ.

The used nabla operator can only be applied for simple discontinuities that can be represented by Dirac delta functions. The field response on such discontinuity is represented by a Green's function. Three different kinds of these discontinuities can occur.

- Oscillatory point-like discontinuities. This requires an oscillating trigger.
- Persistent point-like discontinuities. These can still move around in the field.
- Transient point-like discontinuities. These can still be grouped. These groups can move as a group.
    - The grouping can result in a coherent swarm
    - The grouping can result in a linear string

The kind of discontinuities will influence the characteristics of the field.

The definition of the quaternionic differential as

$$\phi = \nabla \varphi \tag{1}$$

defines this formula as a differential continuity equation. In fact the quaternionic second order partial differential equation represents the combination of two continuity equations

$$\zeta = \nabla^* \phi \tag{2}$$

$$\phi = \nabla \varphi \tag{3}$$

$$\zeta = \nabla^* \nabla \varphi \tag{4}$$

The phenomena are all solutions of the second order partial differential equation.

Thus the discontinuities can be interpreted as sources, as drains, as oscillatory triggers, as charges or as transient embedding events.

Examples are [5]:

- Electric charges. These can be interpreted as persistent sources or drains. These objects may move around.
- Elementary particles. Stochastic processes control the recurrent transient embedding of point-like artifacts that together form a coherent swarm and a hopping path. The swarm is characterized by a continuous location density distribution that conforms to the squared modulus of the wave function of the particle.
- Linear strings of moving artifacts. The fronts that represent the Green's functions of the artifacts move with constant speed along the path of the string and may rotate around the axis of the string. These strings may represent photons.

8.1 Coupling equation

The coupling equation represents a peculiar property of the quaternionic differential equation.

We start with two normalized functions ψ and φ and a normalizable function $\Phi = m\,\varphi$.

Here m is a fixed quaternion. Function φ can be adapted such that m becomes a real number.

$$\|\psi\| = \|\varphi\| = 1 \tag{1}$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla \psi = m\,\varphi \tag{2}$$

$\Phi = \nabla\psi$ defines the ***differential equation***. (3)

$\nabla\psi = \Phi$ formulates a differential ***continuity equation***. (4)

All quaternionic functions ψ and ψ that obey $\|\psi\| = \|\varphi\| = 1$, will also obey the ***coupling equation***.

$$\nabla\psi = m\,\varphi \tag{5}$$

9 Conclusion

Great similarities, but also essential differences exist between quaternionic differential equations and Maxwell based differential equations. In the quaternionic differential calculus the differential can be seen as a product between the four-component differential operator ∇ and the four-component field φ. That simple interpretation is not possible in Maxwell based differential calculus. It is not possible to interpret the Maxwell field as a function of a parameter space that directly corresponds to a number system. In the Maxwell approach, the parameter space has a Minkowski signature and does not form a division ring. In quaternionic function theory the parameter space has a Euclidean signature. This shows in the structure of the second order partial differential equations of the two approaches. These equations have solutions that differ between the two approaches. However, the Poisson part of the two second order partial differential equations is similar. This does not hold for the screened Poisson equation. The corresponding Green's functions are similar. Both homogeneous second order partial differential equations have solutions in the form of one dimensional and three dimensional fronts that keep their shape. The one dimensional fronts also keep their amplitude. Between the two approaches, these fronts have different mathematical representations. In applications, the fronts can act as carriers of information or energy. Only the Maxwell based version supports harmonic oscillations in the form of waves.

This paper replaces in the Maxwell based differential calculus the usage of gauges by the introduction of an extra scalar field $\varkappa$. This results in the same form of the Maxwell based wave equation as the Lorentz gauge delivers, but the non-homogeneous equation applies different charge and current distributions. The impact of the difference in charges and currents is not treated here.

The two approaches offer different views on the same basic field. These views reveal different phenomena of that basic field. They might also split basic fields in categories. One category reveals their properties with Maxwell based differential calculus and another category reveals their properties with quaternionic differential calculus. EM fields fit better in the first category. The field that describes our living space fits better in the second category.

9.1 Extra

Maxwell based differential calculus can be implemented with complex numbers. In that way it does not have to cope so intensively with non-commutative operators. As a consequence, gauges can be implemented easily.

A disadvantage of Maxwell based differential calculus is that spacetime based dynamic geometric data must first be dismantled into real numbers or complex numbers before Hilbert spaces can handle them.

Quaternionic differential calculus must circumvent most gauges. On the other hand, the quaternionic approach offers compensating advantages.

Hilbert spaces can directly cope with quaternions as eigenvalues of operators. This holds for separate quaternions and for quaternionic continuums [1][4].

Since proper time is Lorentz invariant, all quaternionic differential equations are inherently Lorentz invariant.

Due to the fact that quaternions form a number system with a non-commutative product, they can implement rotations:

$$c = a\, b/a$$

In this way they can implement the functionality of gluons [4]. This is not possible with parameters of the Maxwell based “field”.

10 References

[1] “Quaternions and quaternionic Hilbert spaces”; http://vixra.org/abs/1411.0178

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[3] The relation between spinor notation and quaternionic formulation is shown in “The Dirac Equation in Quaternionic Format”; http://vixra.org/abs/1505.0149

[4] “Foundations of a Mathematical Model of Physical Reality”; http://vixra.org/abs/1502.0186

[5] “On the Origin of Electric Charge”; http://vixra.org/abs/1507.0185 .